QUANTUM FIELD THEORY

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Preface

This is half a book on quantum field theory. Non-Abelian gauge theory including the electro-weak theory and more recent developments are not covered. On the other hand, the topics which are included are treated in a completely modern manner. For example, the functional integral representation is used as the basis for the theory, and it is presented in the first chapter. The nature of spontaneous symmetry breakdown is illustrated early on by the example of superfluid helium. Dimensional regularization is used throughout, and the renormalization group is introduced as soon as renormalization is discussed. The final chapter on quantum electrodynamics does contain some calculations whose strong-interaction QCD (non-Abelian) counterparts are discussed. In general, the material which is included is thoroughly covered starting from first principles. This has made for a rather long volume even though its range is limited. Perhaps someday I will write a second volume to make a whole book, but the remaining topics are treated in the references which are listed at the end of this volume.

A set of problems appears at the end of each chapter. The role of these problems should be explained. Many are exercises (some simple, some not) which help the reader gain facility with the ideas and techniques introduced in the text. Other problems review material which is assumed to be known in the text such as those which describe determinants at the end of Chapter 1 and special relativity at the end of Chapter 3. But many problems introduce additional topics that are not treated in the text, such as those which discuss chiral symmetry in Chapter 6 and the nature of the infrared divergence in Chapter 8. The problems which are not exercises contain the basic description of the ideas involved, but leave the details to be worked out by the reader. To master the material, one should work out all of the problems; at the very least one should read though them.

This book presents my views and understanding of quantum field theory resulting from years of thinking about it. It is a personal amalgam of many people's work. I have absorbed and then forgotten the origins of what I now understand in my own way. I believe that the resulting presentation has been enhanced by a consequent uniform style which does not make reference to the historical development. Indeed, following the historical order would, I believe, often set up notions that later have to be unlearned, which is poor pedagogy. I must, however, apologize to the many authors who will find parts of their work adopted in the text without reference. The reader who wants to track the historical development with proper credits can find it in the books listed in the references at the end of the text.

I am indebted to many people for their help and advice on quantum field theory. I owe a great deal to David Boulware who has shared his wisdom with me over many years; in particular, the simplified wave-packet treatment of scattering in Chapter 3, Section 4 and several of the proofs about the graphical structure of the theory given in Chapter 4, Section 4 are due to him. Laurence Yaffe read several of the chapters, and these have been much improved by his critical but insightful comments. Many improvements in the text have been brought about by the comments and suggestions of other colleagues and former students. I should like to thank the following for such aid: Pankaj Agrawal, Jeremy Bernstein, Marc Bernstein, Michael Boulware, Keith Clay, Scott Dodelson, Sangyong Jeon, Taeyeon Lee, Rafael Nepomechie, Stephen Sharpe, Donald Yennie, and Chengxing Zhai. The writing of this book was completed during a sabbatical leave, and I would like to acknowledge the hospitality of the Physics Department at Columbia University and the Theoretical Division at the Los Alamos National Laboratory where this was done.

This paperback edition has a number of minor errors corrected. I am indebted to Steven Strauss and especially to Yan Zhang for pointing them out to me.

1
Quantum mechanics, functional integrals

Quantum field theories are quantum mechanical systems with an infinite number of degrees of freedom. We start this book with a review of quantum mechanics in a form that is particularly suited for the field theory generalization. This form utilizes the Heisenberg picture and the functional integral representation of probability amplitudes — the transformation functions of Dirac. Some time will be spent developing and discussing the functional or path integral representation of quantum mechanics since this will enable us to learn a new formalism in a simple and familiar physical context — we won't have to learn too many new things all at once. The functional formulation will be illustrated by applying it at some length to the harmonic oscillator which is not only the simplest (and exactly soluble) non-trivial quantum system but which is also the dynamics of a single mode of a free field theory. The purpose of this review of quantum mechanics is to introduce in the simplest form much of the notation and methods used later in the book. Throughout this chapter we will explicitly work only with systems that have a single degree of freedom (one coordinate variable). The extension to systems with several degrees of freedom is straightforward.

1.1 Quantum mechanics

There is essentially only one question that one can ask in quantum mechanics.

The Only Quantum Question:
Suppose a physical system is prepared at time t_1, so that an observable property represented by the Hermitian operator $\zeta(t_1)$ has the value* ζ'.

* We shall often follow the convention of Dirac in denoting the eigenvalue of an operator with a prime.

The initial state is represented by the (unit-norm) eigen-ket $|\zeta', t_1\rangle$,

$$\zeta(t_1)|\zeta', t_1\rangle = |\zeta', t_1\rangle\zeta' . \tag{1.1.1}$$

Then what is the probability $\mathcal{P}$ that the observation of the system at a later time t_2 will yield the value ξ' for some property represented by the Hermitian operator $\xi(t_2)$?

Answer:

With the (unit-norm) eigen-bra $\langle\xi', t_2|$ defined by

$$\langle\xi', t_2|\ \xi(t_2) = \xi'\langle\xi', t_2| , \tag{1.1.2}$$

the probability is given by

$$\mathcal{P} = |\langle\xi', t_2|\zeta', t_1\rangle|^2 . \tag{1.1.3}$$

The probability amplitude $\langle\xi', t_2|\zeta', t_1\rangle$ is the bra(c)ket (Dirac's pun) or transformation function.[†]

We turn now to develop methods to compute transformation functions. In the Heisenberg picture, the time derivative of any operator is given by the commutator of that operator with the Hamiltonian,

$$\dot{\xi}(t) = \frac{d}{dt}\xi(t) = i[H, \xi(t)] . \tag{1.1.4}$$

This assumes that ξ has no explicit time dependence — that it contains no time-dependent numerical parameters. Note that

$$\begin{aligned}\xi' i\frac{d}{dt}\langle\xi', t| =& i\frac{d}{dt}[\langle\xi', t|\xi(t)] \\ =& i\Big(\frac{d}{dt}\langle\xi', t|\Big)\xi(t) + \langle\xi', t|[\xi(t), H] \\ =& i\Big(\frac{d}{dt}\langle\xi', t|\Big)\xi(t) + \xi'\langle\xi', t|\ H - \langle\xi', t|\ H\xi(t) .\end{aligned} \tag{1.1.5}$$

This is consistent with the definition of the dynamical evolution of a state vector which is defined to be

$$i\frac{d}{dt}\langle\xi', t|\ = \langle\xi', t|\ H . \tag{1.1.6}$$

If H does not contain the time t explicitly (if the Hamiltonian does not contain time-dependent external forces), one has the formal operator solution

$$\xi(t) = e^{iHt}\xi e^{-iHt} , \tag{1.1.7}$$

[†] This is the most detailed question that one may ask and answer. One can, of course, work with less detailed descriptions involving statistical mixtures of initial states and/or calculate average values of final observables. Note that in the Heisenberg picture which we employ, the time-dependent states appear as a basis of eigenstates of the time-dependent Heisenberg operators.

and

$$\langle\xi', t| = \langle\xi'|e^{-iHt} . \tag{1.1.8}$$

Here as usual, quantities with no time label refer to a standard, fixed time which is conventionally taken to be $t = 0$. In this case of time-independent dynamics

$$\langle\xi', t_2|\zeta', t_1\rangle = \langle\xi'|e^{-iH(t_2-t_1)}|\zeta'\rangle . \tag{1.1.9}$$

The familiar time-dependent wave function is the transformation function

$$\psi(q', t) = \langle q', t|\psi\rangle . \tag{1.1.10}$$

It is the probability amplitude of finding the eigenvalue q' of the coordinate operator $q(t)$ at time t starting with an initial state labeled by ψ.

Associated with the coordinate operator $q(t)$ is the canonically conjugate momentum at time t, $p(t)$. In the natural units that we (almost always) employ where $\hbar = 1$, this pair obeys the equal-time canonical commutation relation

$$[q(t), p(t)] = i . \tag{1.1.11}$$

Let us temporarily suppress the time variable t and define

$$U = 1 + i\delta q' p , \tag{1.1.12}$$

where $\delta q'$ is an infinitesimal numerical parameter. To infinitesimal order, this is a unitary operator,

$$U^\dagger U = 1 , \tag{1.1.13}$$

and hence when U acts on a state vector it preserves the norm (or length) of this vector. In view of the canonical commutation relation (1.1.11),

$$UqU^{-1} = q + i[p, q]\delta q' = q + \delta q' , \tag{1.1.14}$$

and so U is the operator which performs infinitesimal translations on the coordinate operator q (but leaves p unchanged). Therefore

$$\begin{aligned}\langle q'|Uq =& \langle q'|UqU^{-1}U = \langle q'|(q + \delta q')U \\ =& (q' + \delta q')\langle q'|U ,\end{aligned} \tag{1.1.15}$$

which implies that

$$\langle q'|U = \langle q' + \delta q'| . \tag{1.1.16}$$

The action of U on $\langle q'|$ is to infinitesimally translate the eigenvalue q' to $q' + \delta q'$, with $\langle q'|$ and $\langle q' + \delta q'|$ having the same norm. It should, however, be noted that in writing this equality a standard phase convention between

the two vectors $\langle q'|$ and $\langle q'+\delta q'|$ has been adopted. With

$$\langle q'|\left[1+i\delta q' p\right] = \langle q'+\delta q'| = \left[1+\delta q'\frac{\partial}{\partial q'}\right]\langle q'|\,, \tag{1.1.17}$$

we conclude that

$$\langle q'|p = \frac{1}{i}\frac{\partial}{\partial q'}\langle q'|\,. \tag{1.1.18}$$

The abstract momentum operator p when acting to the left on a coordinate eigenstate has the concrete realization as the derivative with respect to the eigenvalue q'. In an entirely similar way, defining

$$V = 1 - i\delta p' q\,, \tag{1.1.19}$$

produces

$$VpV^{-1} = p + \delta p'\,, \tag{1.1.20}$$

and leads to the conclusion that

$$\langle p'|q = -\frac{1}{i}\frac{\partial}{\partial p'}\langle p'|\,. \tag{1.1.21}$$

With these tools in hand, it is easy to derive the Schrödinger equation. In view of Eqs. (1.1.6) and (1.1.18),

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(q',t) &= \langle q',\, t|H\left(q(t),p(t);t\right)|\psi\rangle \\ &= H\left(q',\frac{1}{i}\frac{\partial}{\partial q'};t\right)\langle q',\, t|\psi\rangle \\ &= H\left(q',\frac{1}{i}\frac{\partial}{\partial q'};t\right)\psi(q',\,t)\,. \end{aligned} \tag{1.1.22}$$

It follows from our development that the ordering of q' and its derivatives in $H\left(q',\partial/i\partial q';t\right)$ is the same as that of the $q(t)$'s and $p(t)$'s in[‡] $H\left(q(t),p(t);t\right)$.

The commutator of one operator with the sum of two other operators is a sum of the separate commutators. The commutator of an operator with a product of other operators obeys the ordered distributive law:

$$\begin{aligned} [AB\cdots X, Z] = &[A,Z]\,B\cdots X + A\,[B,Z]\cdots X \\ &+ \ldots + AB\cdots[X,Z]\,. \end{aligned} \tag{1.1.23}$$

These are just the rules obeyed by the derivative and, in view of the canonical commutation relation (1.1.11), they imply that the commutator of an arbitrary function of the canonical variables q, p with p (or q) is the properly ordered derivative of that function with respect to $-iq$ (or ip). Thus the general equation of motion (1.1.4) applied to the canonical variables yields the Hamiltonian equations of motion, but in operator form:

$$\dot{q}(t) = i\left[H\left(q(t),p(t)\right),q(t)\right] = \frac{\partial H\left(q(t),p(t)\right)}{\partial p(t)}\,, \tag{1.1.24}$$

and

$$\dot{p}(t) = i\left[H\left(q(t),p(t)\right),p(t)\right] = -\frac{\partial H\left(q(t),p(t)\right)}{\partial q(t)}\,. \tag{1.1.25}$$

The $\langle q'|p'\rangle$ transformation function is a fundamental quantity. It is easy to compute:

$$\langle q'|p'\rangle p' = \langle q'|p|p'\rangle = \frac{1}{i}\frac{\partial}{\partial q'}\langle q'|p'\rangle\,, \tag{1.1.26}$$

with the integration of this differential equation giving

$$\langle q'|p'\rangle = \frac{1}{\sqrt{2\pi}}e^{iq'p'}\,. \tag{1.1.27}$$

To see why the integration constant was chosen to be $(2\pi)^{-1/2}$, we note that

$$\langle p'|q'\rangle = \langle q'|p'\rangle^* = \frac{1}{\sqrt{2\pi}}e^{-iq'p'}\,, \tag{1.1.28}$$

and so

$$\begin{aligned} \int dp'\langle q'|p'\rangle\langle p'|q''\rangle &= \int\frac{dp'}{2\pi}e^{ip'(q'-q'')} \\ &= \delta(q'-q'') = \langle q'|q''\rangle\,, \end{aligned} \tag{1.1.29}$$

which is what one must have to yield the resolution of the identity given by

$$\int dp'|p'\rangle\langle p'| = 1\,. \tag{1.1.30}$$

Similarly

$$\int dq'\langle p'|q'\rangle\langle q'|p''\rangle = \langle p'|p''\rangle\,, \tag{1.1.31}$$

[‡] This remark raises the question of the proper ordering of the operators $q(t)$ and $p(t)$ which appear in the original quantum Hamiltonian $H\left(q(t),p(t);t\right)$. The form of the quantum Hamiltonian is generally chosen so as to agree with its classical counterpart, and there is no difficulty for the simple form $H = p^2/2m + V(q)$. The classical correspondence does not, however, determine the operator ordering in more general systems except that the q's and p's must appear in some symmetrical fashion so as to give an Hermitian Hamiltonian. Thus there is no ambiguity to order $\hbar$ (which would involve the imaginary unit i and give a non-Hermitian term), but an ambiguity may arise in order $\hbar^2$, for example in the difference between qp^2q and $(qpqp+pqpq)/2$. In such cases, the correct operator ordering must be determined by some intrinsically quantum-mechanical principle.

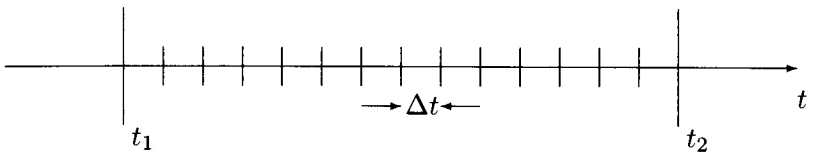


Fig. 1.1. The functional integral representation is obtained by dividing up the time interval in the transformation function into a large number of short intervals and inserting a resolution of the identity at each of these time slices.

and

$$\int dq'|q'\rangle\langle q'| = 1\,. \tag{1.1.32}$$

The knowledge of the $\langle q',\, t_2|q'',\, t_1\rangle$ transformation function enables one to compute any quantum amplitude, for

$$\begin{aligned}\langle \xi',\, t_2|\zeta',\, t_1\rangle &= \int dq'dq''\langle \xi',\, t_2|q',\, t_2\rangle\langle q',\, t_2|q'',\, t_1\rangle\langle q'',\, t_1|\zeta',\, t_1\rangle \\ &= \int dq'dq''\langle \xi'|q'\rangle\langle q',\, t_2|q'',\, t_1\rangle\langle q''|\zeta'\rangle\,. \end{aligned}\tag{1.1.33}$$

The second equality follows from the time-independence of the end-point wave functions, for example

$$i\frac{\partial}{\partial t}\langle q',\, t|\zeta',\, t\rangle = \langle q',\, t|(H-H)|\zeta',\, t\rangle = 0\,. \tag{1.1.34}$$

This time independence implies that we can, at any time, resolve the identity in terms of q or p eigenstates,

$$1 = \int dq'|q',\, t\rangle\langle q',\, t| = \int dp'|p',\, t\rangle\langle p',\, t|\,, \tag{1.1.35}$$

as was done in Eq. (1.1.33).

1.2 Functional integral representation

We shall now express the basic transformation function $\langle q',\, t_2|q'',\, t_1\rangle$ in terms of a functional integral. First, we divide up the time interval $t_2 - t_1$ into a large number of small bits,

$$T = t_2 - t_1 = (N+1)\Delta t\,, \tag{1.2.1}$$

as illustrated in Fig. 1.1. Inserting N resolutions of the identity in the q-representation, one for each of these time slices, yields

$$\begin{aligned}\langle q',\, t_2|q'',\, t_1\rangle &= \langle q',\, t_2|1\cdot 1\cdot 1\cdot 1\cdot\,\cdots\,\cdot 1\,|q'',\, t_1\rangle \\ &= \int \prod_{k=1}^{N} dq_k\langle q',\, t_2|q_N,\, t_2-\Delta t\rangle\langle q_N,\, t_2-\Delta t|q_{N-1},\, t_2-2\Delta t\rangle \\ &\quad \langle q_{N-1},\, t_2-2\Delta t|q_{N-2},\, t_2-3\Delta t\rangle\cdots\langle q_1,\, t_1+\Delta t|q'',\, t_1\rangle\,. \end{aligned}\tag{1.2.2}$$

Each of the transformation functions in this decomposition involves only the small time difference Δt. They may be computed in an approximation that becomes exact in the limit $\Delta t \to 0$. This limit is obtained by taking $N \to \infty$ which produces a functional integral — an integral over an infinite number of variables. It is a "functional" integral because, as we shall soon see, it essentially involves integrating over all possible functions defined in the interval $t_1\,,\ t_2$.

To compute the transformation functions for the short time interval Δt, we introduce a complete set of intermediate momentum eigenstates and write a typical factor as

$$\langle q_k,\, t+\Delta t|q_{k-1},\, t\rangle = \int dp_k\langle q_k,\, t+\Delta t|p_k,\, t\rangle\langle p_k,\, t|q_{k-1},\, t\rangle\,. \tag{1.2.3}$$

Here

$$\langle p_k,\, t|q_{k-1},\, t\rangle = \langle p_k|q_{k-1}\rangle = \frac{1}{\sqrt{2\pi}}\exp\{-ip_kq_{k-1}\}\,, \tag{1.2.4}$$

while the other factor is determined by the Schrödinger equation

$$i\frac{\partial}{\partial\Delta t}\langle q_k, t+\Delta t|p_k, t\rangle = \langle q_k, t+\Delta t|H\left(q(t+\Delta t), p(t+\Delta t); t+\Delta t\right)|p_k, t\rangle\,. \tag{1.2.5}$$

To solve this, we first write the Hamiltonian in an ordered form

$$H\left(q(t+\Delta t), p(t+\Delta t);\, t+\Delta t\right) = \tilde{H}\left(q(t+\Delta t); p(t+\Delta t);\, t+\Delta t\right) \tag{1.2.6}$$

in which all of the q-operators in $\tilde{H}$ appear to the left of all the p-operators. Although $\tilde{H}$ and H are exactly the same operator, their functional form in q and p may differ because of this ordering process that involves the canonical commutator (1.1.11). Having made this ordering in $\tilde{H}$, the $q(t+\Delta t)$ operators act directly on their eigenvector $\langle q_k, t+\Delta t|$, and they may be replaced by their eigenvalue q_k. To deal with the $p(t+\Delta t)$ operator, we use its equation of motion (1.1.25) to write

$$p(t+\Delta t) = p(t) - \frac{\partial H\left(q(t), p(t)\right)}{\partial q(t)}\Delta t + \cdots\,. \tag{1.2.7}$$

Since we are are concerned only with the limit $\Delta t \to 0$, it would appear

that we may simply place $p(t+\Delta t) = p(t)$. Then the $p(t)$ in the ordered version of the Hamiltonian act on $|p_k t\rangle$, and they may be replaced by their eigenvalues p_k. However, we must integrate over all p_k in Eq. (1.2.3). Neglecting operator ordering contributions and replacing the operators $p(t)$ in Eq. (1.2.7) by their eigenvalues p_k, we see that the Δt correction in Eq. (1.2.7) could become large. This could happen if the Hamiltonian contains velocity-dependent potentials which involve products of the operators $q(t)$ and $p(t)$, in which case $\dot{p}(t)$ involves $p(t)$ as well as $q(t)$. In most cases of interest the Hamiltonian involves the momentum in the following form:

$$H(q(t),p(t);t) = \frac{p(t)^2}{2m} + \frac{1}{2}[p(t)A(q(t);t) + A(q(t);t)p(t)] + V(q(t);t)\,. \tag{1.2.8}$$

Although the method we are about to explain can be extended to more general Hamiltonians, we shall restrict the discussion to the structure given in Eq. (1.2.8). The ordered form of the Hamiltonian reads

$$\tilde{H}(q(t);p(t);t) = \frac{p(t)^2}{2m} + A(q(t);t)p(t) - \frac{i}{2}\frac{\partial A(q(t);t)}{\partial q(t)} + V(q(t);t)\,. \tag{1.2.9}$$

The kinetic energy term $p_k^2\Delta t/2m$ in the exponential of the solution (1.2.15) that we are about to derive effectively cuts off* the p_k-integrals for momentum on the order of $p_k \sim \sqrt{m/\Delta t}$. Thus only the first Δt correction in Eq. (1.2.7) is significant and only its momentum-dependent part,

$$p(t+\Delta t) = p(t) - p(t)\frac{\partial A(q(t);t)}{\partial q(t)}\Delta t + \cdots\,, \tag{1.2.10}$$

is important in the kinetic energy term. Moreover, the extra commutator terms brought about by operator ordering in addition to that done in Eq. (1.2.9) vanish in the $\Delta t \to 0$ limit. Thus the ordered version of the Hamiltonian at the time $t+\Delta t$ may be written with sufficient accuracy as

$$\begin{aligned}\tilde{H}(q(t+\Delta t);p(t+\Delta t);t+\Delta t) \simeq{}& \frac{p(t)^2}{2m} - \frac{\partial A(q(t+\Delta t);t+\Delta t)}{\partial q(t+\Delta t)}\frac{p(t)^2}{m}\Delta t \\ &+ A(q(t+\Delta t);t+\Delta t)p(t) - \frac{i}{2}\frac{\partial A(q(t+\Delta t);t+\Delta t)}{\partial q(t+\Delta t)} \\ &+ V(q(t+\Delta t);t+\Delta t)\,.\end{aligned} \tag{1.2.11}$$

* If Δt were continued to a negative imaginary number, there would be an obvious Gaussian damping factor. With Δt real, damping is obtained in a more subtle way by rapid phase oscillation.

With the result (1.2.11) placed in the Schrödinger equation (1.2.5), the operators $q(t+\Delta t)$ and $p(t)$ act directly on their eigenvectors, and they may be replaced by their eigenvalues. Hence

$$i\frac{\partial}{\partial \Delta t}\langle q_k,\, t+\Delta t|p_k,\, t\rangle \simeq \tilde{H}\,(q_k;p_k;\,t+\Delta t)\,\langle q_k,\, t+\Delta t|p_k,\, t\rangle\,. \tag{1.2.12}$$

The time-variation of the parameters in the Hamiltonian gives a higher-order correction in Δt, and it may be neglected in the integration of this differential equation. The integration of the Δt correction in Eq. (1.2.11) involves $\Delta t^2/2$, and so

$$\langle q_k,\, t+\Delta t|p_k,\, t\rangle \simeq \exp\left\{-i\tilde{H}'(q_k;p_k;\,t+\Delta t)\Delta t\right\}\langle q_k|p_k\rangle\,, \tag{1.2.13}$$

where

$$\begin{aligned}\tilde{H}'(q_k,p_k;t+\Delta t) = {}& \frac{p_k^2}{2m} - \frac{\partial A(q_k;t+\Delta t)}{\partial q_k}\frac{p_k^2}{2m}\Delta t + A(q_k;t+\Delta t)p_k \\ &- \frac{i}{2}\frac{\partial A(q_k;t+\Delta t)}{\partial q_k} + V(q_k;t+\Delta t)\,.\end{aligned} \tag{1.2.14}$$

Using the explicit form of $\langle q_k|p_k\rangle$ gives

$$\langle q_k,\, t+\Delta t|p_k,\, t\rangle = \frac{1}{\sqrt{2\pi}}\exp\left\{iq_kp_k - i\tilde{H}'(q_k;p_k;\,t+\Delta t)\Delta t\right\}\,. \tag{1.2.15}$$

It should be emphasized that although Δt is small, the complete exponential form is needed to obtain convergence in the p_k integration. As stated above, the exponential form cuts off the p_k-integrals for momentum on the order of $p_k \sim \sqrt{m/\Delta t}$. The remaining evaluation of the short-time transformation function $\langle q_k,\, t+\Delta t|p_k,\, t\rangle$ is rather a shaggy-dog story. To the order we need, the correction involving the Δt factor in Eq. (1.2.14) alters the kinetic energy term to read $\bar{p}_k^2/2m$, where

$$\bar{p}_k = p_k\left[1 - \frac{1}{2}\frac{\partial A(q_k;t+\Delta t)}{\partial q_k}\Delta t\right]\,. \tag{1.2.16}$$

In the other terms in Eq. (1.2.15), p_k may be replaced by $\bar{p}_k$, since the difference vanishes in the $\Delta t \to 0$ limit. Thus we may simply change integration variables from p_k to $\bar{p}_k$ and then drop the bar. The net result of this Δt correction is to produce a Jacobian factor

$$\frac{dp_k}{d\bar{p}_k} = \left[1 + \frac{1}{2}\frac{\partial A(q_k;t+\Delta t)}{\partial q_k}\Delta t\right] \simeq \exp\left\{\frac{1}{2}\frac{\partial A(q_k;t+\Delta t)}{\partial q_k}\Delta t\right\}\,. \tag{1.2.17}$$

This just cancels the imaginary term in Eq. (1.2.14) brought about by the

operator ordering of pA, and thus the final result is given simply by

$$\langle q_k, t+\Delta t|q_{k-1}, t\rangle \simeq \int \frac{dp_k}{2\pi} \exp\{ip_k(q_k - q_{k-1}) - iH(q_k, p_k; t+\Delta t)\Delta t\}\,, \tag{1.2.18}$$

where

$$H(q_k, p_k; t) = \frac{p_k^2}{2m} + A(q_k; t)p_k + V(q_k; t)\,. \tag{1.2.19}$$

Inserting the evaluation given by Eq. (1.2.18) into the previous decomposition (1.2.2) of the transformation function yields

$$\langle q',\, t_2|q'',\, t_1\rangle = \int \frac{dp_{N+1}}{2\pi} \int \prod_{k=1}^{N} \frac{dq_k dp_k}{2\pi}$$
$$\exp\left\{ i \sum_{k=1}^{N+1} p_k(q_k - q_{k-1}) - i \sum_{k=1}^{N+1} H(q_k, p_k; t_1 + k\Delta t)\Delta t \right\}. \tag{1.2.20}$$

Here

$$q_{N+1} = q', \quad q_0 = q'', \tag{1.2.21}$$

and the limit $N \to \infty$ is to be taken. Note that, except for an extra final momentum integration (which appears because a q-q rather than a p-q or q-p transformation function is evaluated), Eq. (1.2.20) involves a product of phase space integrations which are invariant under canonical transformations.

Some notation is needed in order to express this limit in a compact form. We write

$$q_k = q(t_k) = q(t_1 + k\Delta t)\,, \qquad p_k = p(t_k) = p(t_1 + k\Delta t)\,, \tag{1.2.22}$$

and, in the limit $N \to \infty$,

$$\sum_{k=1}^{N+1} p_k(q_k - q_{k-1}) = \sum_{k=1}^{N+1} p(t_k) \frac{q(t_k) - q(t_k - \Delta t)}{\Delta t} \Delta t$$
$$= \int_{t_1}^{t_2} dt\, p(t)\dot{q}(t)\,, \tag{1.2.23}$$

$$\sum_{k=1}^{N+1} H(q_k, p_k;\, t_1 + k\Delta t)\Delta t = \sum_{k=1}^{N+1} H(q(t_k), p(t_k);\, t_k)\Delta t$$
$$= \int_{t_1}^{t_2} dt\, H(q(t), p(t);\, t)\,, \tag{1.2.24}$$

and

$$\frac{dp_{N+1}}{2\pi} \prod_{k=1}^{N} \frac{dq_k dp_k}{2\pi} = [dq][dp]\,. \tag{1.2.25}$$

In terms of this elegant notation

$$\langle q',\, t_2|q'',\, t_1\rangle = \int [dq][dp] \exp\left\{ i \int_{t_1}^{t_2} dt \left[p(t)\dot{q}(t) - H(q(t), p(t);\, t) \right] \right\}. \tag{1.2.26}$$

One must keep in mind that here $q(t)$ and $p(t)$ are numbers, not operators. Moreover, the time-integral which appears in the exponential must be considered to be defined as shown in Eqs. (1.2.23) and (1.2.24), with the number of steps N being precisely the number of integration variables N used in the functional measure (1.2.25). Although it is neat, one should not be misled by the simplicity of the notation. Now

$$L = p(t)\dot{q}(t) - H(q(t), p(t)\,; t) \tag{1.2.27}$$

is the classical Lagrangian in first order form[†] so the result is simply

$$\langle q',\, t_2|q'',\, t_1\rangle = \int [dq][dp] \exp\left\{ i \int_{t_1}^{t_2} dt L \right\}. \tag{1.2.28}$$

This is the functional integral formulation of quantum mechanics. We shall explain in some detail in Section 9 at the end of this chapter how this functional integral can also be viewed as a path integral or a sum over histories. Here we shall simply note that the integral is appropriately called a "functional" integral because it sums over all the values of $q(t)$ and $p(t)$ at every time t, and thus over all functions $q(t)$ and $p(t)$ in the interval t_1, t_2.

Suppose that there are no velocity-dependent forces, with the Hamiltonian given simply by

$$H = \frac{p^2}{2m} + V(q;\, t)\,. \tag{1.2.29}$$

In this case all the p-integrations can be done. Going back to the definition of the p-functional integral, completing the square in the exponential, shifting the p-integration variables, and performing the resulting Gaussian

[†] The first-order form of the Lagrangian is a function of the pair of canonically conjugate variables q and p, with only a single time derivative of one these variables appearing. The requirement that the corresponding action be stationary under variations of both of these variables yields the pair of Hamiltonian equations of motion, the classical counterpart of Eqs. (1.24) and (1.25). The second-order form of the Lagrangian contains only the variable q, with a term of the form $\dot{q}^2$. The stationary action principal applied to it produces the second-order in time Lagrange equation of motion. We shall often need to make this distinction between these two forms of the Lagrangian and action principles.

integrals gives

$$\int [dp] \exp\left\{ i \int dt [p\dot{q} - p^2/2m] \right\}$$
$$= \int \prod_{k=1}^{N+1} \frac{dp_k}{2\pi} \exp\left\{ -\frac{i\Delta t}{2m} \sum_{k=1}^{N+1} \left[(p_k - m\dot{q}_k)^2 - m^2 \dot{q}_k^2 \right] \right\}$$
$$= \left[\int \frac{dp}{2\pi} \exp\left\{ -\frac{i\Delta t}{2m} p^2 \right\} \right]^{N+1} \exp\left\{ i \sum_{k=1}^{N+1} \Delta t \frac{1}{2} m \dot{q}_k^2 \right\}$$
$$= \left[\frac{1}{2\pi} \sqrt{\frac{2\pi m}{i\Delta t}} \right]^{N+1} \exp\left\{ i \int_{t_1}^{t_2} dt \frac{1}{2} m \dot{q}(t)^2 \right\} . \qquad (1.2.30)$$

This yields the "second order form" of the functional integral representation of the transformation function,

$$\langle q', t_2 | q'', t_1 \rangle = \int [dq] \exp\left\{ i \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \dot{q}(t)^2 - V(q(t), t) \right] \right\} , \qquad (1.2.31)$$

where the measure now appears as

$$[dq] = \sqrt{\frac{m}{2\pi i \Delta t}} \prod_{k=1}^{N} \sqrt{\frac{m}{2\pi i \Delta t}} dq_k . \qquad (1.2.32)$$

Since $\Delta t = (t_2 - t_1)(N+1)^{-1}$, one could write the measure in the rather alarming form

$$[dq] = \left[\sqrt{\frac{m(N+1)}{2\pi i (t_2 - t_1)}} \right]^{N+1} \prod_{k=1}^{N} dq_k ,$$

with the overall factor out in front diverging very badly in the limit $N \to \infty$. However, the N-fold multiple q-integral produces a compensating factor to yield a finite and well-defined result in the $N \to \infty$ limit. The new form of the functional integral involves

$$L = \frac{1}{2} m \dot{q}^2 - V(q; t) , \qquad (1.2.33)$$

which is the classical Lagrangian in second order form, and the result can be written in the compact form

$$\langle q', t_2 | q'', t_1 \rangle = \int [dq] \exp\left\{ i \int_{t_1}^{t_2} dt L \right\} . \qquad (1.2.34)$$

It is easy to generalize this result to the case in which the Hamiltonian H contains in addition a velocity-dependent force obtained by the replacement $p \to p - A(q)$ such as that which arises when an external magnetic field is introduced; the resulting second order Lagrangian is given by the replacement $L \to L + A(q)\dot{q}$.

1.3 External forces

We turn now to explain the significance of an external force $f(t)$ for a quantum mechanical system. First we need to introduce the concept of a functional derivative. For a linear functional we write

$$\frac{\delta}{\delta f(t)} \int dt' f(t') g(t') = g(t) . \qquad (1.3.1)$$

Writing the integral as the (implicit) limit of a discrete sum,

$$\frac{\delta}{\delta f(t)} \int dt' f(t') g(t') = \frac{\delta}{\delta f(t)} \sum \Delta t f(t_k) g(t_k) , \qquad (1.3.2)$$

gives the correspondence

$$\frac{\delta}{\delta f(t)} \longleftrightarrow \frac{1}{\Delta t} \frac{\partial}{\partial f(t_l)} . \qquad (1.3.3)$$

In the general case there is a functional of the external force of the form

$$F[f] = \int dt_1 K_1(t_1) f(t_1) + \frac{1}{2} \int dt_1 dt_2 K_2(t_1, t_2) f(t_1) f(t_2)$$
$$+ \frac{1}{3!} \int dt_1 dt_2 dt_3 K_3(t_1, t_2, t_3) f(t_1) f(t_2) f(t_3) + \cdots , \qquad (1.3.4)$$

where $K_n(t_1 \cdots t_n)$ is completely symmetrical in $t_1 \cdots t_n$. (Any antisymmetrical part of a K would not contribute to the symmetrical integration.) In this general case

$$\frac{\delta}{\delta f(t)} F[f] = K_1(t) + \int dt_2 K_2(t, t_2) f(t_2)$$
$$+ \frac{1}{2} \int dt_2 dt_3 K_3(t, t_2, t_3) f(t_2) f(t_3) + \cdots . \qquad (1.3.5)$$

With an external force adjoined to the dynamics, the previous Lagrangian, which is now denoted as L_0, is augmented to read

$$L = L_0 + f(t) q(t) . \qquad (1.3.6)$$

To examine a use that can be made of an external force on a quantum-mechanical system, we first compute

$$\frac{1}{i} \frac{\delta}{\delta f(t)} \langle q', t_2 | q'', t_1 \rangle^f$$
$$= \int [dq] e^{\, i \int_{t_1}^{t_2} dt' L_0} \frac{1}{i} \frac{\delta}{\delta f(t)} e^{\, i \int_{t_1}^{t_2} dt' f(t') q(t')}$$
$$= \int [dq] q(t) e^{\, i \int_{t_1}^{t_2} dt' [L_0 + fq]} . \qquad (1.3.7)$$

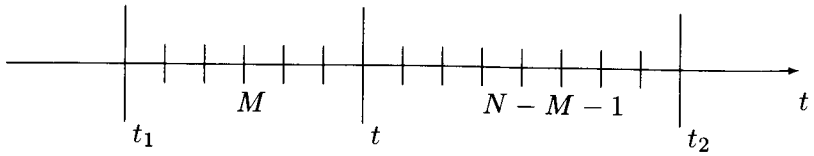


Fig. 1.2. Partition of the times in Eq. (3.8).

This functional integral may be decomposed into two parts, with one part involving all the time intervals before the time t and the other part all the time intervals after the time t:

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|q'', t_1\rangle^f = \int dq(t) \int [dq] e^{i\int_t^{t_2} dt'[L_0+fq]} q(t) \int [dq] e^{i\int_{t_1}^{t} dt'[L_0+fq]} . \quad (1.3.8)$$

The measure in Eq. (1.3.7) is given by

$$[dq] = \left[\sqrt{\frac{m}{2\pi i\Delta t}}\right]^{N+1} \prod_{k=1}^{N} dq_k . \quad (1.3.9)$$

On the other hand, the discrete partition of the time intervals in Eq. (1.3.8) appears as shown in Fig. 1.2. This partition corresponds to writing

$$\underbrace{\int_{t_1}^{t_2}[dq]}_{N \text{ factors}} = \int dq(t) \underbrace{\int_{t}^{t_2}[dq]}_{N-M-1 \text{ factors}} \underbrace{\int_{t_1}^{t}[dq]}_{M \text{ factors}} . \quad (1.3.10)$$

Since $[\sqrt{\ }]^{N+1} = [\sqrt{\ }]^{N-M} \times [\sqrt{\ }]^{M+1}$, all the square root factors in the measure work out. This corresponds to the fact that the functional integral must correctly reproduce the composition law

$$\langle q', t_2|q'', t_1\rangle^f = \int dq'''\langle q', t_2|q''', t\rangle^f \langle q''', t|q'', t_1\rangle^f . \quad (1.3.11)$$

Indeed, it is precisely this composition law that was used in the first place to derive the functional integral representation. Since each of the functional integrals defines a transformation function,

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|q'', t_1\rangle^f = \int dq(t)\langle q', t_2|q(t), t\rangle^f q(t)\langle q(t), t|q'', t_1\rangle^f . \quad (1.3.12)$$

One may be a little confused by the notation: $q(t)$ is an eigenvalue, not

an operator. This is rectified by replacing

$$q(t) \longrightarrow \bar{q}' , \quad (1.3.13)$$

and writing

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|q'', t_1\rangle^f = \int d\bar{q}'\langle q', t_2|\bar{q}', t\rangle^f \bar{q}'\langle \bar{q}', t|q'', t_1\rangle^f . \quad (1.3.14)$$

Using

$$\int d\bar{q}'\, |\bar{q}', t\rangle\bar{q}'\langle \bar{q}', t| = \int d\bar{q}'|\bar{q}', t\rangle\langle \bar{q}', t|q(t) = q(t) , \quad (1.3.15)$$

the result may be written as

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|q'', t_1\rangle^f = \langle q', t_2|q(t)|q'', t_1\rangle^f . \quad (1.3.16)$$

We find that a functional derivative with respect to the external force coupled to $q(t)$ constructs the matrix element of the Heisenberg operator $q(t)$. The functional derivative "inserts" the operator $q(t)$ in the matrix element.

Having taken one functional derivative, let us take another. On the one hand we have

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\langle q', t_2|q'', t_1\rangle^f = \int [dq] q(t)q(t') e^{i\int_{t_1}^{t_2} d\bar{t}[L_o+fq]} , \quad (1.3.17)$$

while on the other

$$\begin{aligned}\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\langle q', t_2|q'', t_1\rangle^f &= \frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|q(t')|q'', t_1\rangle^f \\ &= \frac{1}{i}\frac{\delta}{\delta f(t)}\int d\bar{q}'\langle q', t_2|\bar{q}', t'\rangle^f \bar{q}'\langle \bar{q}', t'|q'', t_1\rangle^f \\ &= \int d\bar{q}'\left(\frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|\bar{q}', t'\rangle^f\right)\bar{q}'\langle \bar{q}', t'|q'', t_1\rangle^f \\ &\quad + \int d\bar{q}'\langle q', t_2|\bar{q}', t'\rangle^f \bar{q}'\frac{1}{i}\frac{\delta}{\delta f(t)}\langle \bar{q}', t'|q'', t_1\rangle^f . \end{aligned} \quad (1.3.18)$$

Since a transformation function depends only upon the values of an external force during the time interval encompassed by the transformation function, the functional derivatives appearing in Eq. (1.3.18) have the forms

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle q', t_2|\bar{q}', t'\rangle^f = \begin{cases} \langle q, t_2|q(t)|\bar{q}', t'\rangle^f , & t > t' \\ 0, & t < t' , \end{cases} \quad (1.3.19)$$

and

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle \bar{q}',\, t'|q'',\, t_1\rangle^f = \begin{cases} 0, & t > t' \\ \langle \bar{q}',\, t'|q(t)|q'',\, t_1\rangle^f, & t < t' . \end{cases} \tag{1.3.20}$$

Removing the $\bar{q}'$-integration by using

$$\int d\bar{q}'|\bar{q}',\, t'\rangle\bar{q}'\langle \bar{q}',\, t'| = q(t') \tag{1.3.21}$$

thus yields

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\langle q',\, t_2|q'',\, t_1\rangle^f = \langle q',\, t_2|T(q(t)q(t'))|q'',\, t_1\rangle^f , \tag{1.3.22}$$

in which $T(\cdots)$ is the time-ordered product defined by

$$T(q(t)\, q(t')) = \begin{cases} q(t)q(t'), & t > t' , \\ q(t')q(t), & t' > t . \end{cases} \tag{1.3.23}$$

Note that the T symbol simply denotes an ordering process; it is not any kind of operator.

These results generalize with the definition

$$T(q(t)q(t')q(t'')\cdots) = q(\,)q(\,)q(\,)\cdots , \tag{1.3.24}$$

where the q-operators appear in an ordered form with ascending times to the left. In terms of this general time-ordered product,

$$\begin{aligned} &\frac{1}{i}\frac{\delta}{\delta f(t)}\,\frac{1}{i}\frac{\delta}{\delta f(t')}\,\frac{1}{i}\frac{\delta}{\delta f(t'')}\cdots\langle q',\, t_2|q'',\, t_1\rangle^f \\ &\quad = \langle q',\, t_2|T(q(t)q(t')q(t'')\cdots)|q'',\, t_1\rangle^f \\ &\quad = \int [dq]q(t)q(t')q(t'')\cdots\exp\Big\{i\int_{t_1}^{t_2} d\bar{t}\,[L_0 + fq]\Big\} . \end{aligned} \tag{1.3.25}$$

The addition of an external force to the Lagrangian in the functional integral produces a *generating functional* for matrix elements values containing the time-ordered product of an arbitrary number of coordinate operators.* Now that f has done its job, it may be taken to vanish, $f = 0$, to obtain

$$\langle q',\, t_2|Tq(t)q(t')\cdots|q'',\, t_1\rangle^{f=0} = \int [dq]q(t)q(t')\cdots\exp\Big\{i\int_{t_1}^{t_2} dt\, L_0\Big\} . \tag{1.3.26}$$

* Again it should be noted that the meaning of the symbol $q(t)$ depends upon where it appears. In the matrix element, it is an operator; in the functional integral, it is the (functional) integration variable.

Therefore, if $F[q]$ is any functional,

$$\langle q',\, t_2|T\ F[q]|q'',\, t_1\rangle = \int [dq]F[q]\exp\Big\{i\int_{t_1}^{t_2} dt\, L_0\Big\} , \tag{1.3.27}$$

where the matrix element on the left involves the time-ordering of the operator functional $F[q]$. In particular, the transformation function with an external force, the generating functional for the time-ordered products, is the force-free matrix element of the time-ordered form of an exponential:

$$\langle q',\, t_2|q'',\, t_1\rangle^f = \langle q',\, t_2|T\ e^{i\int_{t_1}^{t_2} dt\, q(t)f(t)}|q'',\, t_1\rangle^{f=0} . \tag{1.3.28}$$

This, of course, must be the case, for we have now gone about a circle in our logic. Indeed, the time-ordered exponential may be defined by time-ordering the operators that appear in its Taylor series development and we return to the first equality in Eq. (1.3.25). Note that since the operators are time-ordered, the time-ordered operator exponential obeys the group multiplication property

$$T\ e^{i\int_{t_1}^{t_3} dt\, q(t)f(t)} = T\ e^{i\int_{t_2}^{t_3} dt\, q(t)f(t)} T\ e^{i\int_{t_1}^{t_2} dt\, q(t)f(t)} , \tag{1.3.29}$$

for $t_3 > t_2 > t_1$. Taking a matrix element of this result and inserting a complete set of intermediate states recovers the composition law for the transformation function $\langle\cdots|\cdots\rangle^f$.

It is easy to see that the results generalize to the first-order form with, for example,

$$\langle q',\, t_2|T\ F[q,p]|q'',\, t_1\rangle = \int [dq][dp]\ F[q,p]e^{i\int_{t_1}^{t_2} dt\, L_0} . \tag{1.3.30}$$

As a simple application, suppose that the Hamiltonian is broken into two pieces,

$$H = H_0(q,p) + H_1(q,p) . \tag{1.3.31}$$

Then

$$\begin{aligned} \langle q',\, t_2|q'',\, t_1\rangle^{H_0+H_1} &= \int [dq][dp]e^{i\int_{t_1}^{t_2} dt\,[p\dot{q}-H_0-H_1]} \\ &= \int [dq][dp]e^{-i\int_{t_1}^{t_2} dt\, H_1}\ e^{i\int_{t_1}^{t_2} dt\, L_0} \\ &= \langle q',\, t_2|Te^{-i\int_{t_1}^{t_2} dt\, H_1}\ |q'',\, t_1\rangle^{H_0} . \end{aligned} \tag{1.3.32}$$

This is the interaction picture result. It expresses the time development of a transformation function under the full Hamiltonian H in terms of a matrix element of a time-ordered product involving the perturbation $H_1(q(t),p(t);t)$ in which the $q(t),p(t)$ are Heisenberg operators whose time development is governed by the unperturbed Hamiltonian H_0 . If

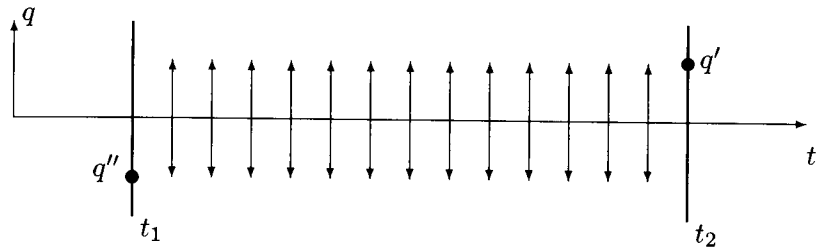


Fig. 1.3. The functional integral is obtained by integrating over $q(t)$ at each time slice but with $q(t)$ fixed on the boundaries.

H_1 is small, this result can be expanded in powers of H_1 to give the usual time-dependent perturbation theory.

1.4 Classical limit

The functional integral is constructed by integrating the $q(t)$ over the line $-\infty$ to $+\infty$ at each time slice, with the values on the initial and final boundaries fixed. This is shown in Fig. 1.3. It is often useful to approximate the functional integral by evaluating each time slice by the stationary phase method. Temporarily reverting to ordinary units that involve Planck's constant $\hbar$ in this section, the phase appears as

$$\exp\left\{\frac{i}{\hbar}\int_{t_1}^{t_2} dtL\right\} = \exp\left\{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\left[\frac{1}{2}m\dot{q}^2 + \cdots\right]\right\}. \tag{1.4.1}$$

The stationary phase approximation corresponds to the formal classical limit $\hbar \to 0$. The kinetic energy term

$$\dot{q}(t)^2\Delta t = \frac{(q(t_{k+1}) - q(t_k))^2}{\Delta t} \tag{1.4.2}$$

in the phase correlates adjacent terms and makes adjacent stationary points near to each other as shown in Fig. 1.4. In the limit $N \to \infty$ the stationary path is determined by

$$\delta\int_{t_1}^{t_2} dt\ L = 0\,, \tag{1.4.3}$$

with the endpoints fixed, $q(t_1) = q''$, $q(t_2) = q'$. The stationary condition (1.4.3) yields the classical Euler-Lagrange equations of motion that define the classical trajectory $q_{\rm cl}(t)$ between the initial coordinate q'' and the final coordinate q'. Thus the stationary phase path is given by the classical trajectory,

$$q(t)\Big|_{\text{stationary phase}} = q_{\rm cl}(t)\,. \tag{1.4.4}$$

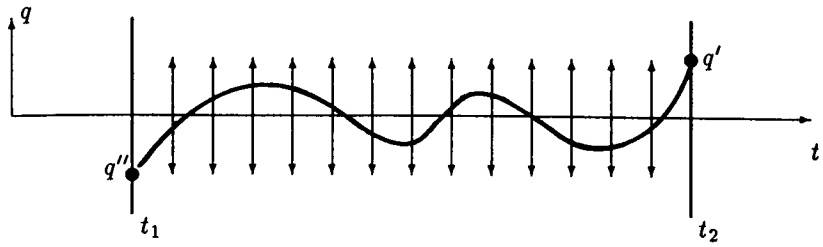


Fig. 1.4. The points of stationary phase of the functional integral give a continuous path which is just the classical trajectory as shown in the curve.

Note that the boundary conditions which have been employed reflect the character of the unique quantum question. They refer to the initially prepared value and to the final value to be observed. The boundary conditions refer symmetrically to the start and to the finish in contrast to the classical case where initial values of coordinate and velocity are specified.

The stationary phase — classical trajectory — provides a zeroth order approximation to the functional integral. This leading approximation is valid in the classical limit, and it provides the starting point for a systematic series much used in quantum field theory (the "loop expansion"). To see how this goes, we expand about the classical solution,

$$q(t) = q_{\rm cl}(t) + x(t)\,, \tag{1.4.5}$$

with

$$[dq] = [dx]\,, \tag{1.4.6}$$

since $q_{\rm cl}(t)$ plays the role of an additive constant at each time slice. Hence

$$\langle q',\, t_2|q'',\, t_1\rangle = e^{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, L_{\rm cl}}\int [dx] e^{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, L_2}\,, \tag{1.4.7}$$

where $L_{\rm cl} = L(q_{\rm cl})$ is the Lagrangian evaluated at the classical trajectory, and

$$L_2 = L(q_{\rm cl} + x) - L(q_{\rm cl})\,. \tag{1.4.8}$$

Since the action is stationary for variations about the classical solution, the action

$$\int dt\, L(q_{\rm cl} + x) \tag{1.4.9}$$

has no linear term in the fluctuation variable $x(t)$. Thus (neglecting total time derivatives which are of no significance) L_2 is at least of quadratic or-

der in $x(t)$. For a general potential function $V(q)$ (rather than the simple quadratic form of a harmonic oscillator), L_2 depends upon the classical solution $q_{\rm cl}(t)$ and is thus explicitly time-dependent. The functional integral over $x(t)$ has the boundary conditions $x(t_1) = 0 = x(t_2)$. The order $\hbar^0$ correction is exhibited by setting $x(t) = \hbar^{1/2}\bar{x}(t)$. Since L_2 starts out at quadratic order in $x(t)$, its leading term is now of order $\hbar$, which cancels the $1/\hbar$ factor in the exponential, indeed giving an order $\hbar^0$ correction. Thus this correction is given by the dynamics of a quadratic Lagrangian — a harmonic oscillator, but one with a time-dependent spring constant. This provides one more reason to study the harmonic oscillator, as will be done in the next section.

As discussed in the previous section, the addition of a time-dependent force,

$$L = L_0 + qf\,, \tag{1.4.10}$$

yields a generating functional for matrix elements of time-ordered products of the coordinate operators with, in particular,

$$\langle q',\, t_2|q(t)|q'',\, t_1\rangle^f = \frac{\hbar}{i}\,\frac{\delta}{\delta f(t)}\langle q',\, t_2|q'',\, t_1\rangle^f\,. \tag{1.4.11}$$

A normalized "expectation" value is defined by

$$\langle q(t)\rangle^f = \frac{\langle q',\, t_2|q(t)|q'',\, t_1\rangle^f}{\langle q',\, t_2|q'',\, t_1\rangle^f}\,. \tag{1.4.12}$$

It should be noted that this is *not* the usual "diagonal" expectation value which gives the average value for the measurement of an operator at a later time for some initially prepared state. Instead, this is a generalized "expectation" value which is related to the response of the system to small changes in an external force. The course of our work will prove the utility of such "expectation" values. The previous expansion about the classical trajectory (1.4.7) yields

$$\langle q(t)\rangle^f = \frac{\delta}{\delta f(t)}\int d\bar{t} L_{\rm cl} + \frac{\hbar}{i}\,\frac{\delta}{\delta f(t)}\ln\left\{\int [dx]\exp\left\{\frac{i}{\hbar}\int d\bar{t}\, L_2\right\}\right\}\,. \tag{1.4.13}$$

Here the second, quantum correction, results from the fact that L_2 depends upon $q_{\rm cl}(\bar{t})$, which in turn is a functional of $f(t)$. Neglecting this quantum correction gives

$$\begin{aligned}\langle q(t)\rangle^f &= \frac{\delta}{\delta f(t)}\int d\bar{t}\, L_{\rm cl}\\ &= q_{\rm cl}(t) + \int dt'\,\frac{\delta q_{\rm cl}(t')}{\delta f(t)}\,\frac{\delta}{\delta q_{\rm cl}(t')}\int d\bar{t}\, L_{\rm cl}\,.\end{aligned} \tag{1.4.14}$$

The functional derivative of the action with respect to the external force contains two parts: The explicit derivative with respect to this force — which produces $q_{\rm cl}(t)$ — plus the implicit derivative given by the second term which represents the chain rule for functional differentiation. In our case

$$\frac{\delta}{\delta q_{\rm cl}(t')}\int d\bar{t}\, L_{\rm cl} = 0\,, \tag{1.4.15}$$

by the Euler-Lagrange equations of motion.* Thus, in the classical limit,

$$\langle q(t)\rangle^f = q_{\rm cl}(t)\,. \tag{1.4.16}$$

1.5 Harmonic oscillator

We turn now to illustrate the methods that have been developed with the forced harmonic oscillator described by the Lagrangian

$$L = \frac{1}{2}m[\dot{q}^2 - \omega^2 q^2] + qf\,. \tag{1.5.1}$$

There are many reasons for studying this simple system. One was noted in the previous section in connection with the classical limit. Another arises from the fact that it describes the motion of a normal mode of a free field theory — a field with no self interactions. Since the equations of motion are linear, the quantum oscillator problem may be completely solved in closed form. This simplicity of the quadratic form of the oscillator Lagrangian is most directly exploited by using the formulation of the previous section. Since now

$$L_2 = \frac{1}{2}m\,[\dot{x}^2 - \omega^2 x^2] \tag{1.5.2}$$

is independent of the classical trajectory, the functional integral over the quantum fluctuation produces

$$\int [dx] e^{i\int_{t_1}^{t_2} dt\, L_2} = \langle q' = 0, t_2|q'' = 0, t_1\rangle^{f=0}\,, \tag{1.5.3}$$

which, since there is no external force, is time-translation invariant and thus is a function only of the time difference $t_2 - t_1$. Thus the transformation function for the driven harmonic oscillator has the structure

$$\langle q',\, t_2|q'',\, t_1\rangle^f = e^{i\int_{t_1}^{t_2} dt\, L_{\rm cl}}\langle q' = 0,\, t_2|q'' = 0,\, t_1\rangle^{f=0}\,. \tag{1.5.4}$$

The explicit form of the classical action is needed. First we note that the classical equation of motion is given by

$$m\left[\frac{d^2}{dt^2} + \omega^2\right] q_{\rm cl}(t) = f(t)\,. \tag{1.5.5}$$

* There could be a $t' = t_1$ or $t' = t_2$ end-point contribution, but $\delta q_{\rm cl}/\delta f$ vanishes at these end-points since here the classical trajectory is constrained by the boundary conditions.

Since

$$\int_{t_1}^{t_2} dt\, \dot{q}_{\rm cl}^2 = \int_{t_1}^{t_2} dt \Big\{ \frac{d}{dt}(q_{\rm cl}\dot{q}_{\rm cl}) - q_{\rm cl}\ddot{q}_{\rm cl} \Big\}\,, \tag{1.5.6}$$

using the classical equations of motion the action may be written as

$$\int_{t_1}^{t_2} dt\, L_{\rm cl} = \frac{m}{2}\big[q'\dot{q}_{\rm cl}(t_2) - q''\dot{q}_{\rm cl}(t_1)\big] + \frac{1}{2}\int_{t_1}^{t_2} dt\, f\, q_{\rm cl}\,, \tag{1.5.7}$$

which expresses the action in terms of the velocities of the classical trajectory $q_{\rm cl}(t)$ at the boundaries plus a simple integral over this trajectory.

The classical equation of motion (1.5.5) is solved with the Green's function defined by

$$m\Big[\frac{d^2}{dt^2} + \omega^2\Big] G(t,t') = \delta(t-t')\,. \tag{1.5.8}$$

It follows from this Green's function equation that

$$q_{\rm cl}(t) = \int_{t_1}^{t_2} dt' G(t,t') f(t') + \frac{q'\sin\omega(t-t_1) + q''\sin\omega(t_2-t)}{\sin\omega(t_2-t_1)} \tag{1.5.9}$$

obeys the harmonic oscillator equation of motion. The terms in addition to the integral of the Green's function with the external force in Eq. (1.5.9) are solutions to the free, undriven equation of motion. They ensure that the boundary conditions,

$$q_{\rm cl}(t_1) = q''\,, \quad q_{\rm cl}(t_2) = q'\,, \tag{1.5.10}$$

are obeyed if the Green's function is chosen to obey the boundary conditions,

$$G(t_2,t') = 0 = G(t_1,t')\,. \tag{1.5.11}$$

The Green's function obeying these boundary conditions is given by

$$G(t,t') = \frac{1}{2m\omega}\sin\omega|t-t'| + \frac{1}{2m\omega\sin\omega(t_2-t_1)} \times \left[\cos\omega(t_2-t_1)\cos\omega(t-t') - \cos\omega(t_1+t_2-t-t')\right]\,. \tag{1.5.12}$$

It is easy to verify that this function obeys the Green's function equation (1.5.8) when $t \neq t'$. The absolute value $|t-t'|$ has a discontinuous slope at $t = t'$ which yields the inhomogeneous $\delta(t-t')$ in the Green's function equation. To prove that this is indeed the case, one integrates the Green's function equation (1.5.8) in t over a small region from $t'-\epsilon$ to $t'+\epsilon$. In the limit $\epsilon \to 0$, this yields the requirement

$$m\frac{d}{dt}G(t,t')\Big|_{t=t'+\epsilon} - m\frac{d}{dt}G(t,t')\Big|_{t=t'-\epsilon} = 1\,, \tag{1.5.13}$$

which is obeyed by the solution (1.5.12). The additional terms, which are solutions of the homogeneous counterpart of the Green's function

equation, enforce the boundary conditions. Using simple trigonometric identities, terms can be combined to produce

$$G(t,t') = \frac{\cos\omega(t_2-t_1-|t-t'|) - \cos\omega(t_1+t_2-t-t')}{2m\omega\sin\omega(t_2-t_1)}\,. \tag{1.5.14}$$

The final and initial velocities are readily calculated from Eqs. (1.5.9) and (1.5.14):

$$\dot{q}_{\rm cl}(t_2) = \int_{t_1}^{t_2} dt' \frac{\sin\omega(t'-t_1)}{m\sin\omega(t_2-t_1)} f(t') + \omega\frac{q'\cos\omega(t_2-t_1) - q''}{\sin\omega(t_2-t_1)}\,, \tag{1.5.15}$$

and

$$\dot{q}_{\rm cl}(t_1) = \int_{t_1}^{t_2} dt' \frac{\sin\omega(t'-t_2)}{m\sin\omega(t_2-t_1)} f(t') + \omega\frac{q' - q''\cos\omega(t_2-t_1)}{\sin\omega(t_2-t_1)}\,. \tag{1.5.16}$$

Inserting these expressions into the previous form (1.5.7) of the action yields

$$\begin{aligned}\int_{t_1}^{t_2} dt\, L_{\rm cl} = &\int_{t_1}^{t_2} dt \frac{q'\sin\omega(t-t_1) - q''\sin\omega(t-t_2)}{\sin\omega(t_2-t_1)}\, f(t) \\ &+ \frac{m\omega}{2}\,\frac{(q'^2+q''^2)\cos\omega(t_2-t_1) - 2q''q'}{\sin\omega(t_2-t_1)} \\ &+ \frac{1}{2}\int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' f(t) G(t,t') f(t')\,.\end{aligned} \tag{1.5.17}$$

This is the expression to put into

$$\langle q',\, t_2 | q'',\, t_1\rangle^f = e^{i\int_{t_1}^{t_2} dt\, L_{\rm cl}} \langle q'=0,\, t_2 | q''=0,\, t_1\rangle^{f=0}\,. \tag{1.5.18}$$

The second factor on the right-hand side of this equation, the "quantum fluctuation factor", is only a function of the time difference $t_2 - t_1$, which makes no reference to the external force f. Hence the classical limit approximation of the previous section is now exact:

$$\begin{aligned}\langle q(t)\rangle^f &= \frac{\langle q',\, t_2 | q(t) | q'',\, t_1\rangle^f}{\langle q',\, t_2 | q'',\, t_1\rangle^f} \\ &= \frac{1}{i}\frac{\delta}{\delta f(t)} \ln\Big\{\langle q',\, t_2 | q'',\, t_1\rangle^f\Big\} \\ &= \frac{\delta}{\delta f(t)}\int_{t_1}^{t_2} d\bar{t}\, L_{\rm cl} = q_{\rm cl}(t)\,.\end{aligned} \tag{1.5.19}$$

Note that the explicit form for the classical action given in Eq. (1.5.17) confirms that

$$\begin{aligned}\frac{\delta}{\delta f(t)}\int_{t_1}^{t_2} d\bar{t}\, L_{\rm cl} &= \frac{q'\sin\omega(t-t_1) - q''\sin\omega(t-t_2)}{\sin\omega(t_2-t_1)} + \int_{t_1}^{t_2} dt' G(t,t') f(t') \\ &= q_{\rm cl}(t)\,.\end{aligned} \tag{1.5.20}$$

The double variational derivative

$$\frac{\delta}{\delta f(t)}\,\frac{\delta}{\delta f(t')}\int_{t_1}^{t_2} d\bar{t} L_{\rm cl} = \frac{\delta}{\delta f(t)} q_{\rm cl}(t') = G(t,t') \tag{1.5.21}$$

produces the Green's function. The connection of the classical trajectory to the quantum expectation value yields

$$\begin{aligned}\frac{\delta q_{\rm cl}(t')}{\delta f(t)} &= \frac{\delta}{\delta f(t)}\,\frac{\langle q',\, t_2|q(t')|q'',\, t_1\rangle^f}{\langle q',\, t_2|q'',\, t_1\rangle^f} \\ &= i\frac{\langle q',\, t_2|Tq(t)q(t')|q'',\, t_1\rangle^f}{\langle q',\, t_2|q'',\, t_1\rangle^f} \\ &\quad - i\frac{\langle q',\, t_2|q(t)|q'',\, t_1\rangle^f}{\langle q',\, t_2|q'',\, t_1\rangle^f}\,\frac{\langle q',\, t_2|q(t')|q'',\, t_1\rangle^f}{\langle q',\, t_2|q'',\, t_1\rangle^f}\,.\end{aligned} \tag{1.5.22}$$

Comparing this with the second equality in Eq. (1.5.21) shows that the Green's function describes the time-ordered correlation of the quantum mechanical fluctuations:

$$G(t,t') = i\Big\langle T\Big(q(t) - \langle q(t)\rangle\Big)\Big(q(t') - \langle q(t')\rangle\Big)\Big\rangle^f\,. \tag{1.5.23}$$

The final factor in Eq. (1.5.18) remains to be evaluated.* Using

$$\langle q'=0,\, t_2|q''=0,\, t_1\rangle^{f=0} = \int [dq] \exp\left\{ i\int_{t_1}^{t_2} dt \left[\frac{1}{2}m(\dot{q}^2 - \omega^2 q^2)\right]\right\}, \tag{1.5.24}$$

we first compute the change in this transformation function when the "spring constant" ω^2 is altered:

$$\begin{aligned}\frac{\partial}{\partial\omega^2}\langle\cdots|\cdots\rangle^{f=0} &= \int [dq]\left(-\frac{i}{2}m\int_{t_1}^{t_2} dt\; q(t)^2\right)\exp\{\ \} \\ &= -\frac{i}{2}m\int_{t_1}^{t_2} dt \langle\cdots|q(t)^2|\cdots\rangle^{f=0} \\ &= -\frac{1}{2}m\int_{t_1}^{t_2} dt\; G(t,t)\langle\cdots|\cdots\rangle^{f=0},\end{aligned} \tag{1.5.25}$$

where in the last equality we have used the connection of the Green's function with expectation value of two coordinate operators given in Eq. (1.5.23) for the case of a vanishing external force. This entails the remark that with $f = 0$, $\langle q(t)\rangle = 0$ for the $q' = 0 = q''$ states: Since $\langle q(t)\rangle$ obeys the free classical harmonic oscillator equation of motion and

* This factor could be found by placing the structure (5.18) into the Schrödinger equation, which gives a simple first-order differential equation that determines it. We follow more roundabout methods because they will be needed in the later field theory work.

is constrained to vanish at the start and end of its trajectory for arbitrary initial and final times t_1, t_2, it vanishes identically. The final factor $\langle\cdots|\cdots\rangle$ in Eq. (1.5.25) appears because the Green's function is the normalized expectation value; dividing by this factor gives

$$\frac{\partial}{\partial\omega^2}\ln\Big\{\langle q'=0\,, t_2|q''=0\,, t_1\rangle^{f=0}\Big\} = -\frac{1}{2}m\int_{t_1}^{t_2} dt\; G(t,t)\,. \tag{1.5.26}$$

Now

$$\begin{aligned}\int_{t_1}^{t_2} dt\; G(t,t) &= \int_{t_1}^{t_2} dt\;\frac{\cos\omega(t_2-t_1) - \cos\omega(t_1+t_2-2t)}{2m\,\omega\sin\omega(t_2-t_1)} \\ &= \frac{(t_2-t_1)\cos\omega(t_2-t_1) - \omega^{-1}\sin\omega(t_2-t_1)}{2m\,\omega\sin\omega(t_2-t_1)} \\ &= \frac{1}{m}\,\frac{\partial}{\partial\omega^2}\left\{\ln\sin\omega(t_2-t_1) - \ln\omega\right\},\end{aligned} \tag{1.5.27}$$

and so Eq. (1.5.26) is easily integrated to obtain

$$\langle q'=0\,,\, t_2|q''=0\,,\, t_1\rangle^{f=0} = (\text{const.})\cdot\sqrt{\frac{\omega}{\sin\omega(t_2-t_1)}}\,. \tag{1.5.28}$$

The overall constant of integration may be obtained from the free particle limit,

$$\lim_{\omega\to 0}\langle\cdots|\cdots\rangle = \langle q'=0|e^{-i\frac{p^2}{2m}\,(t_2-t_1)}|q''=0\rangle = (\text{const.})\,\sqrt{\frac{1}{t_2-t_1}}\,. \tag{1.5.29}$$

Using ordinary quantum mechanical techniques we calculate

$$\begin{aligned}&\langle q'=0|e^{-i\,(p^2/2m)\,(t_2-t_1)}|q''=0\rangle \\ &\quad = \int dp'\langle q'=0|e^{-i\,(p^2/2m)\,(t_2-t_1)}|p'\rangle\langle p'|q''=0\rangle \\ &\quad = \int dp' e^{-i\,(p'^2/2m)\,(t_2-t_1)}|\langle q'=0|p'\rangle|^2 \\ &\quad = \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i(t_2-t_1)}}\,,\end{aligned} \tag{1.5.30}$$

since $|\langle q'=0|p'\rangle|^2 = 1/2\pi$. Hence the constant which appears in Eq. (1.5.28) is given by

$$(\text{const.}) = \sqrt{\frac{m}{2\pi i}}\,. \tag{1.5.31}$$

We can now put everything together:

$$\langle q',\, t_2|q'',\, t_1\rangle^f = \sqrt{\frac{m\omega}{2\pi i \sin\omega(t_2-t_1)}}\ \exp\left\{i\int_{t_1}^{t_2} dt\, L_{\rm cl}\right\}. \qquad (1.5.32)$$

For the force-free case, $f = 0$, the solution becomes

$$\langle q',\, t_2|q'',\, t_1\rangle^{f=0} = \sqrt{\frac{m\omega}{2\pi i \sin\omega(t_2-t_1)}}$$
$$\exp\left\{\frac{i}{2}m\omega\,\frac{(q'^2+q''^2)\cos\omega(t_2-t_1)-2q''q'}{\sin\omega(t_2-t_1)}\right\}. \qquad (1.5.33)$$

We turn now to describe, within the context of the harmonic oscillator, a general method for obtaining the energy eigenvalues and their corresponding wave functions from the $\langle q',\, t_2|q'',\, t_1\rangle$ transformation function. First we recall that, for a general system whose Hamiltonian has no explicit (external) time dependence,

$$\langle q',\, t_2|q'',\, t_1\rangle = \langle q'|e^{-iH(t_2-t_1)}|q''\rangle\,. \qquad (1.5.34)$$

The eigenvalue spectrum of the Hamiltonian H must have a lower bound — there must be a lowest energy eigenvalue. If this were not the case the system would be unstable and be continually decaying into lower energy states through some small coupling to its external environment (such as the coupling to the radiation field for the electrons in an atom), a coupling that can be neglected for stable systems, but which (physically) cannot be neglected for unstable systems. A constant may always be added to the energy so there is no loss of generality in requiring that none of the energy eigenvalues are negative, which is to say that the Hamiltonian is a non-negative operator, $H \geq 0$. Having done this, the time difference $t_2 - t_1$ in the transformation function (1.5.34) may be analytically continued into the complex plane so long as its imaginary part remains negative so as to give a damped exponential. In the course of our work, we shall often make use of such an analytic continuation of the time which gives well-defined, damped exponential terms. It is convenient to analytically continue the time interval to a purely negative imaginary number,

$$t_2 - t_1 \to -iT\,. \qquad (1.5.35)$$

Introducing a complete set of energy eigenstates,

$$1 = \sum_{n=0}^{\infty} |n\rangle\langle n|\,, \qquad H|n\rangle = |n\rangle E_n\,, \qquad (1.5.36)$$

expresses

$$\langle q'|e^{-HT}|q''\rangle = \sum\nolimits_n \langle q'|e^{-HT}|n\rangle\langle n|q''\rangle$$
$$= \sum\nolimits_n e^{-E_nT}\langle q'|n\rangle\langle n|q''\rangle$$
$$= \sum\nolimits_n e^{-E_nT}\psi_n(q')\psi_n(q'')^*\,. \qquad (1.5.37)$$

Here $\psi_n(q')$ is just the time-independent Schrödinger wave function for the state with energy eigenvalue E_n. Suppose that the overall (imaginary) time T is large. Then with the energy eigenvalues ordered as $E_0 < E_1 < E_2 < \cdots$,

$$e^{-E_0T} \gg e^{-E_1T} \gg e^{-E_2T} \gg \cdots\,, \qquad (1.5.38)$$

and the individual terms in the expansion (1.5.37) are easily identified, which is the reason for our use of imaginary time.†

For the harmonic oscillator case, the analytic continuation to imaginary time (1.5.35) in Eq. (1.5.33) produces

$$\langle q'|e^{-HT}|q''\rangle = e^{-\frac{1}{2}\omega T}\sqrt{\frac{m\omega}{\pi}}\ \frac{1}{\sqrt{1-e^{-2\omega T}}}$$
$$\exp\left\{-\frac{m\omega}{2}\,\frac{(q'^2+q''^2)(1+e^{-2\omega T})-4q'q''e^{-\omega T}}{1-e^{-2\omega T}}\right\}. \qquad (1.5.39)$$

Taking the limit $T \to \infty$, and comparing with the expansion (1.5.37), it is easy to see that the leading term gives

$$\langle q'|e^{-HT}|q''\rangle \simeq e^{-E_0T}\psi_0(q')\psi_0(q'')^*$$
$$= e^{-\frac{1}{2}\omega T}\sqrt{\frac{m\omega}{\pi}}\exp\left\{-\frac{1}{2}m\omega(q'^2+q''^2)\right\}, \qquad (1.5.40)$$

from which one concludes that

$$E_0 = \frac{1}{2}\omega\,, \qquad (1.5.41)$$

and, with an overall phase convention,

$$\psi_0(q') = \left(\frac{m\omega}{\pi}\right)^{1/4}\exp\left\{-\frac{1}{2}m\omega q'^2\right\}. \qquad (1.5.42)$$

These are the familiar ground state energy and ground state wave function for the harmonic oscillator. This is an example of the general result that for large $T = i(t_2 - t_1)$ any transformation function $\langle\cdots t_2|\cdots t_1\rangle$ involves only ground state quantities. Going on to the next order in examining

† In the general case, some of the energy eigenvalues may be degenerate, and additional labels are needed to specify the state. Usually these are provided by obvious symmetries (such as rotational invariance) whose generators are operators that commute with the Hamiltonian. These operators may be simultaneously diagonalized and their eigenvalues together with the energy eigenvalue completely specify the state. If further degeneracies persist, they may be removed by the addition of a small perturbation to the Hamiltonian which can then be removed after the individual states are identified.

the transformation function (1.5.39) gives

$$\langle q'|e^{-HT}|q''\rangle \simeq e^{-\frac{1}{2}\omega T}\sqrt{\frac{m\omega}{\pi}}\,[1+2m\omega q'q''e^{-\omega T}]\exp\Big\{-\frac{1}{2}m\omega(q'^2+q''^2)\Big\}\,. \tag{1.5.43}$$

Comparing this with the decomposition (1.5.37) shows that the energy of the first excited state is given by

$$E_1 = \frac{3}{2}\omega\,, \tag{1.5.44}$$

with the wave function of this state being

$$\psi_1(q') = \sqrt{2m\omega}\,q'\psi_0(q')\,. \tag{1.5.45}$$

It is easy to see from the expansion of harmonic oscillator transformation function (1.5.39) that the energy of the n-th state is given by

$$E_n = \Big(n+\frac{1}{2}\Big)\omega\,, \tag{1.5.46}$$

but some work is needed (which will not be done here) to show that the corresponding wave function $\psi_n(q')$ can be identified as a Hermite polynomial times $\exp\Big\{-\frac{1}{2}m\omega q'^2\Big\}$, as is well-known.

The extension of these considerations to include an external force f is worth describing since it will appear in our later field theory work. Let us first require that the force $f(t)$ vanishes outside the interval $-t_0$, $+t_0$, with the initial and final measurement times lying outside of this interval, $t_1 < -t_0$ and $t_2 > +t_0$. In this configuration, a complete set of energy eigenstates may be introduced in the initial and final force-free regions to obtain

$$\langle q', t_2|q'', t_1\rangle^f = \sum_{n,n'}\langle q', t_2|n, +t_0\rangle^{f=0}\langle n, +t_0|n', -t_0\rangle^f\langle n', -t_0|q'', t_1\rangle^{f=0}\,. \tag{1.5.47}$$

In the limit $t_1 \to +i\infty$, $t_2 \to -i\infty$, the leading contribution is given by the intermediate ground states, $n = 0 = n'$. It is convenient to remove the zero-point energy by the redefinition

$$H \to H - \omega/2\,, \tag{1.5.48}$$

so that the ground state now has zero energy and is time-independent. With this redefinition, the limit becomes

$t_1 \to +i\infty$, $t_2 \to -i\infty$:

$$\langle q', t_2|q'', t_1\rangle^f = \psi_0(q')\langle 0, +t_0|0, -t_0\rangle^f\psi_0^*(q'')\,. \tag{1.5.49}$$

This infinite imaginary time limit of the transformation function solution (1.5.32), with the action (1.5.17), reproduces the structure shown in Eq. (1.5.49) and identifies

$$\langle 0, +t_0|0, -t_0\rangle^f = \exp\Big\{\frac{i}{2}\int_{-t_0}^{+t_0}dt\int_{-t_0}^{+t_0}dt'\,f(t)G_0(t,t')f(t')\Big\}\,. \tag{1.5.50}$$

In this limit, Eq. (1.5.14) yields

$$G_0(t,t') = \frac{i}{2m\omega}e^{-i\omega|t-t'|}\,. \tag{1.5.51}$$

One may now take the further limit $t_0 \to \infty$ to include the effect of any force $f(t)$, so long as this force vanishes in the remote past and distant future. It should be emphasized that the $t_2 \to \infty$, $t_1 \to -\infty$ limit of the functional integral produces essentially this result, so long as the limit is taken along a line in the complex plane which makes a small negative angle with the real axis. Moreover, one can be cavalier about the boundary conditions on the functional integral — the initial and final coordinate values. These correspond to initial and final wave function factors which are removed by a trivial renormalization to make the transformation function reduce to unity when the force vanishes — such factors may be (cavalierly) hidden in the functional integration measure since their precise value is never needed. We have described the character of this ground-state transformation function in some detail because its generalization plays a very important role in field theory.

1.6 Determinants

Previously we resorted to a differential ruse to compute

$$\langle q'=0\,,\,t_2|q''=0\,,\,t_1\rangle^{f=0} = \int[dq]\exp\Big\{i\int_{t_1}^{t_2}dt\frac{1}{2}m[\dot{q}^2-\omega^2q^2]\Big\}\,. \tag{1.6.1}$$

Here we shall directly calculate the functional integral in Eq. (6.1). We do this because, as with almost all of our work on quantum mechanics, the techniques that are developed here in a simple context will be used in our later work in field theory. In particular, we will now encounter infinite or Fredholm determinants that will often appear in our later work.

To perform the direct evaluation of the functional integral, we first recall that the Green's function obeys

$$m\Big[\frac{d^2}{dt^2}+\omega^2\Big]G(t,t') = \delta(t-t')\,. \tag{1.6.2}$$

Let us view the Green's function as an (infinite) matrix whose rows and columns are labeled by the (continuous) indices t and t'. This is equivalent to viewing the Green's function as the matrix element of an operator G,

$$G(t,t') = \langle t|G|t'\rangle\,. \tag{1.6.3}$$

Here, although the Dirac bra and ket notation is used, it should be emphasized that this is just a convenient notation and has nothing to do with physical states. Since the δ-function obeys

$$\int d\bar{t}\delta(t-\bar{t})\delta(\bar{t}-t') = \delta(t-t')\,, \tag{1.6.4}$$

it appears as the matrix element of the identity operator, and we write

$$\delta(t-t') = \langle t|1|t'\rangle\,. \tag{1.6.5}$$

This is consistent with the completeness statement

$$\int dt|t\rangle\langle t| = 1\,. \tag{1.6.6}$$

Finally, we define

$$m\Big[\frac{d^2}{dt^2}+\omega^2\Big]\delta(t-t') = \langle t|G^{-1}|t'\rangle\,. \tag{1.6.7}$$

The Green's function equation (1.6.2) can now be expressed as

$$\int d\bar{t}\langle t|G^{-1}|\bar{t}\rangle\langle\bar{t}|G|t'\rangle = \langle t|1|t'\rangle\,, \tag{1.6.8}$$

or, using the completeness of the integral over the intermediate vectors,

$$\langle t|G^{-1}G|t'\rangle = \langle t|1|t'\rangle\,, \tag{1.6.9}$$

which implies that, as an operator statement,

$$G^{-1}G = 1\,. \tag{1.6.10}$$

The harmonic oscillator action can be written in a compact operator notation. Since $q(t)$ vanishes at the boundary times, we may integrate by parts,

$$\int_{t_1}^{t_2} dt\frac{1}{2}m[\dot{q}^2-\omega^2q^2] = -\frac{1}{2}\int_{t_1}^{t_2} dt\, qm\Big[\frac{d^2}{dt^2}+\omega^2\Big]q\,, \tag{1.6.11}$$

to write

$$\int_{t_1}^{t_2} dt\frac{1}{2}m[\dot{q}^2-\omega^2q^2] = -\frac{1}{2}qG^{-1}q\,. \tag{1.6.12}$$

Thus

$$\langle q'=0\,,t_2|q''=0\,,t_1\rangle^{f=0} = \int[dq]e^{-\frac{i}{2}qG^{-1}q}\,. \tag{1.6.13}$$

Returning to a discrete break up of the time interval $t_1 < t < t_2$, the functional integral on the right-hand side of Eq. (1.6.13) is of the general form of a multidimensional Gaussian integral,

$$\int\prod_{n=1}^{N} dx_n \exp\left\{-\frac{1}{2}\sum_{kl=1}^{N} x_kM_{kl}x_l\right\}\,. \tag{1.6.14}$$

Clearly the matrix M may be taken to be symmetrical,

$$M_{kl} = M_{lk}\,, \tag{1.6.15}$$

since any antisymmetric part of M does not contribute to the symmetric sum over k,l. This symmetrical matrix can be diagonalized,

$$M = RDR^T\,, \tag{1.6.16}$$

$$D = \begin{pmatrix} d_1 & & 0 & \\ & d_2 & & \\ 0 & & d_3 & \\ & & & \ddots \end{pmatrix}\,, \tag{1.6.17}$$

where R is an orthogonal matrix

$$R^TR = 1\,. \tag{1.6.18}$$

The transposed matrix is the matrix inverse, $R^T = R^{-1}$. In component form

$$M_{kl} = \sum\nolimits_m R_{km}R_{lm}d_m\,, \tag{1.6.19}$$

and

$$\sum\nolimits_{kl} x_kM_{kl}x_l = \sum\nolimits_{klm} x_kR_{km}x_lR_{lm}d_m\,. \tag{1.6.20}$$

This leads us to make the variable change

$$y_m = \sum\nolimits_k x_kR_{km} \tag{1.6.21}$$

to get

$$\int\Big(\prod dx_n\Big)e^{-\frac{1}{2}\sum_{kl}x_kM_{kl}x_l} = \int\Big(\prod dy_n\Big)\,\frac{\partial(x)}{\partial(y)}e^{-\frac{1}{2}\sum_k y_k^2d_k}\,. \tag{1.6.22}$$

The integration variable change entails the Jacobian determinant

$$\frac{\partial(x)}{\partial(y)} = \left[\frac{\partial(y)}{\partial(x)}\right]^{-1} \tag{1.6.23}$$

with

$$\frac{\partial(y)}{\partial(x)} = \det\,\frac{\partial y_k}{\partial x_l} = \det\,R\,. \tag{1.6.24}$$

Since, in general, the determinant of any matrix is equal to the determinant of the transposed matrix and the product of two determinants is the determinant of the product of the two matrices while, in particular, the matrix R is orthogonal,

$$(\det R)^2 = \det R\det R^T = \det RR^{-1} = \det 1 = 1\,. \tag{1.6.25}$$

Since the integration volume must be positive, it is, in fact, the absolute value of the determinant that must be used,

$$|\det R| = +1\,, \tag{1.6.26}$$

and so

$$\int \left(\prod dx_n\right) e^{-\frac{1}{2}\sum_{kl} x_k M_{kl} x_l} = \prod_{k=1}^{N} \int dy_k e^{-\frac{1}{2} d_k y_k^2}$$
$$= \frac{(2\pi)^{N/2}}{\sqrt{\prod_{k=1}^{N} d_k}}\,. \tag{1.6.27}$$

Here

$$\prod d_k = \det D = \det R^T M R = \det R^T \det R \det M = \det M\,. \tag{1.6.28}$$

Thus the multiple Gaussian integral may be expressed in terms of a determinant that is independent of the particular coordinate system which was used to evaluate the integral:

$$\int \prod_{n=1}^{N} dx_n \exp\left\{-\frac{1}{2}\sum_{kl=1}^{N} x_k M_{kl} x_l\right\} = \frac{(2\pi)^{N/2}}{[\det M]^{1/2}}\,. \tag{1.6.29}$$

We find that we must understand determinants. We recall that

$$\sum\nolimits_m M_{km} C_{ml}(M) = \delta_{kl} \det M\,, \tag{1.6.30}$$

where $C_{ml}(M)$ is the transposed, signed, cofactor matrix,

$$C_{ml}(M) = (-1)^{l+m}[\det M] \left.\begin{matrix} m-\text{th column} \\ \ell-\text{th row} \end{matrix}\right\} \text{omitted}\,. \tag{1.6.31}$$

This is the formula which gives the determinantal construction of the matrix inverse,

$$(M^{-1})_{kl} = \frac{C_{kl}(M)}{\det M}\,. \tag{1.6.32}$$

Taking $k = l$ in Eq. (1.6.30) gives

$$\det M = \sum\nolimits_m M_{km} C_{mk}(M)\,, \tag{1.6.33}$$

and so since $C_{mk}(M)$ does not contain M_{km},

$$\frac{\partial}{\partial M_{km}} \det M = C_{mk} = (M^{-1})_{mk} \det M\,, \tag{1.6.34}$$

or

$$\frac{\partial}{\partial M_{km}} \ln \det M = (M^{-1})_{mk}\,. \tag{1.6.35}$$

Accordingly,

$$\delta \ln \det M = \sum\nolimits_{km} \delta M_{km} \frac{\partial \ln \det M}{\partial M_{km}}$$
$$= \sum\nolimits_{km} \delta M_{km} (M^{-1})_{mk}$$
$$= \mathrm{tr}\, \delta M M^{-1} = \mathrm{tr}\, M^{-1} \delta M\,, \tag{1.6.36}$$

where in the last equality the cyclic symmetry of the trace has been used. This variational statement can, in fact, be taken as the *defining equation* for $\ln \det M$ along with the boundary condition that $\ln \det 1 = \ln 1 = 0$.*

Determinants which involve a matrix of the form $M = 1 - X$, where X is "small" will often be encountered. In this case

$$\delta \ln \det (1 - X) = -\mathrm{tr} \frac{1}{1 - X}\, \delta X$$
$$= -\mathrm{tr}\{1 + X + X^2 + X^3 + \ldots\} \delta X\,. \tag{1.6.37}$$

According to the cyclic symmetry of the trace,

$$\delta \mathrm{tr} X^2 = \mathrm{tr}(\delta X X + X \delta X) = 2\mathrm{tr} X \delta X\,, \tag{1.6.38}$$

$$\delta\ \mathrm{tr} X^3 = \mathrm{tr}(\delta X X^2 + X \delta X X + X^2 \delta X) = 3\mathrm{tr} X^2 \delta X\,, \tag{1.6.39}$$

with, in general,

$$\delta\, \mathrm{tr}\, X^n = n\, \mathrm{tr}\, X^{n-1} \delta X\,. \tag{1.6.40}$$

Hence

$$\delta \ln \det(1 - X) = -\delta\, \mathrm{tr} \left\{X + \frac{1}{2} X^2 + \frac{1}{3} X^3 + \ldots\right\}$$
$$= \delta\, \mathrm{tr} \ln(1 - X)\,, \tag{1.6.41}$$

and we conclude that

$$\det(1 - X) = \exp\{\mathrm{tr}\, \ln(1 - X)\}\,. \tag{1.6.42}$$

The Gaussian functional integral that motivated the discussion involved a kernel of the form

$$K(t, t') = \langle t | K | t' \rangle\,, \tag{1.6.43}$$

which can be viewed as an infinite matrix whose indices are the continuous variables t and t'. The matrix traces now become

$$\mathrm{Tr} K = \int dt K(t, t)\,, \tag{1.6.44}$$

* For further details see Problem 2.

$$\mathrm{Tr}K^2 = \int dtdt' K(t,t')K(t',t)\,, \tag{1.6.45}$$

and so forth. In this way an infinite or Fredholm determinant can be defined:

$$\mathrm{Det}\,[1-K] = \exp\{\mathrm{Tr}\,\ln(1-K)\} = \exp\{-\mathrm{Tr}\,K - \frac{1}{2}\mathrm{Tr}\,K^2 - \frac{1}{3}\mathrm{Tr}\,K^3 \ldots\}\,. \tag{1.6.46}$$

The power series in the exponential converges if K is sufficiently small. On the other hand if, for example, K has an eigenvalue λ that is larger than unity, the series will not converge since it contains $\ln(1-\lambda)$. But even if λ exceeds unity,

$$\exp\{\ln(1-\lambda)\} = 1-\lambda \tag{1.6.47}$$

is well defined. This remark suggests that even if the power series of traces of ascending powers of K in the exponent does not converge, the expansion of the determinant itself,

$$\mathrm{Det}\,[1-K] = 1 - \mathrm{Tr}\,K + \frac{1}{2}[(\mathrm{Tr}\,K)^2 - \mathrm{Tr}\,K^2] + \cdots\,, \tag{1.6.48}$$

may well converge. Problem 2 deals with the properties of infinite, Fredholm determinants with some rigor and shows that this direct expansion does indeed converge provided that

$$|\mathrm{Tr}K| = \left|\int dtK(t,t)\right| < \infty\,, \tag{1.6.49}$$

and

$$\mathrm{Tr}KK^\dagger = \int dtdt'|K(t,t')|^2 < \infty\,. \tag{1.6.50}$$

We now have an explicit, albeit formal, evaluation of the harmonic oscillator transformation function that motivated our discussion of determinants, namely

$$\langle q'=0,\,t_2|q''=0,\,t_1\rangle^{f=0} = \int [dq]e^{-\frac{i}{2}qG^{-1}q} = \frac{1}{\sqrt{\mathrm{Det}\,G^{-1}}}\,. \tag{1.6.51}$$

One might well ask about what has happened to factors like $(2\pi i)^{\infty/2}$ which have been glossed over in this evaluation. To put the result in a well-defined form, we note again that the free particle transformation function can be expressed as a limit of an oscillator with vanishing spring constant, $\omega = 0$,

$$\langle q'=0,\,t_2|q''=0,\,t_1\rangle_0 = \frac{1}{\sqrt{\mathrm{Det}\,G_0^{-1}}}\,, \tag{1.6.52}$$

and so we can write

$$\begin{aligned}\langle q'=0,t_2|q''=0,t_1\rangle &= \langle q'=0,t_2|q''=0,t_1\rangle_0 \frac{\sqrt{\mathrm{Det}\,G_0^{-1}}}{\sqrt{\mathrm{Det}\,G^{-1}}} \\ &= \langle q'=0,t_2|q''=0,t_1\rangle_0 \frac{1}{\sqrt{\mathrm{Det}\,G^{-1}G_0}}\,.\end{aligned} \tag{1.6.53}$$

Here, since

$$G^{-1} = G_0^{-1} + m\omega^2\,, \tag{1.6.54}$$

the transformation function is expressed in terms of a well-defined determinant,

$$\mathrm{Det}\,G^{-1}G_0 = \mathrm{Det}\,[1-K]\,, \tag{1.6.55}$$

where $K = -m\omega^2 G_0$. To calculate this infinite determinant, we use

$$\begin{aligned}\frac{\partial}{\partial\omega^2}\,\ln\,\mathrm{Det}\,G^{-1}G_0 &= \mathrm{Tr}\,(G^{-1}G_0)^{-1}\frac{\partial G^{-1}}{\partial\omega^2}\,G_0 \\ &= \mathrm{Tr}\,G_0^{-1}G\,\frac{\partial G^{-1}}{\partial\omega^2}\,G_0 \\ &= \mathrm{Tr}\,G\,\frac{\partial G^{-1}}{\partial\omega^2}\,,\end{aligned} \tag{1.6.56}$$

where the last equality is obtained from the cyclic symmetry of the trace to move G_0 over to get $G_0G_0^{-1} = 1$. Since

$$\frac{\partial}{\partial\omega^2}\,G^{-1} = \frac{\partial}{\partial\omega^2}\,m\Big[\frac{d^2}{dt^2} + \omega^2\Big] = m1\,, \tag{1.6.57}$$

the logarithmic derivative in Eq. (1.6.56) is given by

$$\mathrm{Tr}\;G\,\frac{\partial G^{-1}}{\partial\omega^2} = m\;\mathrm{Tr}\;G = m\int_{t_1}^{t_2} dt\;G(t,t)\,. \tag{1.6.58}$$

This is equivalent to the previous result for $\partial\ln\langle\cdots|\cdots\rangle/\partial\omega^2$ given in Eq. (1.5.25). Thus the previous result is reproduced, and there is no need to carry the analysis any further.

1.7 Coherent states

It is worth examining the creation and annihilation operators of the harmonic oscillator in some detail because these operators are closely related to the field operators of non-relativistic quantum field theory, the theory of many-particle systems. Eigenstates of the annihilation operator may be constructed. These are called coherent states; they are closely related to

the classical limit of the harmonic oscillator, and so they are of interest in their own right. Moreover, these states are needed to construct the functional integral representation that will be presented in the next section which will then be generalized in the next chapter to treat many-particle systems that provide the first example of a quantum field theory.

In view of the structure of the oscillator Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\omega^2 m}{2} q^2 , \tag{1.7.1}$$

it is convenient to make the canonical transformation

$$p = \sqrt{m\omega}\, \bar{p} , \quad q = \frac{1}{\sqrt{m\omega}}\, \bar{q} . \tag{1.7.2}$$

This is a canonical transformation because it preserves the commutation relation

$$[\bar{q}, \bar{p}] = i . \tag{1.7.3}$$

The Hamiltonian has a simple form in terms of these new variables,

$$H = \frac{1}{2}\omega[\bar{p}^2 + \bar{q}^2] . \tag{1.7.4}$$

For simplicity, the overbar will now be omitted. The creation and annihilation operators defined, respectively, by

$$a^\dagger = \frac{1}{\sqrt{2}}\,(q - ip) , \tag{1.7.5}$$

and

$$a = \frac{1}{\sqrt{2}}\,(q + ip) , \tag{1.7.6}$$

obey

$$[a, a^\dagger] = 1 , \tag{1.7.7}$$

and put the Hamiltonian in the form

$$H = \frac{1}{2}\omega\left[aa^\dagger + a^\dagger a\right] = \omega\left[a^\dagger a + \frac{1}{2}\right] . \tag{1.7.8}$$

The algebra is simplified by adding a constant to the energy so as to remove the zero-point energy $\omega/2$:

$$H \to H = \omega a^\dagger a . \tag{1.7.9}$$

We know from our previous work on the energy spectrum that the lowest energy eigenstate, or ground state, $|0\rangle$ obeys

$$H|0\rangle = 0 . \tag{1.7.10}$$

Hence

$$0 = \langle 0|H|0\rangle = \langle 0|a^\dagger a|0\rangle = |a|0\rangle|^2 , \tag{1.7.11}$$

and we conclude that

$$a|0\rangle = 0 , \tag{1.7.12}$$

which is the reason that a is called the annihilation operator.

To start on the construction of the coherent state, we first consider

$$a(z) = e^{-za^\dagger} a e^{za^\dagger} , \tag{1.7.13}$$

which obeys

$$\begin{aligned} \frac{\partial}{\partial z} a(z) &= e^{-za^\dagger}(aa^\dagger - a^\dagger a)e^{za^\dagger} \\ &= e^{-za^\dagger} e^{za^\dagger} = 1 . \end{aligned} \tag{1.7.14}$$

Hence the operator similarity transformation of Eq. (1.7.13) produces a numerical translation of the annihilation operator,

$$a(z) = a(0) + z = a + z . \tag{1.7.15}$$

With this result in hand, we define the coherent state by

$$|z\rangle = e^{za^\dagger}|0\rangle . \tag{1.7.16}$$

It is an eigenstate of the annihilation operator because

$$\begin{aligned} a|z\rangle &= e^{za^\dagger} e^{-za^\dagger} a e^{za^\dagger}|0\rangle \\ &= e^{za^\dagger}(a + z)|0\rangle = |z\rangle z . \end{aligned} \tag{1.7.17}$$

These eigenstates have many interesting and useful properties. As shown in Problem 3, they are classical limit states in the sense that they give a minimum dispersion $\Delta q \Delta p = \frac{1}{2}$. They are the states produced when the oscillator, initially in the ground state, is driven by an external force (Problem 4). Thus these states, for example, are the one-dimensional analog of the state of the electromagnetic field produced by a radio transmitter. As we shall soon see, the time-dependent transformation functions for these states is easy to compute and provides a convenient generating function for the energy eigenstates.

To proceed with our development, we note that, in view of Eq. (1.7.16) the role of the creation operator on these states appears as

$$\frac{\partial}{\partial z}|z\rangle = a^\dagger|z\rangle . \tag{1.7.18}$$

The adjoint of Eq. (1.7.16) defines

$$(|z\rangle)^\dagger = \langle 0|e^{z^* a} = \langle z^*| . \tag{1.7.19}$$

The adjoint of Eq. (1.7.17) shows that this is an eigenstate of the creation operator (but acting to the left),

$$\langle z^*|a^\dagger = z^*\langle z^*|\,. \tag{1.7.20}$$

The adjoint of Eq. (1.7.18) yields

$$\frac{\partial}{\partial z^*}\langle z^*| = \langle z^*|a\,. \tag{1.7.21}$$

The transformation function between two coherent states may be obtained following the method [Eq. (1.1.26)] used to construct the transformation function $\langle q'|p'\rangle$, namely by writing

$$\langle z_1^*|z_2\rangle z_2 = \langle z_1^*|a|z_2\rangle = \frac{\partial}{\partial z_1^*}\langle z_1^*|z_2\rangle\,. \tag{1.7.22}$$

The solution of this differential equation is

$$\langle z_1^*|z_2\rangle = e^{z_1^* z_2} f(z_2)\,, \tag{1.7.23}$$

where $f(z_2)$ is an arbitrary function. In a similar fashion,

$$z_1^*\langle z_1^*|z_2\rangle = \langle z_1^*|a^\dagger|z_2\rangle = \frac{\partial}{\partial z_2}\langle z_1^*|z_2\rangle\,, \tag{1.7.24}$$

which has the solution

$$\langle z_1^*|z_2\rangle = e^{z_1^* z_2} g(z_1^*)\,, \tag{1.7.25}$$

with $g(z_1^*)$ arbitrary. The two solutions given in Eqs. (1.7.23) and (1.7.25) are consistent only if the two arbitrary functions are identical constants, and the boundary condition $\langle 0|0\rangle = 1$ requires that

$$\langle z_1^*|z_2\rangle = e^{z_1^* z_2}\,. \tag{1.7.26}$$

The corresponding time-dependent transformation function may be written as

$$\langle z_1^*,\, t_2|z_2,\, t_1\rangle = \langle 0,\, t_2|e^{a(t_2)z_1^*}e^{a(t_1)^\dagger z_2}|0,\, t_1\rangle\,. \tag{1.7.27}$$

Since the ground state $|0,t\rangle$ is annihilated by the Hamiltonian, $H|0,t\rangle = 0$, it is time independent, and we can write simply $\langle 0,\, t_2| = \langle 0|$ and $|0,\, t_1\rangle = |0\rangle$. To compute the time-dependent operators on the right-hand side of Eq. (1.7.27), we note that the general Heisenberg equation of motion,

$$\frac{d}{dt}a(t) = i[H, a(t)]\,, \tag{1.7.28}$$

gives

$$\begin{aligned}\frac{d}{dt}a(t) &= i\omega[a^\dagger(t)a(t), a(t)] = i\omega[a^\dagger(t), a(t)]\,a(t)\\ &= -i\omega a(t)\,.\end{aligned} \tag{1.7.29}$$

Integrating this differential equation produces

$$a(t) = e^{-i\omega t}a(0) = e^{-i\omega t}a\,, \tag{1.7.30}$$

and the adjoint

$$a(t)^\dagger = e^{i\omega t}a^\dagger\,. \tag{1.7.31}$$

Accordingly,

$$\begin{aligned}\langle z_1^*,\, t_2|z_2,\, t_1\rangle &= \langle 0|\exp\left\{az_1^*e^{-i\omega t_2}\right\}\exp\left\{a^\dagger z_2 e^{i\omega t_1}\right\}|0\rangle\\ &= \langle e^{-i\omega t_2}z_1^*|e^{i\omega t_1}z_2\rangle\\ &= \exp\left\{z_1^*z_2e^{-i\omega(t_2-t_1)}\right\}\,.\end{aligned} \tag{1.7.32}$$

Following the general method discussed before, a complete set of energy eigenstates may be inserted in the time-dependent transformation function,

$$\begin{aligned}\langle z_1^*,\, t_2|z_2,\, t_1\rangle &= \langle z_1^*|e^{-iH(t_2-t_1)}|z_2\rangle\\ &= \sum\nolimits_n e^{-iE_n(t_2-t_1)}\langle z_1^*|n\rangle\langle n|z_2\rangle\,,\end{aligned} \tag{1.7.33}$$

to compute the energy eigenvalues and "wave functions". Comparing this sum over energy eigenstates with the expansion of the exponential in Eq. (1.7.32), we conclude that

$$E_n = n\omega \tag{1.7.34}$$

and, with a phase convention, that

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\,. \tag{1.7.35}$$

Now

$$\langle n|z\rangle = \langle n|e^{a^\dagger z}|0\rangle = \sum\nolimits_{n'}\frac{z^{n'}}{n'!}\langle n|(a^\dagger)^{n'}|0\rangle\,, \tag{1.7.36}$$

which, in comparison with Eq. (1.7.35), requires that

$$\langle n|(a^\dagger)^{n'}|0\rangle = \begin{cases} 0\,, & n' \neq n\,,\\ \sqrt{n!}\,, & n' = n\,.\end{cases} \tag{1.7.37}$$

Since the vectors $\{|n\rangle\}$ form a complete and orthonormal set, this implies that the n-th state has the construction

$$|n'\rangle = \frac{(a^\dagger)^{n'}}{\sqrt{n'!}}|0\rangle\,. \tag{1.7.38}$$

This is, of course, the well-known construction which can be obtained by simple operator manipulations.

1.8 Coherent state functional integrals

The previous development of the functional integral representation utilized q and p eigenstates. We turn now to cast this functional formulation in terms of coherent states. This we do because it will eventually lead to a convenient functional integral representation for many-particle systems that are the non-relativistic limit of quantum field theories. In order to construct functional integrals employing the states $|z\rangle$, $\langle z^*|$, we need a resolution of the identity constructed with these states. Writing

$$z = \frac{1}{\sqrt{2}}(q' + ip')\,, \quad z^* = \frac{1}{\sqrt{2}}(q' - ip')\,, \tag{1.8.1}$$

such a resolution of the identity is given by

$$\int \frac{dq'dp'}{2\pi}\, |z\rangle e^{-\frac{1}{2}(q'^2+p'^2)}\langle z^*| = 1\,. \tag{1.8.2}$$

Note that the integral involves the phase-space volume $dq'dp'/2\pi$ multiplied by the weighting factor $\exp\{-\frac{1}{2}(q'^2+p'^2)\} = \exp\{-z^*z\}$. To prove the validity of this resolution of the identity, we first note that expanding the exponentials in the coherent state constructions (1.7.16) and (1.7.19) produces

$$\int \frac{dq'dp'}{2\pi}\, |z\rangle e^{-z^*z}\langle z^*| = \int \frac{dq'dp'}{2\pi} e^{-z^*z} \sum\nolimits_{n,n'} \frac{(za^\dagger)^n}{n!}|0\rangle\langle 0|\frac{(az^*)^{n'}}{n'!}\,. \tag{1.8.3}$$

The integrals are now easily performed by introducing "polar coordinates"

$$q' = \sqrt{2\rho}\cos\theta\,, \quad p' = \sqrt{2\rho}\sin\theta\,, \tag{1.8.4}$$

which give

$$dq'dp' = d\theta\,\sqrt{2\rho}\,d\sqrt{2\rho} = d\theta d\rho\,, \tag{1.8.5}$$

and

$$z = \sqrt{\rho}e^{i\theta}\,, \quad z^* = \sqrt{\rho}e^{-i\theta}\,. \tag{1.8.6}$$

With these coordinates we encounter the integrals

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(n-n')\theta} = \delta_{n,n'}\,, \tag{1.8.7}$$

and

$$\int_0^\infty d\rho\rho^n e^{-\rho} = n!\,. \tag{1.8.8}$$

Hence,

$$\int \frac{dq'dp'}{2\pi}\, |z\rangle e^{-z^*z}\langle z^*| = \sum\nolimits_n \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\langle 0|\frac{a^n}{\sqrt{n!}}$$

$$= \sum\nolimits_n |n\rangle\langle n|\,, \tag{1.8.9}$$

where the last line follows from Eq. (1.7.38). Since the vectors $\{|n\rangle\}$ form a complete and orthonormal set, this proves the validity of the resolution of the identity (1.8.2).

It is worthwhile to present another proof of the coherent state resolution of the identity which will bring out aspects that we will need in our later work. Consider the $\langle z_1^*|\cdots|z_2\rangle$ matrix element of Eq. (1.8.2),

$$\langle z_1^*| \int \frac{dq'dp'}{2\pi} e^{-z^*z}|z\rangle\langle z^*|z_2\rangle = \langle z_1^*|z_2\rangle\,. \tag{1.8.10}$$

The $|z_2\rangle$ and $\langle z_1^*|$ form a complete set of states since the energy eigenstates can be constructed from them and the latter are complete. Thus the resolution of the identity (1.8.2) is proved if one can show that Eq. (1.8.10) holds for all z_1^* and z_2. Using the explicit form of the transformation functions

$$\langle z^{*\prime}|z''\rangle = e^{z^{*\prime}z''}\,, \tag{1.8.11}$$

Eq. (1.8.10) reads

$$\int \frac{dq'dp'}{2\pi} e^{-z^*z+z_1^*z+z^*z_2} = e^{z_1^*z_2}\,. \tag{1.8.12}$$

All that we need do is to prove this evaluation of the integral.

Before directly computing the integral in Eq. (1.8.12), let us make some general remarks. First we note that in working with the coherent state representation, the transformation function $\langle z^*|z\rangle$ and the integration weight $\exp\{-z^*z\}$ can both obviously be extended to be analytic functions in the separate, distinct variables z^* and z. This is evident in the integrand in Eq. (1.8.12). In view of this extended analyticity, one can, in general, make independent translations

$$z \to z + a\,, \quad z^* \to z^* + b^*\,, \tag{1.8.13}$$

where a and b^* are arbitrary complex numbers. Referring back to Eq. (1.8.1), this is done by the complex shifts

$$q' \to q' + \frac{1}{\sqrt{2}}(a + b^*)\,, \quad p' \to p' + \frac{1}{\sqrt{2}i}(a - b^*)\,, \tag{1.8.14}$$

which do not alter the value of the integral. In particular, shifting $z \to z + z_2$ and $z^* \to z^* + z_1^*$ produces

$$\int \frac{dq'dp'}{2\pi} e^{-z^*z+z_1^*z+z^*z_2} = e^{z_1^*z_2} \int \frac{dq'dp'}{2\pi} e^{-z^*z}\,. \tag{1.8.15}$$

Writing $\exp\{-z^*z\} = \exp\{-\frac{1}{2}(q'^2+p'^2)\}$ and performing the trivial Gaus-

sian integral now gives

$$\int \frac{dq'dp'}{2\pi} e^{-z^*z} = 1\,, \tag{1.8.16}$$

and completes the proof.

The variables z and z^* may also be subjected to separate scale transformations

$$z \to \bar{z} = \lambda z\,, \quad z^* \to \bar{z}^* = \kappa^* z^*\,, \tag{1.8.17}$$

with again λ and κ^* arbitrary complex constants. In view of Eq. (1.8.1), this is done by the transformations

$$q' \to \bar{q}' = \frac{1}{2}(\lambda + \kappa^*)q' + \frac{i}{2}(\lambda - \kappa^*)p'\,, \tag{1.8.18}$$

$$p' \to \bar{p}' = -\frac{i}{2}(\lambda - \kappa^*)q' + \frac{1}{2}(\lambda + \kappa^*)p'\,. \tag{1.8.19}$$

This may be done so long as the exponential continues to produce a damping factor for large $\bar{q}'$ and $\bar{p}'$ so that the integrals remain convergent. The Jacobian determinant of this transformation is given by

$$\frac{\partial(\bar{q}', \bar{p}')}{\partial(q', p')} = \begin{vmatrix} \partial\bar{q}'/\partial q' & \partial\bar{q}'/\partial p' \\ \partial\bar{p}'/\partial q' & \partial\bar{p}'/\partial p' \end{vmatrix} = \lambda\kappa^*\,. \tag{1.8.20}$$

This is just the factor that one would get from the scale transformation if the integration measure were written as the formal product $dzdz^*$. We again see how, in a large part, z and z^* can be considered to be *independent* variables. To emphasize this fact, we shall write the coherent state resolution of the identity as

$$\int \frac{dzdz^*}{2\pi i}\, e^{-z^*z} |z\rangle\langle z^*| = 1\,. \tag{1.8.21}$$

(The factor of $1/i$ is placed here because it arises in the Jacobian in the formal variable change from q', p' to z, z^*.) We shall write the resolution of the identity in this way because in our later field-theory work we will use the generalization to (infinitely) many z^* and z variables, and it is convenient to be able to make separate transformations amongst these variables.

We may now construct the functional integral representation for the transformation function $\langle z_f^*, t_2 \,|\, z_i, t_1\rangle$ for a system with an arbitrary Hamiltonian which we write in terms of the creation and annihilation of Eqs. (1.7.5), (1.7.6), $H = H(a^\dagger, a)$. Since we have already given a careful treatment of the functional integral in Section 2, we shall now only outline the derivation and omit fine points. As was done previously,

the construction is performed using

$$1 = \int \frac{dzdz^*}{2\pi i}\, e^{-z^*z} |z, t\rangle\langle z^*, t| \tag{1.8.22}$$

at many time slices. As before, the time interval t_1 to t_2 is partitioned into $N+1$ small pieces by writing $t_2 - t_1 = (N+1)\Delta t$. Thus

$$\begin{aligned}\langle z_f^*, t_2 | z_i, t_1\rangle = \int \prod_{k=1}^{N} \left[\frac{dz_k dz_k^*}{2\pi i}\right] \\ \langle z_f^*, t_2 | z_N, t_2 - \Delta t\rangle\, e^{-z_N^* z_N} \langle z_N^*, t_2 - \Delta t | z_{N-1}, t_2 - 2\Delta t\rangle \\ e^{-z_{N-1}^* z_{N-1}} \langle z_{N-1}^*, t_2 - 2\Delta t | z_{N-2}, t_2 - 3\Delta t\rangle \\ \cdots e^{-z_1^* z_1} \langle z_1^*, t_1 + \Delta t | z_i, t_1\rangle\,.\end{aligned} \tag{1.8.23}$$

With Δt small, and $t_k = t_1 + k\Delta t$, we have, as before

$$\begin{aligned}\langle z_{k+1}^*, t_{k+1} | z_k, t_k\rangle &= \langle z_{k+1}^*, t_k | \exp\{-iH(a^\dagger, a; t_k)\Delta t\} | z_k, t_k\rangle \\ &\simeq \exp\{-iH(z_{k+1}^*, z_k; t_k)\Delta t\} \langle z_{k+1}^*, t_k | z_k, t_k\rangle \\ &= \exp\{-iH(z_{k+1}^*, z_k; t_k)\Delta t + z_{k+1}^* z_k\}\,.\end{aligned} \tag{1.8.24}$$

In addition to the sum of Hamiltonians in the exponential, there appears

$$\begin{aligned}\exp\left\{z_f^* z_N - z_N^* z_N + z_N^* z_{N-1} - z_{N-1}^* z_{N-1} + \cdots + z_2^* z_1 - z_1^* z_1 + z_1^* z_i\right\} \\ = \exp\left\{z_f^* z_f - \sum_{k=1}^{N+1} z_k^* \frac{z_k - z_{k-1}}{\Delta t}\Delta t\right\}\,,\end{aligned} \tag{1.8.25}$$

where we have defined

$$z_{N+1}^* = z_f^*\,, \quad z_0 = z_i\,. \tag{1.8.26}$$

We have also written $z_{N+1} = z_f$, but this is a *superfluous* variable which is introduced only to achieve a nice notation. The variable z_f does not, in fact, appear since the last term in the sum of differences contains $-z_{N+1}^* z_{N+1}$, which just cancels the extra initial $z_f^* z_f$ that has been placed in the exponential. In the limit $N \to \infty$, the exponential in Eq. (1.8.25) becomes

$$\exp\{\ \} = \exp\left\{z_f^* z_f - \int_{t_1}^{t_2} dt\, z^*(t)\dot{z}(t)\right\}\,, \tag{1.8.27}$$

while*

$$\exp\left\{-i\sum_{k=0}^{N} H(z_{k+1}^*, z_k; t_k)\Delta t\right\} = \exp\left\{-i\int_{t_1}^{t_2} dt H(z^*(t), z(t); t)\right\}, \tag{1.8.28}$$

and

$$\prod_{k=1}^{N} \frac{dz_k dz_k^*}{2\pi i} = [dz][dz^*]. \tag{1.8.29}$$

In terms of this notation, the coherent state transformation function is given by the functional integral representation

$$\langle z_f^*, t_2 | z_i, t_1\rangle = e^{z_f^* z_f}\int [dz][dz^*] e^{i\int_{t_1}^{t_2} dt L}, \tag{1.8.30}$$

where

$$L = iz^*(t)\dot{z}(t) - H(z^*(t), z(t); t). \tag{1.8.31}$$

Since, except for an additional total time derivative, $iz^*\dot{z} = p\dot{q}$, this Lagrangian is essentially the same as the first-order $p-q$ form. It would appear that the coherent-state functional integral differs only in a trivial way from the $p-q$ phase-space form that we initially introduced — that it differs only by a trivial notational change in the variables employed. This is not quite the case: Using the proper boundary conditions, the functional integral produces the coherent state transformation function. These are the boundary conditions presented in Eq. (1.8.26), which we write as

$$z^*(t_2) = z_f^*, \quad z(t_1) = z_i. \tag{1.8.32}$$

The significance of these boundary conditions can be brought out by considering the classical limit of the simple harmonic oscillator. In this limit, the functional integral is dominated by the stationary phase path. This is the path $z_{\rm cl}(t)$, $z_{\rm cl}^*(t)$ which makes the action

$$S = \int_{t_1}^{t_2} dt\,\{iz^*(t)\dot{z}(t) - \omega z^*(t)z(t)\} \tag{1.8.33}$$

stationary for arbitrary variations of z^* and z. This requires that

$$\dot{z}_{\rm cl}(t) = -i\omega z_{\rm cl}(t), \qquad \dot{z}_{\rm cl}^*(t) = i\omega z_{\rm cl}^*(t), \tag{1.8.34}$$

with the solutions obeying the boundary conditions (1.8.32) given by

$$z_{\rm cl}(t) = z_i e^{-i\omega(t-t_1)}, \quad z_{\rm cl}^*(t) = z_f^* e^{i\omega(t-t_2)}. \tag{1.8.35}$$

* Note that $z_{k+1}^* = z^*(t+\Delta t)$ has been replaced with $z^*(t)$ in the Hamiltonian. This may be done because, as was discussed in Section 2, a careful examination of the integration shows that the error in this replacement vanishes in the limit $\Delta t \to 0$.

Unless z_f^* corresponds to the classical point in phase space that evolves from the initial phase-space point described by z_i, $z_f^* = z_i^* e^{i\omega(t_2-t_1)}$, these paths do not correspond to phase-space trajectories $q(t)$, $p(t)$ with real $q(t)$ and $p(t)$. This indicates again how z and z^* must be considered to be independent variables.†

We conclude this discussion of the coherent state functional integral by examining the effect of adding arbitrary external forces to the Hamiltonian,

$$H \to H - z^*(t)f(t) - z(t)f(t)^*, \tag{1.8.36}$$

in which f and f^* are considered as independent functions,

$$\langle z_f^*, t_2 | z_i, t_1\rangle^{f,f^*} = e^{z_f^* z_f}\int [dz][dz^*] e^{i\int_{t_1}^{t_2} dt[L + z^* f + z f^*]}. \tag{1.8.37}$$

A variation of the force $f(t)$ gives

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle\,|\,\rangle = \langle\,|a^\dagger(t)|\,\rangle, \tag{1.8.38}$$

while a similar variation of $f^*(t)$ produces an insertion of the operator $a(t)$. Just as in the previous discussion of external forces, multiple variations produce the matrix element of time-ordered products of the $a(t)$ and $a^\dagger(t)$ operators. Since the situation here is exactly analogous to that worked out previously, there is no need to give any further detail.

1.9 Path integration

Our previous development of the functional integral formulation of quantum theory was based on a slicing of the time interval $T = t_2 - t_1$ into $N+1$ small pieces of size $\Delta t = T/(N+1)$. The $\langle q', t_2 | q'', t_1\rangle$ transformation function is then represented, in the first-order form, by the product of N ordinary integrals with measure $dqdp/2\pi$ at each time slice times a final integration over a momentum variable. One must then take the $N \to \infty$ limit. This procedure provides a precise definition of the idea of "integrating over all phase space at every time". But this procedure yields trajectories in phase space that are very "jerky" as the by now familiar time-slicing picture in Fig. 1.5 illustrates. In this section,

† In this regard, it is worth considering the variation of the action in the exponential of the functional integral (8.30) when $z(t)$ is varied. To obtain this variation, the time derivative of $z(t)$ must be integrated by parts, and the boundary term $-z^*(t_2)\delta z(t_2) + z^*(t_1)\delta z(t_1)$ appears. Since $z(t_1) = z_i$ is specified and must be held fixed, $\delta z(t_1) = 0$. On the other hand, the remaining boundary term is canceled by the variation of the exponent in the prefactor in Eq. (8.30), $z_f^*\delta z_f$, again showing that $z(t_2) = z_{N+1} = z_f$ is a superfluous variable.

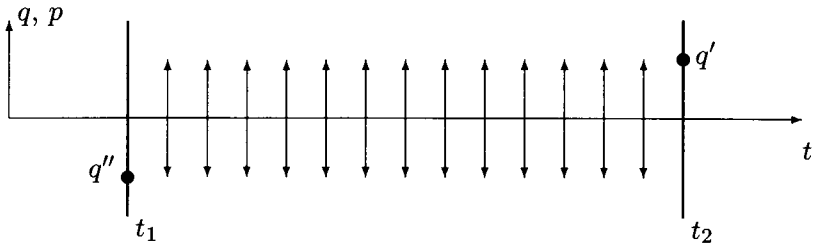


Fig. 1.5. Previous time-slicing definition of the functional integral.

an alternative method of defining the functional integration measure is described which emphasizes, instead of discontinuous trajectories, continuous paths in phase space. To do this, we first approximate all of the q and p integrals at each time slice by finite sums. Then the functional appears as a (huge) sum over many phase-space points at each of the many time slices as illustrated in Fig. 1.6. The limit of infinitely many steps in q, p, and t, $\Delta q \to 0$, $\Delta p \to 0$, $\Delta t \to 0$, defines the functional integral. The previous set of "time-slices" corresponding to Fig. 1.5 is obtained by first taking the limit where the q and p sums at each time slice become integrals. This may be visualized as first connecting the dots in Fig. 1.6 with vertical lines. On the other hand, we may first visualize connecting dots at adjacent times so as to form segmented paths in phase space as shown in Fig. 1.7. In the limit of infinitely many points, the functional representation may thus be considered as a sum over phase space paths or histories as illustrated by the sketch in Fig. 1.8.

The path integral notion is made explicit by changing the integration variables from $q(t)$, $p(t)$ to "mode amplitudes" Q_n, P_n. That is, we expand the functional integration variables

$$p(t) = \sum_{n=0}^{\infty} P_n f_n(t) \tag{1.9.1}$$

and

$$q(t) = \frac{q' - q''}{T} t + q'' + \sum_{n=1}^{\infty} Q_n g_n(t)\,. \tag{1.9.2}$$

Here $f_n(t)$ and $g_n(t)$ are some complete sets of continuous functions. To simplify the notation, we have set $t_1 = 0$ so that $T = t_2$ and the time runs along $T \geq t \geq 0$. One must require that $g_n(T) = 0 = g_n(0)$ in order for $q(t)$ to obey the boundary conditions $q(T) = q'$, $q(0) = q''$. Although the two complete sets of functions $\{f_n(t)\}$ and $\{g_n(t)\}$ may otherwise be chosen arbitrarily, two single, unique sets arise in the simplest description.

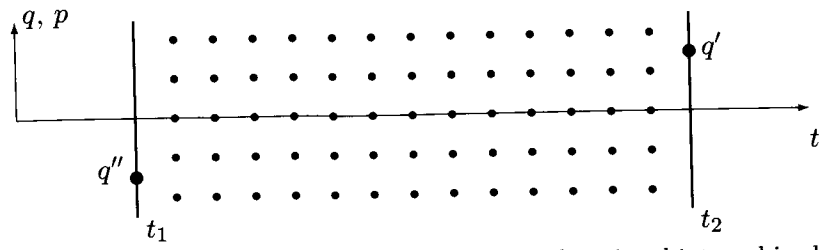


Fig. 1.6. A completely discretized version of the functional integral is obtained if the q and p integrals at each time slice are approximated by finite sums. One then sums over all the points sketched here and then takes the limit of an infinite number of Δq, Δp, and Δt slices.

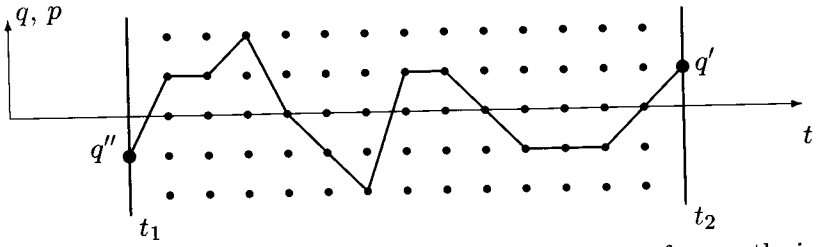


Fig. 1.7. Connecting points at neighboring times so as to form paths in phase space.

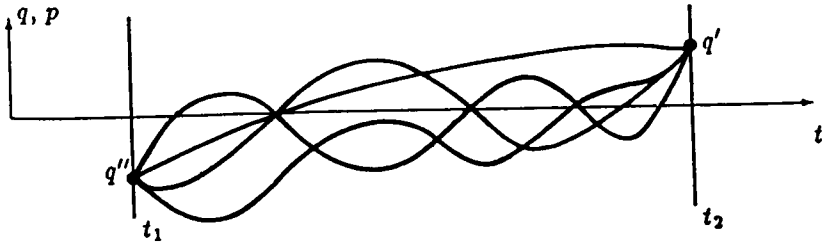


Fig. 1.8. The path integral representation illustrated here is equivalent to the previous functional integral. Instead of integrating over all $q(t)$ and $p(t)$ values at each time slice, one integrates over over all curves with fixed endpoints. This is done by integrating over the infinite number of amplitudes of a complete set of mode functions.

To see this, we first note that the Jacobians of the transformation from the $p(t)$, $q(t)$ variables to the P_n, Q_n coefficients are greatly simplified if the functions $f_n(t)$ and $g_n(t)$ form separate orthogonal sets. Next we note that it is convenient to have $f_0(t) = 1$ and, for $n \geq 1$, $f_n(t)$ and $\dot{g}_m(t)$ orthogonal and normalized such as to obtain the diagonal, "canonical" form

$$\int_0^T dt\, p(t)\dot{q}(t) = (q' - q'')P_0 + \sum_{n=1}^{\infty} P_n Q_n \,. \tag{1.9.3}$$

Since the $f_n(t)$ form a complete set, this requires that $f_n(t)$ be proportional to $\dot{g}_n(t)$ and the orthogonality of the $f_n(t)$ mode functions requires that for $m \neq n$

$$0 = \int_0^T dt\, \dot{g}_m(t)\dot{g}_n(t) = -\int_0^T dt g_m(t)\ddot{g}_n(t) \,. \tag{1.9.4}$$

Since the $g_m(t)$ are a complete and orthogonal set, $\ddot{g}_n(t)$ must be proportional to $g_n(t)$. In view of the boundary constraints $g_n(T) = 0 = g_n(0)$, $g_n(t)$ must be a sine function, and the proportionality of $f_n(t)$ to $\dot{g}_n(t)$ forces $f_n(t)$ to be a set of cosine functions. We choose the normalization so that $f_0(t) = 1$, while for $n \geq 1$

$$f_n(t) = \sqrt{2}\cos\left(\frac{n\pi t}{T}\right) , \tag{1.9.5}$$

and

$$g_n(t) = A_n\sqrt{2}\sin\left(\frac{n\pi t}{T}\right) . \tag{1.9.6}$$

Here $A_n = 1/n\pi$ so as to give the $p\dot{q}$ structure shown before. With this set of mode functions, the functional integral representation is transformed into

$$\langle q', T|q'', 0\rangle = \int [dQ][dP] J \exp\left\{ i(q'-q'')P_0 + i\sum_{n=1}^{\infty} P_n Q_n - i\int_0^T dt H \right\} . \tag{1.9.7}$$

The measure $[dQ][dP]$ that appears here is roughly an infinite product of $dQ_n dP_n$ whose precise definition will be made clear below. The normalization of the mode functions is such that the action associated with the free-particle Hamiltonian,

$$H_0 = \frac{p(t)^2}{2m} , \tag{1.9.8}$$

is given by

$$\int_0^T dt H_0 = \frac{T}{2m}\sum_{n=0}^{\infty} P_n^2 \,. \tag{1.9.9}$$

In general, $\int_0^T dt H(q(t), p(t); t)$ appears as a complicated sum over the Q_n, P_n.

It remains to compute the Jacobian J that appears in this path integral formulation. To do this with some rigor, we must return to the explicit formula

$$\langle q', T|q'', 0\rangle = \int \frac{dp_{N+1}}{2\pi} \int \prod_{k=1}^{N} \frac{dq_k dp_k}{2\pi} \exp\left\{ i\sum_{k=1}^{N+1} p_k(q_k - q_{k-1}) - i\sum_{k=1}^{N+1} H(q_k, p_k; k\Delta t)\Delta t \right\} , \tag{1.9.10}$$

where

$$q_{N+1} = q' , \quad q_0 = q'' , \tag{1.9.11}$$

and the limit $N \to \infty$ is to be taken. In this discrete form, the mode functions $f_n(t_k)$ and $g_n(t_k)$ appear as finite, complete sets of vectors. For simplicity, they will be written as $f_n(k)$ and $g_n(k)$. We shall view k as a vector component index while n, of course, labels the various different orthogonal vectors. The momentum expansion

$$p_k = \sum_{n=0}^{N} P_n f_n(k) \tag{1.9.12}$$

involves the $N+1$ $f_n(k)$ for $k = 1, 2, \ldots, N+1$. The coordinate expansion

$$q_k = \frac{q' - q''}{N+1}k + q'' + \sum_{n=1}^{N} Q_n g_n(k) \tag{1.9.13}$$

utilizes the N $g_n(k)$ for $k = 1, 2, \ldots, N$, with the convention that $g_0(k) = 0$. There is a finite set of discrete vectors $f_n(k)$ and $g_n(k)$ which have have properties analogous to those chosen for the continuous functions $f_n(t)$ and $g_n(t)$ and become these functions in the $N \to \infty$ limit. The vectors are defined by $f_0(k) = 1$, and for $n \geq 1$

$$f_n(k) = \sqrt{2}\cos\left[\frac{n\pi(k - \frac{1}{2})}{N+1}\right] , \tag{1.9.14}$$

while

$$g_n(k) = A_n\sqrt{2}\sin\left[\frac{n\pi k}{N+1}\right] . \tag{1.9.15}$$

Using the finite geometric sum

$$\sum_{k=0}^{N} z^k = \frac{1 - z^{N+1}}{1 - z} , \tag{1.9.16}$$

writing out the sine and cosine functions in exponential form, and doing some algebra it is not hard to prove that

$$\sum_{k=1}^{N+1} f_m(k) f_n(k) = (N+1)\delta_{m,n}\,, \tag{1.9.17}$$

and

$$\sum_{k=1}^{N} g_m(k) g_n(k) = A_n^2 (N+1)\delta_{m,n}\,. \tag{1.9.18}$$

Thus we indeed have a complete set of orthogonal vectors which, in the continuum limit, produce the previous trigonometric mode functions. Moreover, the discrete version of the time derivative involves

$$\begin{aligned} g_n(k) - g_n(k-1) &= A_n\sqrt{2}\left\{\sin\left[\frac{n\pi(k-\frac{1}{2}+\frac{1}{2})}{N+1}\right] - \sin\left[\frac{n\pi(k-\frac{1}{2}-\frac{1}{2})}{N+1}\right]\right\} \\ &= 2\sin\left(\frac{n\pi}{2N+2}\right) A_n f_n(k)\,, \end{aligned} \tag{1.9.19}$$

which is an exact analog of the continuum limit. Thus we choose*

$$A_n = \left[(2N+2)\sin\left(\frac{n\pi}{2N+2}\right)\right]^{-1}. \tag{1.9.20}$$

We now have, without approximation,

$$\sum_{k=1}^{N+1} p_k(q_k - q_{k-1}) = (q' - q'')P_0 + \sum_{n=1}^{N} P_n Q_n\,. \tag{1.9.21}$$

The Jacobian J is the product of the $(N+1)\times(N+1)$ determinant of $\partial p_k/\partial P_n$ with the $N\times N$ determinant of $\partial q_k/\partial Q_n$. Writing

$$\frac{\partial p_k}{\partial P_n} = f_n(k) = \sqrt{N+1}\,F_{nk} \tag{1.9.22}$$

defines, in view of the orthogonality condition (1.9.17) on the $f_n(k)$, an $(N+1)\times(N+1)$ orthogonal matrix F_{nk}. Since the determinant of an orthogonal matrix is unity (except perhaps for a sign that can be neglected

* Since $N+1 = T/\Delta t$, $A_n = [(2T/\Delta t)\sin(n\pi\Delta t/2T)]^{-1}$, which for $\Delta t \to 0$ gives $A_n \to 1/n\pi$, and we recover the previous continuum results. However, as we shall soon see, we cannot be so cavalier in computing the Jacobian J – taking such limits before completing the entire calculation gives the wrong result. It should be emphasized that, upon taking sufficient care, the functional integral can be constructed as a *well-defined* limit of an infinite number of multiple integrals – one is not simply formally manipulating ill-defined and infinite quantities. One of the reasons for the work in this section is to show how this can be done.

in calculating the Jacobian),

$$\det\frac{\partial p_k}{\partial P_n} = (N+1)^{(N+1)/2}\,. \tag{1.9.23}$$

Similarly, in view of Eq. (1.9.18),

$$\frac{\partial q_k}{\partial Q_n} = g_n(k) = \sqrt{N+1}\,A_n\,G_{nk}\,, \tag{1.9.24}$$

defines an $N\times N$ orthogonal matrix G_{nk} with a unit determinant. Hence

$$\det\frac{\partial q_k}{\partial Q_n} = (N+1)^{N/2}\prod_{n=1}^{N} A_n\,, \tag{1.9.25}$$

and we have, in view of Eq. (1.9.20),

$$J = \sqrt{N+1}\prod_{n=1}^{N}\left[2\sin\left(\frac{n\pi}{2N+2}\right)\right]^{-1}. \tag{1.9.26}$$

Since $\sin(\pi-\theta) = \sin\theta$ and $\sin\pi/2 = 1$, the product may be extended to $n = 2N+1$ by writing

$$J^2 = M\prod_{n=1}^{M-1}\left[2\sin\left(\frac{n\pi}{M}\right)\right]^{-1}, \tag{1.9.27}$$

where $M = 2N+2$. Now†

$$2\sin Mx = 2\sin x\prod_{n=1}^{M-1} 2\sin\left(x+\frac{n\pi}{M}\right). \tag{1.9.28}$$

Taking the $x\to 0$ limit of Eq. (1.9.28) shows that

$$J^2 = 1\,. \tag{1.9.29}$$

We have arrived at a precise definition of the path integral. It is the $N\to\infty$ limit of

$$\begin{aligned} \langle q', T|q'', 0\rangle &= \int\frac{dP_0}{2\pi}\int\prod_{n=1}^{N}\frac{dQ_n dP_n}{2\pi} \\ &\exp\left\{i(q'-q'')P_0 + i\sum_{n=1}^{N} P_n Q_n - i\sum_{k=1}^{N+1} H(q_k,p_k;k\Delta t)\Delta t\right\}. \end{aligned} \tag{1.9.30}$$

† Proof: Divide the right-hand side of this equation by $2\sin Mx$ and call the result $F(x)$. It is easy to see that $F(x)$ has no poles; hence it is an entire analytic function. Taking $|x|\to\infty$ it is easy to see that $F(x)$ is bounded; hence it is a constant. Taking, for example, $x\to+i\infty$, writing the sine functions in exponential form, and using $\sum_{n-1}^{M-1} n = \frac{1}{2}M(M-1)$ to compute the phase, it is easy to see that $F(x)\to+1$; hence $F(x) = 1$.

The Hamiltonian is to be written in terms of the mode amplitudes Q_n, P_n and the k-sum calculated using the vector sets $f_n(k)$, $g_n(k)$.

A simple example is provided by the free particle. Here

$$H_0 = \frac{p_k^2}{2m}\,, \tag{1.9.31}$$

and in view of the orthogonality of the $f_n(k)$,

$$\sum_{k=1}^{N+1} H_0 \Delta t = \frac{T}{2m} \sum_{n=0}^{N} P_n^2\,. \tag{1.9.32}$$

Each Q_n integration can be performed using

$$\int \frac{dQ_n}{2\pi} e^{iP_nQ_n} = \delta(P_n)\,, \tag{1.9.33}$$

which removes all of the P_n integrations save that over P_0. Therefore

$$\langle q',\, T|q'',\, 0\rangle = \int \frac{dP_0}{2\pi} e^{i(q'-q'')P_0 - i(P_0^2/2m)T}\,, \tag{1.9.34}$$

which is simply the result of introducing a single resolution of the identity $1 = \int dP_0 |P_0\rangle\langle P_0|$ into the transformation function.

Problems

> A Vulgar Mechanick can practice what he has been taught or seen done, but if he is in an error he knows not how to find it out and correct it, and if you put him out of his road, he is at a stand; Whereas he that is able to reason nimbly and judiciously about figure, force and motion, is never at rest till he gets over every rub.
>
> Isaac Newton to Nathaniel Haws
> 25 May 1694

1. The functional integral representation of the driven quantum oscillator shows quite simply that

$$\langle q',\, t_2|q'',\, t_1\rangle^f = e^{i\int_{t_1}^{t_2} dt L_{\rm cl}} F(t_2 - t_1)\,,$$

but the determination of the function $F(t_1 - t_2)$ is more difficult. Using the explicit form of the classical action, show that the composition law

$$\langle q',\, t_2|q'',\, t_1\rangle^f = \int d\bar{q}\, \langle q',\, t_2|\bar{q},\, t_3\rangle^f \langle \bar{q},\, t_3|q'',\, t_1\rangle^f$$

determines $F(t_2 - t_1)$ up to an overall factor of the form $\exp\{\lambda(t_2 - t_1)\}$, where λ is a constant.

2. Determinants

Suppose that one defines the determinant by the variational statement

$$\delta \ln \det A = \mathrm{tr} A^{-1}\, \delta A$$

together with the boundary condition

$$\det 1 = 1\,.$$

a) Prove that this definition is consistent — that the variational definition is integrable independent of path — that the right-hand side does indeed define a total variation. That is, prove that if $\delta_1 A$ and $\delta_2 A$ are two different variations of the matrix A, then the definition shows that

$$\delta_1 \delta_2 \ln \det A = \delta_2 \delta_1 \ln \det A\,.$$

b) Prove that the definition implies that

$$(\det A)^* = \det A^*\,, \qquad \det A^T = \det A\,,$$

and

$$\det AB = \det A \det B\,.$$

c) Suppose that

$$A = 1 - \lambda K\,.$$

Then the solution to the variational definition is given by

$$\det[1-\lambda K] = \exp\{\mathrm{tr}\ln[1-\lambda K]\}\,.$$

This is fine if λ is small enough, but for large λ the power series defining $\ln[1-\lambda K]$ may not converge. On the other hand, one can show that the power series

$$d(\lambda) = \det[1-\lambda K] = \sum_{n=0}^{\infty} d_n \lambda^n$$

always converges provided that the kernel K is sufficiently "well-behaved". What this entails will be explained in the essay at the end of this problem. Indeed, for an $N \times N$ matrix, $d(\lambda)$ is a finite polynomial in λ of order N. For small λ,

$$\frac{d}{d\lambda}\, d(\lambda) = -d(\lambda) \sum_{m=0}^{\infty} \lambda^m\ \mathrm{tr}K^{m+1}\,.$$

(Why?) Substitute the power series for $d(\lambda)$ into this equation and find a recursion relation relating $d_{n+1}\,, d_n$, and $\mathrm{tr}K^{n+1}$. Prove that this recursion relation and the obvious boundary conditions are solved by

$$d_n = \frac{(-1)^n}{n!} \begin{vmatrix} \mathrm{tr}K & n-1 & & & \\ \mathrm{tr}K^2 & \mathrm{tr}K & n-2 & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \mathrm{tr}K^n & \mathrm{tr}K^{n-1} & \cdot & \cdot & \mathrm{tr}K \end{vmatrix}\,.$$

(One might think that it is silly to express a simple, single determinant in terms of a whole sum of other determinants. However, this formula enables one to define the determinant of an infinite matrix or integration kernel K in a rigorous fashion for all λ.)

d) Suppose that K is a 3×3 matrix. Derive the formula that expresses $\mathrm{tr}K^4$ in terms of $\mathrm{tr}K^3\,, \mathrm{tr}K^2$, and $\mathrm{tr}K$.

e) Suppose that K is the outer product of two vectors, $K = \phi\psi$. Prove that

$$\det[1-\lambda\phi\psi] = 1 - \lambda(\psi,\phi)\,,$$

where (ψ,ϕ) is the inner product of the two vectors.

f) Suppose that

$$K_N = \sum_{a,b=1}^{N} \phi_a k_{ab} \phi_b\,,$$

where

$$(\phi_a\,, \phi_b) = \delta_{ab}\,.$$

Prove that

$$\det[1+K_N] = \det[1+k]\,,$$

where the second determinant is that of the $N \times N$ matrix.

g) Suppose that $K(x,x')$ is a retarded integral kernel,

$$K(x,x') = 0\,, \quad x \le x'\,.$$

Prove that

$$\mathrm{Det}\ [1-\lambda K] = 1\,.$$

Here is the promised essay on the convergence of the power series in λ that rigorously defines the infinite determinant for an arbitrary kernel $K(x\,,x')$. We shall work with the determinant with its first trace removed, which we denote with a prime,

$$d'(\lambda) = e^{\lambda \mathrm{Tr}K}\mathrm{Det}[1-\lambda K]\,.$$

From the exp tr ln form of the determinant, this definition corresponds to deleting the first trace in the expansion of the logarithm. Thus the corresponding power series expansion coefficient d'_n has the previous form, but that appropriate to a traceless matrix — all the terms involving $\mathrm{Tr}K$ are deleted. Using the multiplication property of determinants, one computes

$$|d'(\lambda)|^2 = e^{|\lambda|^2 \mathrm{Tr}KK^\dagger}\ e^{\mathrm{Tr}X}\mathrm{Det}[1-X]\,,$$

in which

$$1 - X = (1-\lambda K)\ (1-\lambda K)^\dagger > 0$$

is a positive, Hermitian operator. Thus, if x is an eigenvalue of X, $x \le 1$, and so

$$x + \ln(1-x) \le 0$$

since the maximum of this from appears at $x = 0$ where it vanishes. Hence

$$e^{\mathrm{Tr}X}\mathrm{Det}[1-X] = e^{\mathrm{Tr}[X+\ln(1-X)]} \le 1\,,$$

and we have the bound

$$|d'(\lambda)| \le e^{\frac{1}{2}|\lambda|^2 \mathrm{Tr}KK^\dagger}\,.$$

The power series coefficients d'_n may be expressed as a contour integral about a circle of radius R:

$$d'_n = \frac{1}{2\pi i}\oint_R \frac{d\lambda}{\lambda^{n+1}}\, d'(\lambda)\,.$$

Hence

$$|d'_n| \le \frac{1}{R^n}|d'(\lambda)|_{\max \text{ on R}} \le \frac{1}{R^n}\, e^{\frac{1}{2}R^2 \mathrm{tr}KK^\dagger}\,.$$

Setting to zero the derivative with respect to R of this latter expression, we obtain the optimum bound with

$$R^2 = n(\mathrm{tr}KK^\dagger)^{-1}\,,$$

which gives

$$|d'_n| \le \left(\frac{\mathrm{tr}KK^\dagger}{n}\right)^{n/2} e^{n/2}\,.$$

Thus, the formal manipulations that we have made yields a bound on $|d'_n|$ that implies that the power series in λ for $d'(\lambda)$ always converges provided only that $\mathrm{tr}KK^\dagger$ is finite. A rigorous proof can be constructed by utilizing a definition of K as the $N \to \infty$ limit of a sequence of kernels K_N of the type examined in part f) above. Since for any finite N the determinant is a finite polynomial of order λ^N, all the previous formal manipulations are correct. One can then show that our bound remains valid in the $N \to \infty$ limit.

3. Coherent States Are Classical Limit States

a) Suppose that

$$z = \frac{1}{\sqrt{2}}\,(q' + ip')\,.$$

Prove that

$$\langle q\rangle = \frac{\langle z^*|q|z\rangle}{\langle z^*|z\rangle} = q'$$

and

$$\langle p\rangle = p'\,.$$

Compute

$$\Delta q^2 = \langle (q - \langle q\rangle)^2\rangle$$

and

$$\Delta p^2 = \langle (p - \langle p\rangle)^2\rangle\,.$$

Note, in particular, that $\Delta q \Delta p = \frac{1}{2}$. Hence these are minimal dispersion or classical limit states.

b) Compute $\langle q'|z\rangle$. [Hint: Consider $\langle q'|p|z\rangle$ or $\langle q'|q|z\rangle$ as was done in the text for the construction of $\langle q'|p'\rangle$.]

c) Use the result of part b) above and the method of the text in deriving Eq. (1.7.32) to compute the time-dependent wave function $\langle q', t_1|z, t_2\rangle$. Examine the absolute square of this wave function and show that its maximum follows the classical trajectory.

d) Suppose that one tries to construct right (ket) eigenstates of the creation operator,

$$a^\dagger|w\rangle = |w\rangle w\,.$$

Show this is nonsense by computing $\langle q'|w\rangle$ and thus showing that $|w\rangle$ has infinite norm — that $\int dq'|\langle q'|w\rangle|^2$ does not converge.

e) Just for the fun of it, prove that

$$\exp\left\{\frac{\partial}{\partial z}\frac{\partial}{\partial z^*}\right\}|z\rangle\langle z^*|\Big|_{z=0=z^*} = |\frac{\partial}{\partial z^*}\rangle\langle z^*|\Big|_{z^*=0} = 1\,.$$

4. More on the Harmonic Oscillator

a) Consider

$$L = \frac{i}{2}(z^*\dot{z} - \dot{z}^* z) - H(z, z^*)\,,$$

in which z^* and z are treated as independent variables. Derive the classical equations of motion for $z(t)$ and $z^*(t)$ from the action principle

$$\delta\int_{t_1}^{t_2} dt\, L = 0\,.$$

b) Consider a forced harmonic oscillator described, in terms of the usual variables, by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 - fq\,.$$

Write H in terms of the z, z^* variables. Write down the classical equations of motion for these variables. Solve these equations with the boundary conditions.

$$z_{\mathrm{cl}}(t_1) = z_2, \quad z^*_{\mathrm{cl}}(t_2) = z_1^*\,.$$

c) Write down the functional integral representation for $\langle z_1^*, t_2|z_2, t_1\rangle^f$ in terms of $[dz]$ $[dz^*]$ integrations.

d) Compute this functional integral using techniques similar to those used in the text in the derivation of $\langle q', t_2|q'', t_1\rangle^f$.

e) Suppose that the system is initially in the ground state. Show that the time-dependent transformation function can be written in terms of an equal-time coherent state transformation function:

$$\langle z^*, t_2|0, t_1\rangle = \langle z^*|Z\rangle f(Z)\,.$$

Explain how this result shows that the initial ground state evolves into a coherent state. (This is an example of the general procedure of writing the Heisenberg transformation function in the Schrödinger representation to describe the final state.)

f) Use $\langle z^*, t_2|0, t_1\rangle^f$ as a generating function to compute $\langle n, t_2|0, t_1\rangle^f$ and hence the probability that the force changes the ground state into the n-th energy eigenstate,

$$p_n = |\langle n, t_2|0, t_1\rangle^f|^2\,.$$

What is the name of the p_n distribution? Prove that

$$\sum\nolimits_n p_n = 1\,,$$

so that the probability of something happening is certain. Compute

$$\langle n\rangle = \sum\nolimits_n n\,p_n\,.$$

How is $\langle n\rangle$ related to the energy gain of a forced classical oscillator? Since each mode of the electromagnetic field is a harmonic oscillator, these results generalize to the description of the electromagnetic field radiated by a radio station.

2
Many-particle systems

Special relativity implies that mass can be created from energy — it implies particle production. A relativistic quantum theory is of necessity a theory of many particles. We shall therefore first describe systems of many identical particles but in the simpler context of non-relativistic theory. As we shall soon see, many-particle systems can be described by field operators that create and destroy particles, with these operators acting in a vastly enlarged vector space that spans states with an arbitrary number of particles, including a state of no particles — the vacuum state. Both Bose and Fermi statistics are handled in this way, and the resulting formalism is compact, convenient, and powerful. For the Bose case, the functional integral representation of the quantum many-particle system is direct generalization of the coherent-state functional integral developed in the previous chapter. To describe the analogous formulation for the Fermi case, we must first develop the theory of anticommuting variables. The power and utility of field operator and functional integral methods will be illustrated by applying it to systems in thermal equilibrium, first to free Fermi and Bose gases, and then to a model of liquid helium which exhibits the phenomenon of spontaneous symmetry breaking, a phenomenon of great importance in condensed matter physics as well as in elementary particle theory.

2.1 Field operators

Although a more general mode expansion could be used, we shall concentrate on the coordinate representation since we wish to emphasize the local field concept. Let us first consider a system of two identical particles. It is described by a wave function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathbf{r}_1, \mathbf{r}_2|\Psi\rangle\,. \tag{2.1.1}$$

Other variables such as spin are suppressed — these may be implicitly included together with a particle's position in the symbol $\mathbf{r}$. There are two kinds of particles: Bosons and Fermions. The two different types have different statistics: wave functions of a Bosonic system are completely symmetrical, wave functions of Fermionic systems are totally antisymmetric. For a two-particle system

$$\Psi(\mathbf{r}_2,\mathbf{r}_1) = \begin{cases} +\Psi(\mathbf{r}_1,\mathbf{r}_2)\,, \text{ Bosons;} \\ \\ -\Psi(\mathbf{r}_1,\mathbf{r}_2)\,, \text{ Fermions.} \end{cases} \tag{2.1.2}$$

This can be expressed as

$$\langle \mathbf{r}_1,\mathbf{r}_2| = \pm \langle \mathbf{r}_2,\mathbf{r}_1|\,, \tag{2.1.3}$$

which gives, with our conventional normalization,

$$\langle \mathbf{r}_1,\mathbf{r}_2|\mathbf{r}_1',\mathbf{r}_2'\rangle = \delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2') \pm \delta(\mathbf{r}_1-\mathbf{r}_2')\delta(\mathbf{r}_2-\mathbf{r}_1')\,. \tag{2.1.4}$$

The sign conventions used here will be employed throughout our work: Upper signs for Bosons, lower signs for Fermions. With n identical particles*, the interchange of any two particle coordinates $\mathbf{r}_a$ and $\mathbf{r}_b$ gives

$$\langle \mathbf{r}_1\cdots\mathbf{r}_a\cdots\mathbf{r}_b\cdots\mathbf{r}_n| = \pm\langle \mathbf{r}_1\cdots\mathbf{r}_b\cdots\mathbf{r}_a\cdots\mathbf{r}_n| \tag{2.1.5}$$

and so

$$\langle \mathbf{r}_1\cdots\mathbf{r}_n|\mathbf{r}_1'\cdots\mathbf{r}_n'\rangle = \sum_{\text{all perms}} (\pm 1)^P \prod_{a=1}^{n} \delta(\mathbf{r}_a - \mathbf{r}'_{\Pi(a)})\,. \tag{2.1.6}$$

Here

$$P = \begin{cases} 0\,, & \text{even permutation,} \\ \\ 1\,, & \text{odd permutation,} \end{cases} \tag{2.1.7}$$

where the permutation sends $a \to \Pi(a)$.

This is a complicated mess of notation. A simple and elegant notation can be achieved, however, if we first enlarge our concepts and, instead of considering

	vector	space	of	one	particle	$\{\|\mathbf{r}\rangle\}$
OR	vector	space	of	two	particles	$\{\|\mathbf{r}_1,\mathbf{r}_2\rangle\}$
OR	vector	space	of	three	particles	$\{\|\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3\rangle\}$
		...	*etc*	...		

* Here we consider systems containing either Bosons or Fermions. The extension to systems containing both types of particles should be clear.

consider instead

	vector	space	of	one	particle	$\{\|\mathbf{r}\rangle\}$
AND	vector	space	of	two	particles	$\{\|\mathbf{r}_1,\mathbf{r}_2\rangle\}$
AND	vector	space	of	three	particles	$\{\|\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3\rangle\}$
		...	*etc*	...		
AND	A	VECTOR	WITH	NO	PARTICLE	$\|0\rangle$.

The idea is to replace the logical *or* with *and* — not to concentrate on the space with a particular number of particles, but rather to consider the whole collection of all the various states with a fixed number of particles. This includes a no-particle or vacuum state $|0\rangle$ which is simply defined as a state of unit norm which is orthogonal to all of the other states. Changing the emphasis from states with a fixed number of particles to the whole collection of states $\{|0\rangle\,, |\mathbf{r}\rangle\,, |\mathbf{r}_1,\mathbf{r}_2\rangle\,, \cdots\}$ is analogous to the extension of the canonical to the grand canonical ensemble in statistical mechanics. It should be emphasized that although we are now extending the mathematics of quantum mechanics, no new physical ideas are being introduced — no new postulates are being made. The theory is just being put in a more convenient form. In this vastly enlarged space, the scalar products are defined by

$$\begin{aligned} &\langle 0|0\rangle = 1\,, \quad \langle 0|\mathbf{r}\rangle = 0\,, \cdots, \\ &\langle \mathbf{r}|0\rangle = 0\,, \quad \langle \mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r}-\mathbf{r}')\,, \quad \langle \mathbf{r}|\mathbf{r}'\mathbf{r}''\rangle = 0\,, \cdots, \end{aligned} \tag{2.1.8}$$

and, in general,

$$\langle \mathbf{r}_1\ldots\mathbf{r}_m|\mathbf{r}_1'\cdots\mathbf{r}_n'\rangle = \delta_{mn}\langle \mathbf{r}_1\cdots\mathbf{r}_m|\mathbf{r}_1'\cdots\mathbf{r}_m'\rangle\,. \tag{2.1.9}$$

A creation operator can be defined in this enlarged space:

$$\psi^\dagger(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int (d^3\mathbf{r}_1)\cdots(d^3\mathbf{r}_n)\,|\mathbf{r}\,\mathbf{r}_1\cdots\mathbf{r}_n\rangle\langle \mathbf{r}_1\cdots\mathbf{r}_n|\,. \tag{2.1.10}$$

To show that this operator does indeed act to create a state with an additional particle, we note that from the rules for the scalar product

$$\begin{aligned} \psi^\dagger(\mathbf{r})\,|0\rangle &= |\mathbf{r}\rangle\langle 0|0\rangle = |\mathbf{r}\rangle\,, \\ \psi^\dagger(\mathbf{r})|\mathbf{r}_1\rangle &= \int (d^3\mathbf{r}')|\mathbf{r}\mathbf{r}'\rangle\langle \mathbf{r}'|\mathbf{r}_1\rangle = |\mathbf{r}\mathbf{r}_1\rangle\,, \end{aligned} \tag{2.1.11}$$

and, in general,

$$\begin{aligned} \psi^\dagger(\mathbf{r})|\mathbf{r}_1\cdots\mathbf{r}_n\rangle &= \frac{1}{n!}\int (d^3\mathbf{r}_1')\cdots(d^3\mathbf{r}_n') \\ &\qquad |\mathbf{r}\,\mathbf{r}_1'\cdots\mathbf{r}'_n\rangle\langle \mathbf{r}_1'\cdots\mathbf{r}_n'|\mathbf{r}_1\cdots\mathbf{r}_n\rangle \\ &= |\mathbf{r}\,\mathbf{r}_1\cdots\mathbf{r}_n\rangle\,. \end{aligned} \tag{2.1.12}$$

In view of Eq. (2.1.6), the transformation function which appears in the first equality involves $n!$ different products of n δ-functions. Each product gives unity when integrated over the n $(d^3\mathbf{r}')$'s. All $n!$ terms add with the same sign because $|\mathbf{r}\,\mathbf{r}'_1\cdots\mathbf{r}'_n\rangle$ is either completely symmetrical or antisymmetrical, matching the symmetry of the δ-function product with the factor of $(\pm1)^P$. Since the action of $\psi^\dagger(\mathbf{r})$ adds another particle to the state, all of the states can be built out of the no-particle state:

$$\begin{aligned}|\mathbf{r}\rangle &= \psi^\dagger(\mathbf{r})|0\rangle\\ |\mathbf{r}_1,\mathbf{r}_2\rangle &= \psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)|0\rangle\\ &\vdots\\ |\mathbf{r}_1\cdots\mathbf{r}_n\rangle &= \psi^\dagger(\mathbf{r}_1)\cdots\psi^\dagger(\mathbf{r}_n)|0\rangle\,. \end{aligned}\tag{2.1.13}$$

Taking the adjoint of Eq. (2.1.10) produces

$$\psi(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{1}{n!}\int(d^3\mathbf{r}_1)\cdots(d^3\mathbf{r}_n)\;|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\langle\mathbf{r}\,\mathbf{r}_1\cdots\mathbf{r}_n|\,,\tag{2.1.14}$$

with the adjoints of Eqs. (2.1.12) and (2.1.13) giving

$$\langle\mathbf{r}_1\cdots\mathbf{r}_n|\psi(\mathbf{r}) = \langle\mathbf{r}\,\mathbf{r}_1\cdots\mathbf{r}_n|\,,\tag{2.1.15}$$

and

$$\langle\mathbf{r}_1\cdots\mathbf{r}_n| = \langle0|\psi(\mathbf{r}_n)\cdots\psi(\mathbf{r}_1)\,.\tag{2.1.16}$$

To find the action of ψ on ket vectors, we note that since ψ does not contain the bra $\langle0|$,

$$\psi(\mathbf{r})|0\rangle = 0\,,\tag{2.1.17}$$

and, in general,

$$\begin{aligned}&\psi(\mathbf{r})\,|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\\ &=\frac{1}{(n-1)!}\int(d^3\mathbf{r}'_1)\cdots(d^3\mathbf{r}'_{n-1})|\mathbf{r}'_1\cdots\mathbf{r}'_{n-1}\rangle\langle\mathbf{r}\,\mathbf{r}'_1\cdots\mathbf{r}'_{n-1}|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\\ &=\frac{1}{(n-1)!}\int(d^3\mathbf{r}'_1)\cdots(d^3\mathbf{r}'_{n-1})|\mathbf{r}'_1\cdots\mathbf{r}'_{n-1}\rangle\\ &\quad\Big\{\delta(\mathbf{r}-\mathbf{r}_1)\sum_{(n-1)!\,\text{perms}}(\pm1)^P\prod\delta(\mathbf{r}'_a-\mathbf{r}_{\Pi(a)})\\ &\quad+(\pm)\delta(\mathbf{r}-\mathbf{r}_2)\sum_{(n-1)!\,\text{perms}}(\pm1)^P\prod\delta(\mathbf{r}'_a-\mathbf{r}_{\Pi(a)})+\cdots\Big\}\\ &=\delta(\mathbf{r}-\mathbf{r}_1)|\mathbf{r}_2\mathbf{r}_3\cdots\mathbf{r}_n\rangle+(\pm)\delta(\mathbf{r}-\mathbf{r}_2)|\mathbf{r}_1\mathbf{r}_3\cdots\mathbf{r}_n\rangle\\ &\quad+\delta(\mathbf{r}-\mathbf{r}_3)|\mathbf{r}_1\mathbf{r}_2\mathbf{r}_4\cdots\mathbf{r}_n\rangle+\cdots\end{aligned}$$

$$=\sum_{a=1}^{n}(\pm1)^{a+1}\delta(\mathbf{r}-\mathbf{r}_a)|\mathbf{r}_1\cdots\mathbf{r}_{a-1}\;\mathbf{r}_{a+1}\cdots\mathbf{r}_n\rangle\,.\tag{2.1.18}$$

This result is obtained because each term with $(n-1)!$ permutations gives $(n-1)!$ identical terms by virtue of the symmetry of the states. For Fermions alternating terms come with alternating signs because they involve an odd permutation. Since ψ acting to the right decreases the number of particles in a state by one, it is called an annihilation operator. The adjoint statements are

$$\langle0|\psi^\dagger(\mathbf{r}) = 0\,,\tag{2.1.19}$$

and

$$\langle\mathbf{r}_1\cdots\mathbf{r}_n|\psi^\dagger(\mathbf{r}) = \sum_{a=1}^{n}(\pm1)^{a+1}\delta(\mathbf{r}-\mathbf{r}_a)\langle\mathbf{r}_1\cdots\mathbf{r}_{a-1}\,\mathbf{r}_{a+1}\cdots\mathbf{r}_n|\,.\tag{2.1.20}$$

The formulae are still very cumbersome. But consider now

$$\begin{aligned}\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')|\mathbf{r}_1\cdots\mathbf{r}_n\rangle &= |\mathbf{r}\,\mathbf{r}'\,\mathbf{r}_1\cdots\mathbf{r}_n\rangle\\ &= (\pm)|\mathbf{r}'\,\mathbf{r}\,\mathbf{r}_1\cdots\mathbf{r}_n\rangle\\ &= (\pm)\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r})|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\,.\end{aligned}\tag{2.1.21}$$

Since the $\{|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\}$ form a complete basis, we conclude that

$$\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')-(\pm)\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r})\equiv[\psi^\dagger(\mathbf{r})\,,\,\psi^\dagger(\mathbf{r}')]_\mp = 0\,.\tag{2.1.22}$$

Here

$$[A,B]_- = AB-BA = [A,B]\tag{2.1.23}$$

is the ordinary commutator, while

$$[A,B]_+ = AB+BA = \{A,B\}\tag{2.1.24}$$

is the anticommutator. Taking the adjoint and using

$$(AB)^\dagger = B^\dagger A^\dagger\,,\tag{2.1.25}$$

gives

$$[\psi(\mathbf{r})\,,\,\psi(\mathbf{r}')]_\mp = 0\,.\tag{2.1.26}$$

Finally, we calculate

$$\begin{aligned}\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')|\mathbf{r}_1\cdots\mathbf{r}_n\rangle &= \psi(\mathbf{r})|\mathbf{r}'\mathbf{r}_1\cdots\mathbf{r}_n\rangle\\ &= \delta(\mathbf{r}-\mathbf{r}')|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\\ &\quad+(\pm)\sum_{a=1}^{n}(\pm1)^{a+1}\,\delta(\mathbf{r}-\mathbf{r}_a)|\mathbf{r}'\mathbf{r}_1\cdots\mathbf{r}_{a-1}\mathbf{r}_{a+1}\cdots\mathbf{r}_n\rangle\,,\end{aligned}\tag{2.1.27}$$

and

$$\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})|\mathbf{r}_1\cdots\mathbf{r}_n\rangle = \psi^\dagger(\mathbf{r}')\sum_{a=1}^{n}(\pm1)^{a+1}\delta(\mathbf{r}-\mathbf{r}_a)|\mathbf{r}_1\cdots\mathbf{r}_{a-1}\ \mathbf{r}_{a+1}\cdots\mathbf{r}_n\rangle$$
$$= \sum_{a=1}^{n}(\pm1)^{a+1}\delta(\mathbf{r}-\mathbf{r}_a)|\mathbf{r}'\mathbf{r}_1\cdots\mathbf{r}_{a-1}\ \mathbf{r}_{a+1}\cdots\mathbf{r}_n\rangle\,. \tag{2.1.28}$$

Subtracting Eq. (2.1.28) from Eq. (2.1.27) for the case of Bosons or adding the two equations for the Fermion case gives

$$[\psi(\mathbf{r})\,,\psi^\dagger(\mathbf{r}')]_\mp\,|\mathbf{r}_1\cdots\mathbf{r}_n\rangle = \delta(\mathbf{r}-\mathbf{r}')|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\,. \tag{2.1.29}$$

Since this is true for the complete set of states,

$$[\psi(\mathbf{r})\,,\psi^\dagger(\mathbf{r}')]_\mp = \delta(\mathbf{r}-\mathbf{r}')\,. \tag{2.1.30}$$

These are the fundamental commutation $(-)$ or anticommutation $(+)$ relations of the many-particle theory. They complete the list of the basic operations of the theory.

A summary is in order:

$$|\mathbf{r}_1\cdots\mathbf{r}_n\rangle = \psi^\dagger(\mathbf{r}_1)\cdots\psi^\dagger(\mathbf{r}_n)|0\rangle\,, \tag{2.1.31}$$

$$\psi(\mathbf{r})|0\rangle = 0\,, \tag{2.1.32}$$

$$[\psi(\mathbf{r}),\psi(\mathbf{r}')]_\mp = 0 = [\psi^\dagger(\mathbf{r}),\psi^\dagger(\mathbf{r}')]_\mp\,, \tag{2.1.33}$$

$$[\psi(\mathbf{r})\,,\psi^\dagger(\mathbf{r}')]_\mp = \delta(\mathbf{r}-\mathbf{r}')\,. \tag{2.1.34}$$

This (plus the adjoint statements involving bras $\langle\cdots|$ in place of kets $|\cdots\rangle$) is all one needs to know. All the properties of the multiparticle states, and the action of the creation and annihilation operators on them, follow from these equations. Having now arrived at Eqs. (2.1.31) — (2.1.34), there will be no further use for the explicit construction (2.1.10) of the creation operator. As an example of the use of these equations, let us compute

$$\begin{aligned}\psi(\mathbf{r})|\mathbf{r}_1\mathbf{r}_2\rangle =& \psi(\mathbf{r})\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)|0\rangle\\ =& [\psi(\mathbf{r})\,,\psi^\dagger(\mathbf{r}_1)]_\mp\psi^\dagger(\mathbf{r}_2)|0\rangle\\ &+(\pm)\psi^\dagger(\mathbf{r}_1)[\psi(\mathbf{r})\,,\psi^\dagger(\mathbf{r}_2)]_\mp|0\rangle\\ &+\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\psi(\mathbf{r})|0\rangle\\ =& \delta(\mathbf{r}-\mathbf{r}_1)|\mathbf{r}_2\rangle \pm \delta(\mathbf{r}-\mathbf{r}_2)|\mathbf{r}_1\rangle + 0\,.\end{aligned} \tag{2.1.35}$$

Extending this result yields the general operator statement derived in Eq. (2.1.18), but with no need for the explicit construction of the operators ψ and $\psi^\dagger$ which was used previously.

2.2 Observables

All the operators needed to describe the multiparticle theory can be constructed from ψ and $\psi^\dagger$. To see how this goes, we first consider the Hermitian operator

$$\rho(\mathbf{r}) = \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) = \rho(\mathbf{r})^\dagger\,. \tag{2.2.1}$$

To investigate the properties of this operator, we note the little lemma

$$[AB,C] = A[B,C]_\mp \pm [A,C]_\mp B\,, \tag{2.2.2}$$

which follows by simply writing out all the terms on both sides of the equation. Using this lemma, we compute

$$\begin{aligned}[\rho(\mathbf{r})\,,\psi^\dagger(\mathbf{r}')] =& \psi^\dagger(\mathbf{r})[\psi(\mathbf{r})\,,\psi^\dagger(\mathbf{r}')]_\mp \pm [\psi^\dagger(\mathbf{r})\,,\psi^\dagger(\mathbf{r}')]_\mp\psi(\mathbf{r})\\ =& \delta(\mathbf{r}-\mathbf{r}')\psi^\dagger(\mathbf{r})\,,\end{aligned} \tag{2.2.3}$$

and the adjoint gives

$$[\rho(\mathbf{r})\,,\psi(\mathbf{r}')] = -\delta(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\,. \tag{2.2.4}$$

Since ψ annihilates the vacuum, so does ρ,

$$\rho|0\rangle = 0\,. \tag{2.2.5}$$

The action of $\rho(\mathbf{r})$ on a multi-particle state is thus given by

$$\begin{aligned}\rho(\mathbf{r})|\mathbf{r}_1\cdots\mathbf{r}_n\rangle =& [\rho(\mathbf{r})\,,\psi^\dagger(\mathbf{r}_1)\cdots\psi^\dagger(\mathbf{r}_n)]|0\rangle\\ =& [\rho(\mathbf{r})\,,\psi^\dagger(\mathbf{r}_1)]\psi^\dagger(\mathbf{r}_2)\cdots\psi^\dagger(\mathbf{r}_n)|0\rangle\\ &+\psi^\dagger(\mathbf{r}_1)[\rho(\mathbf{r})\,,\psi^\dagger(\mathbf{r}_2)]\cdots\psi^\dagger(\mathbf{r}_n)|0\rangle\\ &+\quad\cdots\cdots\\ &+\psi^\dagger(\mathbf{r}_1)\cdots[\rho(\mathbf{r})\,,\psi^\dagger(\mathbf{r}_n)]|0\rangle\\ =& \sum_{a=1}^{n}\delta(\mathbf{r}-\mathbf{r}_a)|\mathbf{r}_1\cdots\mathbf{r}_n\rangle\,.\end{aligned} \tag{2.2.6}$$

This shows that $\rho(\mathbf{r})$ is the particle density operator. Note, incidentally, that the commutator obeys the same distributive law which is obeyed by a derivative — the commutator of an operator X with a product of other operators is the sum of the commutators of X with each of the operators.

In view of Eq. (2.2.6), the volume integral

$$N = \int (d\mathbf{r}) \rho(\mathbf{r}) \tag{2.2.7}$$

obeys

$$N|\mathbf{r}_1 \cdots \mathbf{r}_n\rangle = n|\mathbf{r}_1 \cdots \mathbf{r}_n\rangle \,. \tag{2.2.8}$$

The volume integral N of $\rho(\mathbf{r})$ measures the total number of particles. The integrated version of Eq. (2.2.3) may be written as

$$N\psi^\dagger(\mathbf{r}) = \psi^\dagger(\mathbf{r})(N+1)\,, \tag{2.2.9}$$

which states that that $\psi^\dagger(\mathbf{r})$ acting to the right increases the number of particles by one. Similarly,

$$N\psi(\mathbf{r}) = \psi(\mathbf{r})(N-1)\,, \tag{2.2.10}$$

and the operator $\psi(\mathbf{r})$ acting to the right decreases the number of particles by one.

The operator which measures the total momentum of the many-body system is given by

$$\mathbf{P} = \int (d\mathbf{r}') \psi^\dagger(\mathbf{r}') \frac{1}{i} \nabla' \psi(\mathbf{r}') \,. \tag{2.2.11}$$

Integrating by parts gives

$$\mathbf{P} = \int (d\mathbf{r}') \left(i\nabla' \psi^\dagger(\mathbf{r}') \right) \psi(\mathbf{r}')\,, \tag{2.2.12}$$

which shows that Eq. (2.2.11) defines an Hermitian operator, $\mathbf{P}^\dagger = \mathbf{P}$. To establish that $\mathbf{P}$ is the correct momentum operator, we use Eq. (2.2.11) and the commutator lemma (2.2.2) together with the basic field commutation or anticommutation relation (2.1.34) to derive

$$[\psi(\mathbf{r}), \mathbf{P}] = \frac{1}{i} \nabla \psi(\mathbf{r}) \,. \tag{2.2.13}$$

The Hermitian adjoint gives

$$[\psi^\dagger(\mathbf{r}), \mathbf{P}] = \frac{1}{i} \nabla \psi^\dagger(\mathbf{r})\,, \tag{2.2.14}$$

which also follows directly by using Eq. (2.2.12) and evaluating the resulting field commutators. Since the field operators annihilate the vacuum state, so does the momentum operator,

$$\mathbf{P}|0\rangle = 0 = \langle 0|\mathbf{P} \,. \tag{2.2.15}$$

The vacuum state has zero momentum as it must. To finally establish that $\mathbf{P}$ is indeed the momentum operator we use Eq. (2.2.11) to compute

the action of $\mathbf{P}$ on a general member of a complete set of states, just as was done in Eq. (2.2.6) for the density operator,

$$\begin{aligned} \langle \mathbf{r}_1 \cdots \mathbf{r}_n|\mathbf{P} &= \langle 0|[\psi(\mathbf{r}_n) \cdots \psi(\mathbf{r}_1), \mathbf{P}] \\ &= \sum_{a=1}^{n} \frac{1}{i} \nabla_a \langle \mathbf{r}_1 \cdots \mathbf{r}_n| \,. \end{aligned} \tag{2.2.16}$$

This corresponds to the familiar representation of the total momentum in terms of a sum of gradient operations applied to a many-particle wave function. The angular momentum operator $\mathbf{J}$ has an expression similar to Eq. (2.2.11) which relates it to a volume integral involving $\psi^\dagger \cdots \psi$. This is covered in Problem 1d.

Let us now consider a system of particles *not* interacting with each other but interacting with an external potential. In this case the Hamiltonian for an individual particle is given by

$$H^{(1)} = \frac{\mathbf{p}^2}{2m} + V_1(\mathbf{r}) \,. \tag{2.2.17}$$

We assert that the Hamiltonian for an arbitrary number of particles is given by

$$H = \int (d\mathbf{r})\, \psi^\dagger(\mathbf{r}) \left[-\frac{\nabla^2}{2m} + V_1(\mathbf{r}) \right] \psi(\mathbf{r}) \,. \tag{2.2.18}$$

This defines an Hermitian operator since the adjoint operation interchanges $\psi^\dagger(\mathbf{r}) \leftrightarrow \psi(\mathbf{r})$ and partial integrations change $-\nabla^2$ from acting on $\psi^\dagger(\mathbf{r})$ to act on $\psi(\mathbf{r})$. To prove that Eq. (2.2.18) is the correct form of the Hamiltonian, we compute the commutator of H with ψ, just as was done for the density operator,

$$\begin{aligned} [\psi(\mathbf{r})\,, H] &= \int (d\mathbf{r}') [\psi(\mathbf{r})\,, \psi^\dagger(\mathbf{r}')]_\mp \Big(-\frac{\nabla'^2}{2m} + V_1(\mathbf{r}') \Big) \psi(\mathbf{r}') \\ &= \Big(-\frac{\nabla^2}{2m} + V_1(\mathbf{r}) \Big) \psi(\mathbf{r}) \,. \end{aligned} \tag{2.2.19}$$

Since $\psi^\dagger$ annihilates the vacuum state when acting to the left,

$$\langle 0|H = 0\,, \tag{2.2.20}$$

and the vacuum is a state of zero energy as it should be. Hence the action of the Hamiltonian on a multi-particle state is described by

$$\begin{aligned} \langle \mathbf{r}_1 \cdots \mathbf{r}_n|H|\Psi\rangle &= \langle 0|[\psi(\mathbf{r}_n) \cdots \psi(\mathbf{r}_1)\,, H]|\Psi\rangle \\ &= \langle 0|\psi(\mathbf{r}_n) \cdots \psi(\mathbf{r}_2)[\psi(\mathbf{r}_1)\,, H]|\Psi\rangle \\ &\quad + \langle 0|\psi(\mathbf{r}_n) \cdots [\psi(\mathbf{r}_2)\,, H]\psi(\mathbf{r}_1)|\Psi\rangle \\ &\quad + \cdots + \langle 0|[\psi(\mathbf{r}_n)\,, H] \cdots \psi(\mathbf{r}_1)|\Psi\rangle\,, \end{aligned} \tag{2.2.21}$$

and with Eq. (2.2.19),

$$\langle \mathbf{r}_1 \cdots \mathbf{r}_n | H | \Psi \rangle = \left\{ \sum_{a=1}^{n} \left[-\frac{\nabla_a^2}{2m} + V_1(\mathbf{r}_a) \right] \right\} \langle \mathbf{r}_1 \cdots \mathbf{r}_n | \Psi \rangle \,, \tag{2.2.22}$$

which is precisely the action of the proper Hamiltonian for the n-particle system.

It is easy to generalize this result to systems with two-particle interactions. The pair-interaction potential must be symmetric since the particles are identical,

$$V_2(\mathbf{r}_1, \mathbf{r}_2) = V_2(\mathbf{r}_2, \mathbf{r}_1) \,. \tag{2.2.23}$$

Simple systems have $V_2 = V_2(|\mathbf{r}_1 - \mathbf{r}_2|)$ but with $\mathbf{r}_1$, $\mathbf{r}_2$ implicitly including other degrees of freedom such as spin, V_2 could be more complex. We shall prove that the total Hamiltonian is given by

$$H = \int (d\mathbf{r}) \psi^\dagger(\mathbf{r}) \Big[-\frac{\nabla^2}{2m} + V_1(\mathbf{r}) \Big] \psi(\mathbf{r}) + \frac{1}{2} \int (d\mathbf{r})(d\mathbf{r}') \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') V_2(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) \,. \tag{2.2.24}$$

This is the sum of the previous Hamiltonian describing the motion of particles in an external potential plus the second double integral that describes all the the two-particle or pair interactions. The structure of this term is easy to understand. It is essentially of the form $\frac{1}{2} \int\int \rho V_2 \rho$, where ρ is the density operator defined in Eq. (2.2.1). Thus it describes the pairwise, particle-particle energy, with the factor of $\frac{1}{2}$ to avoid double counting. The interaction term differs from this density operator-density operator structure only by the operator ordering in which the destruction operators ψ all appear on the right. This must be done so that the interaction term vanishes when it acts on a single particle state (which is annihilated by two ψ operators.) Choosing to label the two destruction operators on the right as $\psi(\mathbf{r}')\psi(\mathbf{r})$, then the two creation operators on the left must appear in the order $\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')$ in order to construct an Hermitian Hamiltonian operator – the adjoint operation interchanges these two sets of operators. (The ordering of the $\psi\psi$ and $\psi^\dagger\psi^\dagger$ amongst themselves needs to be specified only for the anticommuting Fermi operators.) To prove that this is indeed the correct Hamiltonian for an interacting system of particles, we proceed as before and first calculate

$$[\psi(\mathbf{r}), H] = \Big(-\frac{\nabla^2}{2m} + V_1(\mathbf{r}) \Big) \psi(\mathbf{r}) + \int (d\mathbf{r}') \psi^\dagger(\mathbf{r}') V_2(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r})$$
$$= \Big(-\frac{\nabla^2}{2m} + V_1(\mathbf{r}) \Big) \psi(\mathbf{r}) + \int (d\mathbf{r}') \rho(\mathbf{r}') V_2(\mathbf{r}', \mathbf{r}) \psi(\mathbf{r}) \,. \tag{2.2.25}$$

Proceeding as in Eq. (2.2.22), the last integral in Eq. (2.2.25) produces a string of terms with the typical member involving

$$\int (d\mathbf{r}') V_2(\mathbf{r}', \mathbf{r}_b) \langle 0 | \psi(\mathbf{r}_n) \cdots \psi(\mathbf{r}_{b+1}) \rho(\mathbf{r}') \psi(\mathbf{r}_b) \cdots \psi(\mathbf{r}_1)$$
$$= \int (d\mathbf{r}') V_2(\mathbf{r}', \mathbf{r}_b) \langle \mathbf{r}_n \cdots \mathbf{r}_{b+1} | \rho(\mathbf{r}') \psi(\mathbf{r}_b) \cdots \psi(\mathbf{r}_1) \,. \tag{2.2.26}$$

According to the adjoint of Eq. (2.2.6), the density operator $\rho(\mathbf{r}')$ acting to the left is equivalent to $\sum_{a=b+1}^{n} \delta(\mathbf{r}_a - \mathbf{r}')$, and the remaining ψ-operators just recompose the bra vector $\langle \mathbf{r}_1 \mathbf{r}_2 \cdots \mathbf{r}_n |$. In this way it is easy to establish that

$$\langle \mathbf{r}_1 \mathbf{r}_2 \cdots \mathbf{r}_n | H | \Psi \rangle = \left\{ \sum_{a=1}^{n} \Big[-\frac{\nabla_a^2}{2m} + V_1(\mathbf{r}_a) \Big] + \sum_{a>b} V_2(\mathbf{r}_a, \mathbf{r}_b) \right\} \langle \mathbf{r}_1 \cdots \mathbf{r}_n | \Psi \rangle \,. \tag{2.2.27}$$

This is indeed the correct operation of the multiparticle Hamiltonian.

Previously we have been working at a fixed time t, which is conventionally taken to be $t = 0$. For general times, in the Heisenberg picture which we always employ, the time development of any operator with no explicit time dependence is given by

$$i \frac{\partial}{\partial t} X = [X, H] \tag{2.2.28}$$

and in particular the destruction operator obeys

$$i \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = [\psi(\mathbf{r}, t), H] \,. \tag{2.2.29}$$

For general times the equal-time commutation relations read

$$[\psi(\mathbf{r}, t), \psi(\mathbf{r}', t)]_\mp = 0 = [\psi^\dagger(\mathbf{r}, t), \psi^\dagger(\mathbf{r}', t)]_\mp \,, \tag{2.2.30}$$

and

$$[\psi(\mathbf{r}, t), \psi^\dagger(\mathbf{r}', t)]_\mp = \delta(\mathbf{r} - \mathbf{r}') \,. \tag{2.2.31}$$

Since $H|0\rangle = 0$, the vacuum state is time independent and it needs no time label. Thus for all times t

$$\langle 0 | \psi^\dagger(\mathbf{r}, t) = 0 \,, \tag{2.2.32}$$

and the n-particle bra vector is constructed by

$$\langle \mathbf{r}_1 \cdots \mathbf{r}_n \,; t | = \langle 0 | \psi(\mathbf{r}_n, t) \cdots \psi(\mathbf{r}_1, t) \,. \tag{2.2.33}$$

Using this construction and the Heisenberg equation of motion (2.2.29) it is easy to check that we have, as we must,

$$i \frac{\partial}{\partial t} \langle \mathbf{r}_1 \cdots \mathbf{r}_n ; t | = \langle \mathbf{r}_1 \cdots \mathbf{r}_n ; t | H \,. \tag{2.2.34}$$

2.3 Bosonic functional integral

Let us now restrict the discussion to Bosons which have the fundamental commutation relation

$$[\psi(\mathbf{r},t)\,,\psi^\dagger(\mathbf{r}',t)] = \delta(\mathbf{r}-\mathbf{r}')\,. \tag{2.3.1}$$

This is clearly akin to the harmonic oscillator commutation relation $[a,a^\dagger] = 1$, and this similarity motivates the construction of the field analog to the coherent states $|z\rangle$ discussed Section 7 of Chapter 1. Just as in the simple oscillator case, we may prove that

$$\exp\left\{-\int(d\mathbf{r}')\psi^\dagger(\mathbf{r}',t)\psi'(\mathbf{r}')\right\}\psi(\mathbf{r},t)\exp\left\{\int(d\mathbf{r}')\psi^\dagger(\mathbf{r}',t)\psi'(\mathbf{r}')\right\} = \psi(\mathbf{r},t)+\psi'(\mathbf{r})\,, \tag{2.3.2}$$

where $\psi'(\mathbf{r})$ is an arbitrary numerical function. The proof follows by taking the functional derivative $\delta/\delta\psi'(\bar{\mathbf{r}})$, and using the basic commutator (2.3.1). Hence field eigenstates are defined by

$$|\psi'\,t\rangle = \exp\left\{\int(d\mathbf{r})\psi^\dagger(\mathbf{r},t)\psi'(\mathbf{r})\right\}|0\rangle\,, \tag{2.3.3}$$

for these obey

$$\psi(\mathbf{r},t)|\psi'\,t\rangle = |\psi'\,t\rangle\psi'(\mathbf{r})\,. \tag{2.3.4}$$

Similarly, the adjoint states

$$\langle\psi'^*\,t| = \langle 0|\exp\left\{\int(d\mathbf{r})\psi(\mathbf{r},t)\psi'^*(\mathbf{r})\right\} \tag{2.3.5}$$

obey

$$\langle\psi'^*\,t|\psi^\dagger(\mathbf{r},t) = \psi'^*(\mathbf{r})\langle\psi'^*\,t|\,. \tag{2.3.6}$$

With ordinary derivatives replaced by functional derivatives, the proof of Eq. (1.7.26) in Chapter 1 is easily generalized to show that

$$\langle\psi_1'^*\,t|\psi_2'\,t\rangle = \exp\left\{\int(d\mathbf{r})\,\psi_1'^*(\mathbf{r})\psi_2'(\mathbf{r})\right\}\,. \tag{2.3.7}$$

To elucidate the structure of this result, we consider a quantum system with N degrees of freedom described by the operators q_k, p_k $k = 1,2,\ldots,N$ or, equivalently, by the N creation and annihilation operators $a_k^\dagger$, a_k. The coherent states for this quantum system are just the direct product states

$$|\{z_k\}\rangle = |z_1\rangle\otimes|z_2\rangle\otimes\cdots\otimes|z_N\rangle \tag{2.3.8}$$

of the coherent states for each degree of freedom. Hence, the overall coherent state transformation function is the product of the result of Eq. (1.7.26) for each degree of freedom:

$$\langle\{z_k^{*(1)}\}|\{z_k^{(2)}\}\rangle = \exp\left\{\sum_{k=1}^N z_k^{*(1)}z_k^{(2)}\right\}\,. \tag{2.3.9}$$

This is the form of Eq. (2.3.7) if the integral there is approximated by a sum

$$\int(d\mathbf{r})\,\psi_1'^*(\mathbf{r})\psi_2'(\mathbf{r}) \simeq \sum_{k=1}^N(d\mathbf{r})\,\psi_1'^*(\mathbf{r}_{(k)})\psi_2'(\mathbf{r}_{(k)})\,, \tag{2.3.10}$$

and the correspondences*

$$z_k^{(2)} \to \sqrt{(d\mathbf{r})}\psi_2'(\mathbf{r}_{(k)})\,, \tag{2.3.11}$$

and

$$z_k^{*(1)} \to \sqrt{(d\mathbf{r})}\psi_1'^*(\mathbf{r}_{(k)})\,, \tag{2.3.12}$$

are made. With these remarks in mind, it should be clear that the coherent state resolution of the identity given in Eq. (1.8.2) is generalized to read

$$1 = \int[d\psi']\,[d\psi'^*]\exp\left\{-\int(d\mathbf{r})\psi'^*(\mathbf{r})\psi'(\mathbf{r})\right\}|\psi'^*\,t\rangle\langle\psi'\,t|\,, \tag{2.3.13}$$

where the measure may be written in a formal way as

$$[d\psi']\,[d\psi'^*] = \prod_{k=1}^N\frac{d\psi'(\mathbf{r}_{(k)})d\psi'^*(\mathbf{r}_{(k)})(d\mathbf{r})}{2\pi i}\,. \tag{2.3.14}$$

As in our previous work with functional integrals, it is understood that the functional integral (2.3.13) is defined by replacing the spatial volume with a large number N of small cells of volume $(d\mathbf{r})$. Then we have an ordinary integration over N variables with the spatial integral in the exponential in Eq. (2.3.13) replaced by the sum (2.3.10). The measure given by Eq. (2.3.14) involves the product of these N variables. Finally, the limit $N\to\infty$ is to be taken.

The expansion of Eq. (2.3.3) shows that these coherent states of the field provide a generating functional for the n-particle states:

* In replacing the volume integral by a sum, one replaces $(d\mathbf{r})\delta(\mathbf{r}_{(k)}-\mathbf{r}'_{(l)})$ by the Kronecker $\delta_{k,l}$, and so the analogous correspondence between the discrete creation and annihilation operators and the field operators converts the field commutator Eq. (3.1) into $[a_k,a_l^\dagger] = \delta_{k,l}$.

$$
\begin{aligned}
|\psi' t\rangle =& |0\rangle + \int (d\mathbf{r}_1)\psi^\dagger(\mathbf{r}_1, t)|0\rangle\psi'(\mathbf{r}_1) \\
&+ \frac{1}{2!}\int (d\mathbf{r}_1)(d\mathbf{r}_2)\psi^\dagger(\mathbf{r}_1, t)\psi^\dagger(\mathbf{r}_2, t)|0\rangle\psi'(\mathbf{r}_1)\psi'(\mathbf{r}_2) + \cdots \\
=& |0\rangle + \int (d\mathbf{r}_1)|\mathbf{r}_1\,; t\rangle\psi'(\mathbf{r}_1) \\
&+ \frac{1}{2!}\int (d\mathbf{r}_1)(d\mathbf{r}_2)|\mathbf{r}_1\mathbf{r}_2\,; t\rangle\psi'(\mathbf{r}_1)\psi'(\mathbf{r}_2) + \cdots . \qquad (2.3.15)
\end{aligned}
$$

Thus

$$
\frac{\delta}{\delta\psi'(\mathbf{r}_1)} \cdots \frac{\delta}{\delta\psi'(\mathbf{r}_n)} |\psi' t\rangle\Big|_{\psi'=0} = |\mathbf{r}_1 \cdots \mathbf{r}_n\,; t\rangle . \qquad (2.3.16)
$$

Since the field eigenstates $|\psi' t\rangle$ give a generating functional for any quantum state, any quantum question is answered by the transformation function $\langle\psi_1^{*\prime} t_2|\psi_2' t_1\rangle$. In view of our previous work on the functional integral for quantum mechanics in the coherent state form, Chapter 1, Section 8, we can immediately write down the answer:

$$
\langle\psi_1^{*\prime} t_2|\psi_2' t_1\rangle = \exp\left\{\int (d\mathbf{r})\psi_1^{*\prime}(\mathbf{r})\psi_1'(\mathbf{r})\right\}\int [d\psi][d\psi^*]\exp\left\{i\int_{t_1}^{t_2} dtL\right\}, \qquad (2.3.17)
$$

where the measure in the functional integral is given by

$$
[d\psi][d\psi^*] = \prod_{l=1}^{M}\prod_{k=1}^{N}\frac{d\psi(\mathbf{r}_{(k)}, t_l)d\psi^*(\mathbf{r}_{(k)}, t_l)\,(d\mathbf{r})}{2\pi i} . \qquad (2.3.18)
$$

The first product is over the M time slices t_l in the interval from t_1 to t_2 with $t_{l+1} - t_l = \Delta t = (t_2 - t_1)/(M+1)$. The second product is over all the spatial points at each time slice. The functional integral is obtained in the limit $M, N \to \infty$. The Lagrangian in the functional integral has the form

$$
L = i\int (d\mathbf{r})\psi^*(\mathbf{r}, t)\frac{\partial}{\partial t}\psi(\mathbf{r}, t) - H , \qquad (2.3.19)
$$

with

$$
\begin{aligned}
H =& \int (d\mathbf{r})\psi^*(\mathbf{r}, t)\Big[-\frac{\nabla^2}{2m} + V_1(\mathbf{r})\Big]\psi(\mathbf{r}, t) \\
&+ \frac{1}{2}\int (d\mathbf{r})(d\mathbf{r}')\psi^*(\mathbf{r}, t)\psi^*(\mathbf{r}', t)V_2(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}, t)\psi(\mathbf{r}', t) . \qquad (2.3.20)
\end{aligned}
$$

2.4 Fermionic variables

We have just worked out the functional integral representation for a many particle Bosonic system by generalizing the treatment of quantum mechanics in terms of creation and annihilation operators of the harmonic oscillator given in Chapter 1, Section 7. The case of identical Fermi particles can be treated in an analogous way if we first develop the quantum mechanics of a "Fermionic harmonic oscillator".

1 Fermion oscillators

Motivated by the Fermi field anticommutation relations, we consider an "oscillator" which is described by a non-Hermitian "complimentary" pair of variables α, $\alpha^\dagger$ that obey the anticommutation relations

$$
\{\alpha, \alpha^\dagger\} = 1 , \qquad (2.4.1)
$$

and

$$
\alpha^2 = 0 = \alpha^{\dagger 2} . \qquad (2.4.2)
$$

It follows from these definitions that

$$
\alpha^\dagger\alpha(1 - \alpha^\dagger\alpha) = \alpha^\dagger\alpha^2\alpha^\dagger = 0 . \qquad (2.4.3)
$$

This is the characteristic equation for the operator $\alpha^\dagger\alpha$, and it is satisfied by the operator's eigenvalues $(\alpha^\dagger\alpha)'$. Hence the eigenvalues are given by $(\alpha^\dagger\alpha)' = 0, 1$. We call the zero eigenstate the "vacuum" state $|0\rangle$,

$$
\alpha^\dagger\alpha|0\rangle = 0 , \qquad (2.4.4)
$$

and take it to be of unit norm,

$$
\langle 0|0\rangle = 1 . \qquad (2.4.5)
$$

Multiplying Eq. (2.4.4) on the left by $\langle 0|$ gives

$$
0 = \langle 0|\alpha^\dagger\alpha|0\rangle = ||\alpha|0\rangle||^2 , \qquad (2.4.6)
$$

which, since the only vector of zero length is the null vector, implies that the vacuum state is annihilated by α,

$$
\alpha|0\rangle = 0 . \qquad (2.4.7)
$$

Taking the adjoint gives

$$
\langle 0|\alpha^\dagger = 0 . \qquad (2.4.8)
$$

The other eigenvector has the construction

$$
|1\rangle = \alpha^\dagger|0\rangle , \qquad (2.4.9)
$$

for

$$\alpha^\dagger\alpha|1\rangle = \alpha^\dagger\{\alpha,\alpha^\dagger\}|0\rangle = |1\rangle\,. \tag{2.4.10}$$

It is also a vector of unit norm,

$$\langle 1|1\rangle = \langle 0|\alpha\alpha^\dagger|0\rangle = \langle 0|\{\alpha,\alpha^\dagger\}|0\rangle = 1\,, \tag{2.4.11}$$

and the two eigenvectors are orthogonal,

$$\langle 1|0\rangle = \langle 0|\alpha|0\rangle = 0\,. \tag{2.4.12}$$

In the Bosonic case there is an asymmetry between the creation and annihilation operators $a^\dagger$ and a . Only the annihilation operator has a zero eigenvalue state, $a|0\rangle = 0$. This asymmetry follows from the minus sign in the commutation relation, and it has the consequence that there is an infinite number of Bosonic oscillator states. In contrast, the anticommutator is symmetric, and there is a complete symmetry between α and $\alpha^\dagger$. Indeed, since $\alpha^{\dagger 2} = 0$,

$$\alpha^\dagger|1\rangle = 0\,, \tag{2.4.13}$$

and so one may interchange $\alpha \leftrightarrow \alpha^\dagger$ if at the same time one interchanges $|0\rangle \leftrightarrow |1\rangle$. Moreover, in the Fermionic case there are only two states (reflecting the Pauli exclusion principle) rather than an infinite number of states as in the Bosonic oscillator. Although there is a symmetry between α and $\alpha^\dagger$, we shall always adopt the convention that α is the annihilation operator with the vacuum state $|0\rangle$.

The non-Hermitian Fermi operators may be expressed in terms of operators ξ_1, ξ_2 which are Hermitian,

$$\xi_k^\dagger = \xi_k\,. \tag{2.4.14}$$

On writing

$$\alpha = \frac{\xi_1 + i\xi_2}{\sqrt{2}}\,, \qquad \alpha^\dagger = \frac{\xi_1 - i\xi_2}{\sqrt{2}}\,, \tag{2.4.15}$$

a little algebra shows that the defining equations (2.4.1) and (2.4.2) are equivalent to the anticommutation statement

$$\{\xi_k,\xi_l\} = \delta_{kl}\,, \tag{2.4.16}$$

for k, $l = 1$, 2. A third Hermitian operator is defined by

$$\xi_3 = -2i\,\xi_1\xi_2 = 2i\,\xi_2\xi_1\,. \tag{2.4.17}$$

It follows from Eq. (2.4.16) that ξ_3 obeys

$$\{\xi_k,\xi_3\} = 0\,, \tag{2.4.18}$$

and

$$\xi_3^2 = 1\,. \tag{2.4.19}$$

The algebra of the three Hermitian ξ-operators is essentially that of the Pauli spin matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{2.4.20}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \tag{2.4.21}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.4.22}$$

Indeed, it is easy to check the correspondence

$$\xi_1 \leftrightarrow \frac{1}{\sqrt{2}}\sigma_1\,, \qquad \xi_2 \leftrightarrow \frac{1}{\sqrt{2}}\sigma_2\,, \tag{2.4.23}$$

and

$$\xi_3 \leftrightarrow \sigma_3\,, \tag{2.4.24}$$

with the matrix elements of the ξ-operators in the $\{|0\rangle, |1\rangle\}$ basis given by the corresponding entries in the Pauli matrices. Although the Pauli matrices do provide a concrete representation of the Fermi operators, it is best to think of them as abstract operators with certain properties because this eases the understanding of the later generalization to an arbitrary number N of complimentary pairs.

2 Grassmann numbers

Just as was done for the Bosonic case, it is useful to introduce "coherent states" for the Fermi operators. The construction of the Bosonic coherent states entailed the use of an operator which performed a translation in phase space. Thus we are led to investigate the nature of a "translation" of the Hermitian ξ_1, ξ_2 which can be implemented by a unitary operator U,

$$U^{-1}\xi_k U = \xi_k + \xi_k'\,. \tag{2.4.25}$$

Multiplying Eq. (2.4.16) on the left with U^{-1} and on the right with U produces

$$\{\xi_k + \xi_k',\, \xi_l + \xi_l'\} = \delta_{kl}\,, \tag{2.4.26}$$

which, again using Eq. (2.4.16), implies that

$$\{\xi'_k, \xi_l\} + \{\xi_k, \xi'_l\} + \{\xi'_k, \xi'_l\} = 0\,. \tag{2.4.27}$$

Since we wish the "translations" ξ'_k to be independent, we conclude that they must obey

$$\{\xi'_k, \xi'_l\} = 0\,, \tag{2.4.28}$$

or $\xi'_1\xi'_2 = -\xi'_2\xi'_1$ and $\xi'^2_1 = 0 = \xi'^2_2$. The remainder of Eq. (2.4.27) is satisfied if we postulate that

$$\{\xi_k, \xi'_l\} = 0\,, \tag{2.4.29}$$

Thus for the description of Fermions we are led to consider quantities ξ'_k that are completely anticommutative – they not only anticommute with the operators ξ_k, $k = 1, 2$, but they also anticommute among themselves. It is easy to construct objects which anticommute with the ξ_k. We write

$$\xi'_k = \xi_3\epsilon_k\,, \tag{2.4.30}$$

where ϵ_k are elements of a Grassmann or "external algebra" that commute with all the ξ-operators,

$$[\epsilon_k, \xi_l] = [\epsilon_k, \xi_3] = 0\,. \tag{2.4.31}$$

Then Eq. (2.4.28) requires that the elements of the external algebra be completely anticommutative,

$$\{\epsilon_k, \epsilon_l\} = 0\,. \tag{2.4.32}$$

The real number system is augmented by the imaginary unit $i = \sqrt{-1}$ so as to keep the square roots of negative numbers within the new system, and to have a closed system. (One should not forget that i is a rather abstract symbol.) In a somewhat analogous way, the generalized "numbers" ϵ_1, ϵ_2 are needed if we are to extend to Fermions the methods which are so convenient in treating Bosons. It suffices to take the ϵ_k to be "real" variables in that they are not changed when the complex conjugation operation is applied to an expression containing them. This we write as

$$\epsilon^*_k = \epsilon_k\,. \tag{2.4.33}$$

On the other hand, we will encounter transformation functions which roughly have the form $\langle\epsilon_1|\epsilon_2\rangle$, and we shall need to have the usual rule $\langle\epsilon_1|\epsilon_2\rangle^* = \langle\epsilon_2|\epsilon_1\rangle$ apply. This motivates the requirement that the operator adjoint obey

$$(A\epsilon_k)^\dagger = \epsilon_k A^\dagger \tag{2.4.34}$$

for an arbitrary operator A and, in particular,

$$(\epsilon_1\epsilon_2)^* = \epsilon_2\epsilon_1\,. \tag{2.4.35}$$

Using again non-Hermitian variables, we note that

$$e^{\alpha^\dagger\zeta} = 1 + \alpha^\dagger\zeta\,, \tag{2.4.36}$$

where ζ is an arbitrary linear combination of the ξ'_1, ξ'_2 with complex coefficients; ζ anticommutes with all Fermionic variables and $\zeta^2 = 0$. Using Eq. (2.4.36) and the basic anticommutator (2.4.1), it is easy to see that

$$e^{-\alpha^\dagger\zeta}\alpha e^{\alpha^\dagger\zeta} = \alpha + \zeta\,. \tag{2.4.37}$$

Thus we have in hand a translation operator, and a Fermionic "coherent state" may be constructed according to

$$|\zeta\rangle = e^{\alpha^\dagger\zeta}|0\rangle\,. \tag{2.4.38}$$

Using the translation property (2.4.37) and the fact that $|0\rangle$ is annihilated by α, it is easy to see that

$$\alpha|\zeta\rangle = e^{\alpha^\dagger\zeta}\zeta|0\rangle\,. \tag{2.4.39}$$

Now ζ contains a factor of ξ_3 multiplying elements of the external algebra, which is made explicit on setting

$$\zeta = \xi_3\zeta'\,. \tag{2.4.40}$$

In view of Eqs. (2.4.15), and (2.4.17), we may write

$$\xi_3 = [\alpha, \alpha^\dagger] = 1 - 2\alpha^\dagger\alpha\,. \tag{2.4.41}$$

Therefore

$$\xi_3|0\rangle = |0\rangle\,, \tag{2.4.42}$$

and Eq. (2.4.39) becomes

$$\alpha|\zeta\rangle = |\zeta\rangle\zeta'\,. \tag{2.4.43}$$

Since ξ_3 is an Hermitian operator,

$$\langle 0|\xi_3 = \langle 0|\,, \tag{2.4.44}$$

with the adjoint statements reading

$$\langle\zeta^*| = \langle 0|e^{\zeta^*\alpha}\,, \tag{2.4.45}$$

and

$$\langle\zeta^*|\alpha^\dagger = \zeta^{*\prime}\langle\zeta^*|\,. \tag{2.4.46}$$

The scalar product of two Fermi coherent states is given by

$$\langle\zeta_1^*|\zeta_2\rangle = \langle 0|[1 + \zeta_1^*\alpha][1 + \alpha^\dagger\zeta_2]|0\rangle$$

$$= \langle 0|[1 + \zeta_1^*(1 - \alpha^\dagger\alpha)\zeta_2]|0\rangle$$
$$= 1 + \zeta_1^{*\prime}\zeta_2' \,. \tag{2.4.47}$$

Writing this as

$$\langle \zeta_1^*|\zeta_2\rangle = e^{\zeta_1^{*\prime}\zeta_2'} \,, \tag{2.4.48}$$

puts the result in exactly the same form as the coherent state Bosonic transformation function given in Eq. (1.7.26), but here one has Grassmann, not ordinary numbers.

To save notational clutter, from now on we shall drop the prime and write simply

$$\langle 0|\zeta^* = \zeta^*\langle 0| \,, \qquad \zeta|0\rangle = |0\rangle\zeta \,, \tag{2.4.49}$$

with the understanding that the Grassmann numbers ζ^*, ζ contain an additional factor of ξ_3 when they appear within matrix elements.

We turn now to construct the resolution of the identity in terms of the Fermionic "coherent states" $|\zeta\rangle$. Since the two eigenvectors $|0\rangle$ and $|1\rangle$ form an orthonormal pair, the identity operator may be expressed as

$$1 = |0\rangle\langle 0| + |1\rangle\langle 1| \,. \tag{2.4.50}$$

This is to be compared with

$$|\zeta\rangle\langle\zeta^*| = [1 + \alpha^\dagger\zeta]|0\rangle\langle 0|[1 + \zeta^*\alpha]$$
$$= |0\rangle\langle 0| + |1\rangle\langle 0|\zeta + \zeta^*|0\rangle\langle 1| + |1\rangle\langle 1|\zeta\zeta^* \,. \tag{2.4.51}$$

We want to express the identity operator in terms of some sort of integration over the Grassmann variables that appear in this outer product.

To this end, we *define* integration over a Grassmann algebra by differentiation! Namely

$$\int d\zeta \, 1 = 0 \,, \qquad \int d\zeta\zeta = 1 \,, \tag{2.4.52}$$

$$\int d\zeta^* \, 1 = 0 \,, \qquad \int d\zeta^*\zeta^* = 1 \,. \tag{2.4.53}$$

Note that these rules imply that the integration of any function is translation invariant:

$$\int d\zeta \, f(\zeta) = \int d\zeta \, f(\zeta + \zeta_0) \,, \tag{2.4.54}$$

and

$$\int d\zeta^* \, g(\zeta^*) = \int d\zeta^* \, g(\zeta^* + \zeta_0^*) \,, \tag{2.4.55}$$

where ζ_0 and ζ_0^* are Grassmann-valued constants. These results follow from the remark that since $\zeta^2 = 0 = \zeta^{*2}$, the most general form of the

arbitrary functions $f(\zeta)$ and $g(\zeta^*)$ are simple linear combinations

$$f(\zeta) = a\zeta + b \,, \quad g(\zeta^*) = c\zeta^* + d \,, \tag{2.4.56}$$

where a, b, c, d are ordinary numbers. With the Grassmann integration rules, one now has

$$\int d\zeta^* d\zeta e^{-\zeta^*\zeta} = \int d\zeta^* d\zeta [1 + \zeta\zeta^*] = \int d\zeta^* \left[\int d\zeta\zeta\right] \zeta^* = 1 \,, \tag{2.4.57}$$

while

$$\int d\zeta^* d\zeta e^{-\zeta^*\zeta} \left\{ \begin{matrix} \zeta \\ \zeta^* \end{matrix} \right\} = 0 \,, \tag{2.4.58}$$

and

$$\int d\zeta^* d\zeta e^{-\zeta^*\zeta}\zeta\zeta^* = 1 \,. \tag{2.4.59}$$

Hence

$$\int d\zeta^* d\zeta e^{-\zeta^*\zeta}|\zeta\rangle\langle\zeta^*| = |0\rangle\langle 0| + |1\rangle\langle 1| = 1 \,, \tag{2.4.60}$$

which is the desired form of the resolution of the identity. Note again that this is exactly the same form as the Bosonic coherent state resolution of the identity given in Eq. (1.8.2), but that the meaning of the integration over the Grassmann variables is very different from that over ordinary numbers. In particular, consistency requires that

$$\int d\zeta^* \int d\zeta F = -\int d\zeta \int d\zeta^* F \,. \tag{2.4.61}$$

We have remarked that integration over Grassmann variables is tantamount to differentiation. Let us consider in detail how one can define differentiation with these peculiar variables. The *left* derivative is defined by the variational statement

$$\delta_\zeta F = \delta\zeta \frac{\partial_\ell}{\partial\zeta} F \,, \tag{2.4.62}$$

where $\delta\zeta$ is an infinitesimal, completely anticommutative (Grassmann) parameter.* Thus

$$\frac{\partial_\ell}{\partial\zeta}(\zeta\zeta^*) = \zeta^* \,, \tag{2.4.63}$$

* The right derivative is defined in a similar fashion with the infinitesimal Grassmann parameter $\delta\zeta$ appearing on the extreme right-hand side of the expression. Here we work entirely with left derivatives, but we shall have occasion to also use right derivatives in our later development.

but

$$\frac{\partial_\ell}{\partial\zeta}(\zeta^*\zeta) = -\zeta^*\,, \tag{2.4.64}$$

for example. The point is that the left derivative is defined by varying ζ wherever it occurs, anticommuting all the $\delta\zeta$ to the extreme left, and then deleting all the $\delta\zeta$. Similarly, with $\delta\zeta^*$ an anticommuting parameter,

$$\delta_{\zeta^*}F = \delta\zeta^*\frac{\partial_\ell}{\partial\zeta^*}F\,. \tag{2.4.65}$$

Thus the integration over the anticommuting variables is indeed identical to a properly ordered set of left derivatives,

$$\int d\zeta^* d\zeta F = +\frac{\partial_\ell}{\partial\zeta^*}\,\frac{\partial_\ell}{\partial\zeta}\,F\,. \tag{2.4.66}$$

Since an arbitrary function is at most linear in a Grassmann variable, any second derivative must vanish:

$$\Big(\frac{\partial_\ell}{\partial\zeta}\Big)^2 F = 0\,. \tag{2.4.67}$$

Hence

$$\int d\zeta^* d\zeta \frac{\partial_\ell}{\partial\zeta}F = 0\,, \tag{2.4.68}$$

and likewise

$$\int d\zeta^* d\zeta \frac{\partial_\ell}{\partial\zeta^*}F = 0\,. \tag{2.4.69}$$

The Grassmann integral of a total derivative vanishes.

This development must be generalized to a system of N oscillators in order to make contact with the Fermi field theory described by the creation and destruction operators $\psi^\dagger(\mathbf{r})$ and $\psi(\mathbf{r})$. Thus we take for $k, l = 1, 2, \ldots, N$,

$$\{\alpha_k, \alpha_l^\dagger\} = \delta_{kl}\,, \tag{2.4.70}$$

and

$$\{\alpha_k, \alpha_l\} = 0 = \{\alpha_k^\dagger, \alpha_l^\dagger\}\,. \tag{2.4.71}$$

The vacuum state $|0\rangle$ is now the direct product of the N vacuum states for each of the N Fermionic oscillator variables, and so it is a state of unit norm

$$\langle 0|0\rangle = 1 \tag{2.4.72}$$

that is annihilated by each of the α_k,

$$\alpha_k|0\rangle = 0\,, \quad k = 1, \ldots, N\,. \tag{2.4.73}$$

We now use N pairs of Grassmann numbers ζ_k , ζ_k^* which are completely anticommutative,

$$\{\zeta_k, \alpha_l\} = 0 = \{\zeta_k, \alpha_l^\dagger\}\,, \tag{2.4.74}$$

$$\{\zeta_k^*, \alpha_l\} = 0 = \{\zeta_k^*, \alpha_l^\dagger\}\,, \tag{2.4.75}$$

$$\{\zeta_k, \zeta_l\} = \{\zeta_k, \zeta_l^*\} = \{\zeta_k^*, \zeta_l^*\} = 0\,. \tag{2.4.76}$$

They are constructed from $2N$ real, anticommutative, external algebra generating elements $\epsilon_1 \cdots \epsilon_{2N}$ which commute with all of the α_k , $\alpha_k^\dagger$. The ζ_k , ζ_k^* also contain a factor of

$$\xi_{2N+1} = \prod_{k=1}^{N} [\alpha_k, \alpha_k^\dagger]\,, \tag{2.4.77}$$

which is the generalization of the ξ_3 given in Eq. (2.4.41). Since $[\alpha_k, \alpha_k^\dagger]$ commutes with α_l and $\alpha_l^\dagger$ for $k \neq l$ while, just as in the case of a single oscillator pair discussed above, $[\alpha_k, \alpha_k^\dagger]$ anticommutes with α_k and $\alpha_k^\dagger$, ξ_{2N+1} anticommutes with all of the α_k , $\alpha_k^\dagger$. The factor of ξ_{2N+1} in the ζ_k and ζ_k^* enable them to also anticommute with all of the α_k , $\alpha_k^\dagger$ variables. Since a pair of Fermi variables commutes with another Fermi variable, the translation operator appears as

$$\exp\Big\{\sum_{k=1}^{N}\alpha_k^\dagger\zeta_k\Big\} = \prod_{k=1}^{N}\Big[1 + \alpha_k^\dagger\zeta_k\Big]\,. \tag{2.4.78}$$

The "coherent" Fermi state is now defined by

$$|\{\zeta\}\rangle = \exp\Big\{\sum_{k=1}^{N}\alpha_k^\dagger\zeta_k\Big\}|0\rangle\,. \tag{2.4.79}$$

The k-th factor of the product (2.4.77) defining ξ_{2N+1} becomes unity when acting on the vacuum state of the k-th oscillator, and so on the direct product state $|0\rangle$

$$\xi_{2N+1}|0\rangle = |0\rangle\,. \tag{2.4.80}$$

Hence, again with the understanding that the factor of ξ_{2N+1} is omitted from ζ_l when it appears outside a matrix element,

$$\alpha_l|\{\zeta\}\rangle = |\{\zeta\}\rangle\zeta_l\,. \tag{2.4.81}$$

Clearly, the single oscillator scalar product generalizes to

$$\langle\{\zeta^*\}|\{\zeta'\}\rangle = \exp\Big\{\sum_{k=1}^{N} \zeta_k^* \zeta_k'\Big\} = \prod_{k=1}^{N} \left[1 + \zeta_k^* \zeta_k'\right], \tag{2.4.82}$$

and the resolution of the identity may be written as

$$1 = \int \prod_{k=1}^{N} (d\zeta_k^*\, d\zeta_k)\; e^{-\sum_{k=1}^{N} \zeta_k^* \zeta_k} |\{\zeta\}\rangle\langle\{\zeta^*\}| \,. \tag{2.4.83}$$

3 Gaussian Grassmann integrals

We shall often have the need to evaluate Grassmann integrals of the form

$$I = \int \prod_{k=1}^{N} (d\zeta_k^*\, d\zeta_k)\; \exp\Big\{-\sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l\Big\} . \tag{2.4.84}$$

To simplify the notation, let us write this as

$$I = \int [d\zeta^* d\zeta]\; e^{-\zeta^* M \zeta} , \tag{2.4.85}$$

where M is an arbitrary matrix.

We shall compute this integral in two different ways. We do this so as to both gain familiarity with Grassmann numbers and to introduce techniques that will prove useful later. But before working out the general case, it is worth examining the integral with just two pairs of Grassmann variables. In this case, the power series expansion of the exponential terminates with the quadratic term since higher powers involve at least the square of one Grassmann variable which vanishes. On the other hand, according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes. Therefore, for the case of two pairs of variables, one has effectively

$$\begin{aligned} e^{-\zeta^* M \zeta} &\to \frac{1}{2!} (\zeta^* M \zeta)^2 \\ &\to \frac{1}{2!} (\zeta_1^* M_{11} \zeta_1 + \zeta_1^* M_{12} \zeta_2 + \zeta_2^* M_{21} \zeta_1 + \zeta_2^* M_{22} \zeta_2)^2 \\ &\to (M_{11} M_{22} - M_{12} M_{21}) \zeta_1^* \zeta_1 \zeta_2^* \zeta_2 , \end{aligned} \tag{2.4.86}$$

where the last line follows from the anticommuting character of the Grassmann numbers. The integration of $\zeta_1^* \zeta_1 \zeta_2^* \zeta_2$ gives unity, and so for this case

$$\int [d\zeta^* d\zeta]\, e^{-\zeta^* M \zeta} = \det M \,. \tag{2.4.87}$$

One should suspect that this result holds in general. Clearly, for the case of N pairs of variables, only the term of order $(\zeta^* M \zeta)^N$ survives in the expansion of the exponential. Moreover, only the terms which are multilinear in all of the ζ_k^*, ζ_k can contribute and, in view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than going through this combinatorial exercise, it is easier to use the methods to which we now turn.

In the first method, we derive and solve a variational equation obeyed by the Gaussian integral. Since $\zeta^* M \zeta$ contains the product of two anticommuting variables and thus is itself a commuting quantity,

$$\delta_M (\zeta^* M \zeta)^n = n (\zeta^* \delta M \zeta)(\zeta^* M \zeta)^{n-1} . \tag{2.4.88}$$

Hence varying M gives

$$\delta I = -\int [d\zeta^* d\zeta]\, \zeta^* \delta M \zeta\, e^{-\zeta^* M \zeta} . \tag{2.4.89}$$

Moreover, again since $\zeta^* M \zeta$ commutes, the left derivative is given by

$$\frac{\partial_\ell}{\partial \zeta_k^*}\, e^{-\zeta^* M \zeta} = -\sum_{l=1}^{N} M_{kl} \zeta_l\, e^{-\zeta^* M \zeta} . \tag{2.4.90}$$

Hence

$$\delta I = \int [d\zeta^* d\zeta]\, \zeta^* \delta M M^{-1}\, \frac{\partial_\ell}{\partial \zeta^*}\, e^{-\zeta^* M \zeta} . \tag{2.4.91}$$

This integral may be evaluated by "integrating by parts". To see how this may be done, we note that the ζ^* variation appears as

$$\begin{aligned} \delta(\zeta_m^* F) &= \delta\zeta_m^* F + \zeta_m^* \delta\zeta_k^* \frac{\partial_\ell}{\partial \zeta_k^*} F \\ &= \delta\zeta_k^* \delta_{km} F - \delta\zeta_k^* \zeta_m^* \frac{\partial_\ell}{\partial \zeta_k^*} F , \end{aligned} \tag{2.4.92}$$

which implies that

$$\frac{\partial_\ell}{\partial \zeta_k^*}\, \zeta_m^* F = \delta_{km} F - \zeta_m^* \frac{\partial_\ell}{\partial \zeta_k^*} F . \tag{2.4.93}$$

Accordingly,

$$\delta I = \int [d\zeta^* d\zeta] \sum_{km} (\delta M M^{-1})_{mk} \Big\{\delta_{km} - \frac{\partial_\ell}{\partial \zeta_k^*}\, \zeta_m^*\Big\} e^{-\zeta^* M \zeta} . \tag{2.4.94}$$

Since the Grassmann integral of a derivative vanishes,

$$\delta I = (\operatorname{tr} \delta M M^{-1}) I , \tag{2.4.95}$$

which gives

$$\delta \ln I = \frac{\delta I}{I} = \operatorname{tr} \delta M \, M^{-1} = \delta \ln \det M \,, \tag{2.4.96}$$

and the solution

$$I = (\text{const.}) \det M \,. \tag{2.4.97}$$

To determine the constant, we set $M \to 1$, in which limit

$$I = \left[\int d\zeta^* d\zeta e^{-\zeta^* \zeta} \right]^N = 1^N = 1 \,. \tag{2.4.98}$$

Hence the constant is unity and the complete evaluation appears as

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} = \det M \,. \tag{2.4.99}$$

This should be compared with the ordinary integration of a Gaussian form which gives $(\det M)^{-1}$.

Let us evaluate this integral by another method. We do this for two reasons: 1) To gain more familiarity with anticommuting variables and 2) To learn a variable transformation that will be useful later. First we note that

$$\int [d\zeta^* d\zeta] \, F = \int \prod_{k=1}^{N} d\zeta_k^* d\zeta_k \, F = \int d\zeta_N^* \cdots d\zeta_1^* \, d\zeta_1 \cdots d\zeta_N \, F \,. \tag{2.4.100}$$

by virtue of the anticommutativity of the integrations. Let us now relabel the "integration" variables

$$\begin{aligned} \zeta_N^* &\to \zeta_1 \qquad \zeta_1 \to \zeta_{N+1} \\ \zeta_{N-1}^* &\to \zeta_2 \qquad \zeta_2 \to \zeta_{N+2} \\ &\vdots \\ \zeta_1^* &\to \zeta_N \qquad \zeta_N \to \zeta_{2N} \,, \end{aligned} \tag{2.4.101}$$

so that

$$\begin{aligned} \int [d\zeta^* d\zeta] F &= \int d\zeta_1 \cdots d\zeta_N \, d\zeta_{N+1} \cdots d\zeta_{2N} \, F \\ &= \frac{1}{(2N)!} \, \epsilon_{a_1 a_2 \cdots a_{2N}} \int d\zeta_{a_1} \cdots d\zeta_{a_{2N}} \, F \,. \end{aligned} \tag{2.4.102}$$

Here ϵ is the completely antisymmetrical symbol in $2N$ dimensions with $\epsilon_{123\cdots(2N)} = 1$, and a sum is implied over all the repeated indices. By virtue of the complete anticommutativity, each of the $(2N)!$ permutations gives an identical contribution and so the prefactor of $1/(2N)!$ normalizes this to the contribution of a single term. Since the integration is tantamount

to differentiation, this can be written as

$$\int [d\zeta^* d\zeta] F = \frac{1}{(2N)!} \, \epsilon_{a_1 \cdots a_{2N}} \frac{\partial_\ell}{\partial \zeta_{a_1}} \cdots \frac{\partial_\ell}{\partial \zeta_{a_{2N}}} \, F \,. \tag{2.4.103}$$

Suppose that a linear transformation of variables is made,

$$\bar{\zeta}_a = S_{ba} \zeta_b \,, \tag{2.4.104}$$

where the repeated b index indicates a sum from $b = 1$ to $b = 2N$. It is easy to see from the definition of the derivative for anticommuting variables that

$$\frac{\partial_\ell}{\partial \zeta_a} \, F = \frac{\partial \bar{\zeta}_b}{\partial \zeta_a} \frac{\partial_\ell}{\partial \bar{\zeta}_b} \, F = S_{ab} \frac{\partial_\ell}{\partial \bar{\zeta}_b} \, F \,. \tag{2.4.105}$$

Hence we have the lemma

$$\begin{aligned} \int [d\zeta^* d\zeta] F &= \frac{1}{(2N)!} \, \epsilon_{a_1 \cdots a_{2N}} \, S_{b_1 a_1} \cdots S_{b_{2N} a_{2N}} \frac{\partial_\ell}{\partial \bar{\zeta}_{b_1}} \cdots \frac{\partial_\ell}{\partial \bar{\zeta}_{b_{2N}}} \, F \\ &= \det S \, \frac{1}{(2N)!} \, \epsilon_{b_1 \cdots b_{2N}} \frac{\partial_\ell}{\partial \bar{\zeta}_{b_1}} \cdots \frac{\partial_\ell}{\partial \bar{\zeta}_{b_{2N}}} \, F \\ &= \det S \int [d\bar{\zeta}^* d\bar{\zeta}] F \,. \end{aligned} \tag{2.4.106}$$

Note the contrast with *ordinary* integration where the change of variables $\bar{x}_a = S_{ab} x_b$ produces

$$\int (dx) F = \int (d\bar{x}) \, \frac{\partial(x)}{\partial(\bar{x})} \, F = (\det S)^{-1} \int (d\bar{x}) F \,. \tag{2.4.107}$$

With the lemma in hand, we can return to evaluate the integral

$$I = \int [d\zeta^* d\zeta] \, e^{\zeta^* M \zeta} \,. \tag{2.4.108}$$

This is now trivial, for the linear transformation

$$\bar{\zeta}_k^* = \zeta_k^* \,, \quad \bar{\zeta}_k = M_{kl} \, \zeta_l \,, \tag{2.4.109}$$

gives

$$I = \det M \int [d\bar{\zeta}^* d\bar{\zeta}] \, e^{-\bar{\zeta}^* \bar{\zeta}} = \det M \,, \tag{2.4.110}$$

which was the previous result.

4 Fermionic functional integrals

Having mastered anticommuting variables, we can turn to Fermion particles which are described by the fundamental anticommutation relation

$$\{\psi(\mathbf{r}, t), \psi^\dagger(\mathbf{r}', t)\} = \delta(\mathbf{r} - \mathbf{r}') \,. \tag{2.4.111}$$

Thus

$$|\psi' t\rangle = \exp\left\{\int (d\mathbf{r})\, \psi^\dagger(\mathbf{r},t)\psi'(\mathbf{r})\right\}|0\rangle \tag{2.4.112}$$

is an eigenvector of the Fermi field,

$$\psi(\mathbf{r},t)|\psi' t\rangle = |\psi' t\rangle\, \psi'(\mathbf{r})\,, \tag{2.4.113}$$

and a resolution of the identity is given by

$$1 = \int [d\psi^{*\prime} d\psi']\ \exp\left\{-\int (d\mathbf{r})\, \psi^{*\prime}(\mathbf{r})\psi'(\mathbf{r})\right\}\ |\psi' t\rangle\langle\psi^{*\prime} t|\,. \tag{2.4.114}$$

Accordingly, exactly as in the Bose case any quantum question is answered by

$$\langle\psi_1^{*\prime} t_2|\psi_2' t_1\rangle = \exp\left\{\int (d\mathbf{r})\psi_1^{*\prime}(\mathbf{r})\psi_1'(\mathbf{r})\right\}\int [d\psi^* d\psi] \exp\left\{i\int_{t_1}^{t_2} dt\ L\right\}, \tag{2.4.115}$$

in which

$$[d\psi^* d\psi] = \prod_t \prod_{\mathbf{r}} d\psi^*(\mathbf{r},t)\, d\psi(\mathbf{r},t)\, (d\mathbf{r})\,, \tag{2.4.116}$$

where the first product is over all the times t in the interval t_1 to t_2 while the second product is over all the spatial points, and

$$L = i\int (d\mathbf{r})\psi^*(\mathbf{r},t)\frac{\partial}{\partial t}\ \psi(\mathbf{r},t) - H\,, \tag{2.4.117}$$

with

$$H = \int (d\mathbf{r})\psi^*(\mathbf{r},t)\Big[-\frac{\nabla^2}{2m} + V_1(\mathbf{r})\Big]\psi(\mathbf{r},t) + \frac{1}{2}\int (d\mathbf{r})(d\mathbf{r}')\psi^*(\mathbf{r},t)\psi^*(\mathbf{r}',t)V_2(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}',t)\psi(\mathbf{r},t). \tag{2.4.118}$$

2.5 Quantum statistical mechanics

Let us at long last make an application of the formalism which we have produced. All of statistical mechanics and much of condensed matter physics is given by the grand canonical partition function. In general, the grand partition function is the sum over all states — the trace — of an exponential containing an arbitrary linear combination of the operators that represent all of the additive* conserved quantities of the system, all the operators which commute with the Hamiltonian and with each

* Additive operators may be expressed as volume integrals of local densities. The additive property is needed for the partition function to define proper intensive quantities.

other. With no "internal" symmetries that give other operators, the set of conserved operators is $\{H, N, \mathbf{P}, \mathbf{J}\}$, where H is the Hamiltonian, N the number operator, $\mathbf{P}$ the total momentum operator, and $\mathbf{J}$ the total angular momentum operator. We shall not consider rotating systems, and so we omit the angular momentum operator and write

$$Z = \mathrm{Tr}\,\exp\{-\beta[H - \mu N - \mathbf{V}\cdot\mathbf{P}]\}\,. \tag{2.5.1}$$

Here $\beta = 1/kT$ with T the temperature and μ is the chemical potential. Because we will make use of it in our later description of superfluid helium, we have retained the velocity parameter $\mathbf{V}$ coupled to the momentum operator $\mathbf{P}$. In the non-relativistic case, the grand partition function describes an ensemble which moves with the velocity $\mathbf{V}$ as seen in the reference "laboratory" system. This will be proved at the end of this section.

To illustrate the use of the grand canonical ensemble, and because we shall need these results later, we pause to outline the thermodynamic relations which follow from it. The grand canonical ensemble defines a statistical mixture of states described by the density operator†

$$\rho = \frac{1}{Z}\exp\{-\beta[H - \mu N - \mathbf{V}\cdot\mathbf{P}]\}\,. \tag{2.5.2}$$

The ensemble average of any observable X is given by

$$\langle X\rangle = \mathrm{Tr}\,\rho X\,. \tag{2.5.3}$$

To show how the thermodynamic relations follow from the grand canonical partition function, we first note the operator relation‡

$$\delta\,\mathrm{Tr}\,e^A = \mathrm{Tr}\,e^A\,\delta A\,. \tag{2.5.4}$$

This result shows that the first derivative (but only the first derivative) of the partition function can be calculated as if the operators that appear within the trace were ordinary numbers. Hence, the average total energy (which is the thermodynamic internal energy), total particle number, and total momentum of the system are given by partial derivatives with respect to the parameters β, $\beta\mu$, and $\beta\mathbf{V}$,

$$\langle H\rangle = -\frac{\partial}{\partial\beta}\ln Z\Big|_{(\beta\mu),(\beta\mathbf{V})\ \text{fixed}}\,, \tag{2.5.5}$$

† The usual convention is followed in denoting the statistical operator by ρ. It should not be confused with the previous particle density operator $\rho(\mathbf{r},t)$.

‡ A simple proof: Write $\exp\{A\}$ as the $N\to\infty$ limit of $[1+A/N]^N$. Then $\delta\exp\{A\}$ appears as N terms of the form $[1+A/N]^{N-M}\delta A/N[1+A/N]^M$. By the cyclic symmetry of the trace, all of these terms are equivalent to $[1+A/N]^N\delta A/N$.

$$\langle N\rangle = \frac{\partial}{\partial(\beta\mu)} \ln Z\Big|_{\beta,(\beta\mathbf{V})\text{ fixed}}, \tag{2.5.6}$$

and

$$\langle \mathbf{P}\rangle = \frac{\partial}{\partial(\beta\mathbf{V})} \ln Z\Big|_{\beta,(\beta\mu)\text{ fixed}}. \tag{2.5.7}$$

Since the logarithm of the partition function plays an essential role, it — except for sign and temperature factor — is called the *thermodynamic* or *grand potential* Ω, where

$$\ln Z = -\beta\Omega. \tag{2.5.8}$$

We shall consider only systems with a large spatial volume V. The total average energy (2.5.5), number (2.5.6), and momentum (2.5.7) are thus energy, number, and momentum densities times the total volume V. In this case, the thermodynamic potential is also proportional to the volume,

$$\Omega = -\mathcal{P}V. \tag{2.5.9}$$

The intensive variable $\mathcal{P}$ is, in fact, the pressure in the system. Some justification for this identification will be given shortly. The entropy S is a measure of disorder of the system given by

$$S = -k\,\mathrm{Tr}\,\rho\ln\rho. \tag{2.5.10}$$

Using the definition (2.5.2) of the density operator ρ, the entropy may be expressed in terms of the quantities introduced previously,

$$S = k\ln Z + k\beta\left[\langle H\rangle - \mu\langle N\rangle - \mathbf{V}\cdot\langle\mathbf{P}\rangle\right]. \tag{2.5.11}$$

A short calculation shows that these terms are produced by the temperature derivative of the thermodynamic potential; thus

$$S = -\frac{\partial}{\partial T}\Omega\Big|_{\mu,\mathbf{V}\text{ fixed}}. \tag{2.5.12}$$

The identifications of the pressure and entropy which have been made may be confirmed by varying all the quantities (including the volume) which appear in Eq. (2.5.11). Another short calculation shows that

$$TdS = \mathcal{P}dV + d\langle H\rangle - \mu\, d\langle N\rangle - \mathbf{V}\cdot d\langle\mathbf{P}\rangle. \tag{2.5.13}$$

This result expresses the conservation of energy: The heat flow into the system $dQ = TdS$ is accounted for by the other energy changes; the work done by the pressure $\mathcal{P}dV$, the change in the internal energy $d\langle H\rangle$, the energy required to add a particle to the system $-\mu d\langle N\rangle$, and finally the energy needed to change its total momentum $-\mathbf{V}\cdot d\langle\mathbf{P}\rangle$.

We turn now to express the partition function in terms of a functional integral. To do this we need to express the trace in terms of the "coherent state" representation that appears in the functional integral formulation. First we write the trace as a diagonal sum over a complete set of orthonormal states and insert the "coherent state" resolution of the identity which holds for either Bosons or Fermions:

$$\begin{aligned}\mathrm{Tr}\,X &= \textstyle\sum_n \langle n|X\,1\,|n\rangle \\ &= \int[d\psi^{*\prime}d\psi']\exp\left\{-\int\psi^{*\prime}\psi'\right\}\textstyle\sum_n\langle n|X|\psi'\rangle\langle\psi^{*\prime}|n\rangle.\end{aligned} \tag{2.5.14}$$

Here the operator X is "Bosonic" — it is composed of an arbitrary number of Boson operators, but only an even number of Fermi operators. For Boson states, the matrix elements are ordinary numbers which commute,

$$\text{Bosons:}\quad \langle n|X|\psi'\rangle\langle\psi^{*\prime}|n\rangle = \langle\psi^{*\prime}|n\rangle\langle n|X|\psi'\rangle, \tag{2.5.15}$$

but for Fermion states the matrix elements contain the anticommuting Grassmann numbers and they may not commute. The integration over Grassmann numbers in Eq. (2.5.14) vanishes unless the terms which appear when the product of the matrix elements is expanded in powers of $\psi^{*\prime}$ and ψ' contain the same number of $\psi^{*\prime}$ and ψ'. In particular, this integration selects only the terms in the matrix element product that are either *both even* in ψ' and $\psi^{*\prime}$ or *both odd* in ψ' and $\psi^{*\prime}$ The matrix elements with an even number of Grassmann variables commute, but those with an odd number of Grassmann variables anticommute. This extra minus sign for the case of an odd number of variables is obtained by changing the sign of ψ' which, of course, has no effect for an even number of variables. Therefore, in either case we may write,

$$\text{Fermions:}\quad \langle n|X|\psi'\rangle\langle\psi^{*\prime}|n\rangle = \langle\psi^{*\prime}|n\rangle\langle n|X|-\psi'\rangle. \tag{2.5.16}$$

So for both Bosons (+) and Fermions (−) the trace may be written as

$$\begin{aligned}\mathrm{Tr}\,X &= \int[d\psi^{*\prime}d\psi']e^{-\int\psi^{*\prime}\psi'}\textstyle\sum_n\langle\psi^{*\prime}|n\rangle\langle n|X|\pm\psi'\rangle \\ &= \int[d\psi^{*\prime}d\psi']e^{-\int\psi^{*\prime}\psi'}\,\langle\psi^{*\prime}|X|\pm\psi'\rangle,\end{aligned} \tag{2.5.17}$$

where in the last equality the completeness of the $|n\rangle$ states has been used.

With this form of the trace,

$$Z = \int[d\psi^{*\prime}d\psi']e^{-\int\psi^{*\prime}\psi'}\langle\psi^{*\prime}|e^{-\beta(H-\mu N-\mathbf{V}\cdot\mathbf{P})}\,|\pm\psi'\rangle. \tag{2.5.18}$$

Let us compare the matrix element which appears here with the transformation function with a time-independent Hamiltonian:

$$\langle a',\,T|b',\,0\rangle = \langle a'|e^{-iTH}|b'\rangle. \tag{2.5.19}$$

The partition function is obtained by a continuation to imaginary time

$$T \to -i\beta\,, \tag{2.5.20}$$

together with the replacement

$$H \to \tilde{H} = H - \mu N - \mathbf{V}\cdot\mathbf{P}\,. \tag{2.5.21}$$

Thus

$$Z = \int [d\psi^{*\prime} d\psi'] \exp\left\{-\int \psi^{*\prime}\psi'\right\} \langle \psi^{*\prime}, -i\beta| \pm \psi', 0\rangle^{(\tilde{H})}\,. \tag{2.5.22}$$

The analytic continuation of the time

$$t \to -i\tau\,, \tag{2.5.23}$$

together with the addition of the extra ("chemical potential") terms to the Hamiltonian replaces the real time Lagrangian L by $-\tilde{L}$, where

$$\tilde{L} = \int (d^3\mathbf{r})\, \psi^{*\prime}(\mathbf{r},\tau)\frac{\partial}{\partial\tau}\,\psi'(\mathbf{r},\tau) + H - \mu N - \mathbf{V}\cdot\mathbf{P}\,, \tag{2.5.24}$$

and produces the functional integral representation

$$\langle \psi^{*\prime}, -i\beta| \pm \psi', 0\rangle = \exp\left\{\int \psi^{*\prime}\psi'\right\} \int [d\psi^* d\psi] \exp\left\{-\int_0^\beta d\tau\, \tilde{L}\right\}. \tag{2.5.25}$$

To quickly present the result, we proceed in a rather formal way and pay no attention to the details which will be explained shortly. The exponential prefactor in Eq. (2.5.18) involving the fields $\psi^{*\prime}\psi'$ cancels the exponential factor in Eq. (2.5.25), and the functional integral representation for the partition function takes the form

$$Z = \int [d\psi^{*\prime} d\psi'] \int [d\psi^* d\psi] \exp\left\{-\int_0^\beta d\tau \tilde{L}\right\}. \tag{2.5.26}$$

Here the first functional integral is over the fields at the end point $\tau = \beta$. The second functional integral includes integrations over all the "time" slices in the interval $0 < \tau < \beta$. We incorporate all these integrations in a single symbol and write

$$Z = \int [d\psi^* d\psi] \exp\left\{-\int_0^\beta d\tau \tilde{L}\right\}. \tag{2.5.27}$$

This is the functional integral representation of the grand canonical partition function.

The work of the preceding paragraph needs further elaboration. In particular, we need to pay attention to the boundary terms at $\tau = 0\,,\ \beta$. The structure of these terms is the same as those in the simplest quantum-mechanical model with a single degree of freedom. Thus, to focus on the

essence of the problem, we consider a system with a pair of variables $\phi^*\,,\ \phi$ which can be either Bosonic or Fermionic. Moreover, the issue is one of kinematics rather than dynamics, and so we do not need to describe the time development explicitly. The essential points can be made by examining the formal trace

$$\mathcal{Z} = \int d\phi_0^* d\phi_0\, e^{-\phi_0^*\phi_0}\, \langle \phi_0^*| \pm \phi_0\rangle\,, \tag{2.5.28}$$

and by examining the contribution of a single intermediate state sum. That is, we write

$$\begin{aligned}\langle \phi_f^*|\phi_i\rangle &= \int d\phi_1^* d\phi_1\, e^{-\phi_1^*\phi_1} \langle \phi_f^*|\phi_1\rangle\langle \phi_1^*|\phi_i\rangle \\ &= \int d\phi_1^* d\phi_1\, e^{S_{f_1}}\,,\end{aligned} \tag{2.5.29}$$

where, since

$$\langle \phi_1^*|\phi_2\rangle = e^{\phi_1^*\phi_2}\,, \tag{2.5.30}$$

the "action" is given by

$$S_{fi} = -\phi_1^*\phi_1 + \phi_f^*\phi_1 + \phi_1^*\phi_i\,. \tag{2.5.31}$$

Writing this "action" in the form

$$S_{fi} = \phi_f^*\phi_f - \phi_f^*(\phi_f - \phi_1) - \phi_1^*(\phi_1 - \phi_i) \tag{2.5.32}$$

corresponds to the form

$$S_{fi} = \phi_f^*\phi_f - \int d\tau \phi^* \frac{\partial\phi}{\partial\tau} \tag{2.5.33}$$

that appears in the functional integral when (infinitely) many intermediate states are introduced into the transformation function. The alternative, algebraic equivalent form

$$S_{fi} = (\phi_f^* - \phi_1^*)\phi_1 + (\phi_1^* - \phi_i^*)\phi_i + \phi_i^*\phi_i \tag{2.5.34}$$

corresponds to

$$S_{fi} = \phi_i^*\phi_i + \int d\tau \frac{\partial\phi^*}{\partial\tau}\phi\,. \tag{2.5.35}$$

The two different forms are related to one another by an "integration by parts". Inserting the single intermediate sum (2.5.29) into the trace (2.5.28) gives

$$\mathcal{Z} = \int d\phi_0^* d\phi_0 d\phi_1^* d\phi_1\, e^S\,, \tag{2.5.36}$$

where we must make the identifications $\phi_f^* = \phi_0^*$ and $\phi_i = \pm\phi_0$. Using Eq. (2.5.32) or Eq. (2.5.34) gives

$$S = -\phi_0^*\phi_0 + \phi_0^*\phi_f - \phi_0^*(\phi_f - \phi_1) - \phi_1^*(\phi_1 \mp \phi_0)\,, \tag{2.5.37}$$

or

$$S = -\phi_0^*\phi_0 \pm \phi_i^*\phi_0 + (\phi_0^* - \phi_1^*)\phi_1 + (\phi_1^* - \phi_i^*)(\pm\phi_0)\,. \tag{2.5.38}$$

Note that ϕ_f and ϕ_i^* may be chosen arbitrarily — the terms involving them cancel each other in Eqs. (2.5.37) and (2.5.38). These superfluous variables were introduced so as to get differences which become derivatives in the continuum limit. However, the proper choice is evident: We choose $\phi_f = \phi_0 = \pm\phi_i$ so that the first two terms in Eq. (2.5.37) cancel and $\phi_i^* = \pm\phi_0^* = \pm\phi_f^*$ so that the first two terms in Eq. (2.5.38) cancel. Thus, only the difference terms remain. Thus the functional integral representation of the trace involves boundary conditions connecting the initial and final variables of the form

$$\phi_f = \pm\phi_i\,, \qquad \phi_f^* = \pm\phi_i^*\,, \tag{2.5.39}$$

with the + sign for Bosons and the − sign for Fermions. Bosonic variables are periodic, Fermionic variables are anti-periodic. In our simple example we have one integration to compose the trace and another integration over an intermediate set of variables. This is equivalent to an integration over $d\phi^* d\phi$ for each of the (two) difference terms in Eqs. (2.5.37) and (2.5.38). It has the obvious generalization: The functional integral representation of the trace involves an integration over each variable pair that appears in the differences which lead to the time derivative in the continuum limit. It is worth noting that the periodic or anti-periodic boundary conditions imply that we may freely integrate by parts with no additional boundary terms as indicated in

$$\int d\tau\, \psi^* \frac{\partial\psi}{\partial\tau} = -\int d\tau\, \frac{\partial\psi^*}{\partial\tau}\psi\,. \tag{2.5.40}$$

A summary is in order. A beautiful and compact expression for both Bose and Fermi statistics has been obtained. The statistics are expressed by the boundary conditions

$$\psi(\mathbf{r},\beta) = \pm\psi(\mathbf{r},0)\,, \qquad \psi^*(\mathbf{r},\beta) = \pm\psi^*(\mathbf{r},0)\,, \tag{2.5.41}$$

with the + sign holding for Bosons, the − sign for Fermions. Bosons are periodic in τ in the interval in $(0,\beta)$ while Fermions are antiperiodic in $(0,\beta)$. This is made manifest if the fields are expressed in terms of Fourier series,

$$\psi(\mathbf{r},\tau) = \sum\nolimits_\nu e^{-2\pi i\nu\tau/\beta}\, \psi_\nu(\mathbf{r})\,, \tag{2.5.42}$$

$$\psi^*(\mathbf{r},\tau) = \sum\nolimits_\nu e^{+2\pi i\nu\tau/\beta}\, \psi_\nu^*(\mathbf{r})\,, \tag{2.5.43}$$

in which for Bosons ν runs over integers while for Fermions ν runs over odd

half integers. These are linear transformations of variables and therefore

$$Z = [\mathrm{Det}\, S]^{\pm 1} \int \prod_\nu [d\psi_\nu^* d\psi_\nu] \exp\left\{-\int_0^\beta d\tau\, \tilde{L}\right\}. \tag{2.5.44}$$

Here S is the infinite dimensional matrix of derivatives of ψ with respect to ψ_ν and ψ^* with respect to ψ_ν^* and $[\mathrm{Det}\, S]^{+1}$ is the constant Jacobian determinant for the transformation of the ordinary integration variables for the Bosons, while the $[\mathrm{Det}\, S]^{-1}$ factor is for transforming the anti-commuting variables for Fermions. The Jacobian $\mathrm{Det}\, S$ can be explicitly computed by generalizing the work of Chapter 1, Section 9 on a single degree of freedom to an infinite number of degrees of freedom. However, this computation can be avoided by simply noting that $\mathrm{Det}\, S$ is essentially an overall constant which is independent of the parameters and the interaction potential of the system. Hence this constant can be found by taking the interaction potential to vanish and by evaluating the grand partition function Z in a suitable limit: One takes $\mathbf{V} = 0$ and μ negative so that the effective Hamiltonian of Eq. (2.5.21) is a positive operator. Then in the zero temperature ($\beta \to \infty$) limit, only the vacuum state contributes to the partition function (2.5.1) and it reduces to unity. The point is that the work of Chapter 1, Section 9 shows that with enough care one can explicitly evaluate such Jacobian factors and that our procedure is, in fact, well defined. But, knowing that it can be done, there is no real need to carry out the calculation explicitly and one can be confident that no errors have been made.

As an example of the utility and simplicity of the functional integral formulation in providing structural relations, we end this section by examining the role of the velocity parameter $\mathbf{V}$ in the grand canonical partition function. The effective Hamiltonian which appears in the functional integral is given by

$$\begin{aligned}\tilde{H} &= H - \mu N - \mathbf{V}\cdot\mathbf{P} \\ &= \int (d\mathbf{r})\psi^*(\mathbf{r},\tau)\Big[-\frac{\nabla^2}{2m} - \mu - \mathbf{V}\cdot\frac{1}{i}\nabla\Big]\psi(\mathbf{r},\tau) + H_2\,,\end{aligned} \tag{2.5.45}$$

where

$$H_2 = \frac{1}{2}\int (d\mathbf{r})(d\mathbf{r}')\psi^*(\mathbf{r},\tau)\psi^*(\mathbf{r}',\tau)V_2(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}',t)\psi(\mathbf{r},\tau)\,. \tag{2.5.46}$$

The local phase rotation

$$\psi(\mathbf{r},\tau) = e^{i\phi(\mathbf{r})}\psi'(\mathbf{r},\tau)\,, \qquad \psi^*(\mathbf{r},\tau) = e^{-i\phi(\mathbf{r})}\psi^{*\prime}(\mathbf{r},\tau)\,, \tag{2.5.47}$$

leaves the functional integration measure invariant. It does not alter the

interaction part H_2 of the Hamiltonian, and a little calculation shows that

$$\tilde{H} = \int (d\mathbf{r}) \psi^{*\prime}(\mathbf{r}, \tau) \Big[-\frac{\nabla^2}{2m} - \Big(\mu + \mathbf{V} \cdot (\nabla\phi) - \frac{1}{2m}(\nabla\phi)^2 + \frac{i}{2m}(\nabla^2\phi) \Big) - \Big(\mathbf{V} - \frac{1}{m}(\nabla\phi) \Big) \cdot \frac{1}{i}\nabla \Big] \psi'(\mathbf{r}, \tau) + H_2 \,. \tag{2.5.48}$$

In particular, the choice

$$\phi(\mathbf{r}) = m\mathbf{V} \cdot \mathbf{r} \tag{2.5.49}$$

removes the original velocity term and replaces it with an effective chemical potential

$$\mu_{\text{eff}} = \mu + \frac{1}{2} m\mathbf{V}^2 \,. \tag{2.5.50}$$

We have thus proven that

$$Z(\beta, \mu, \mathbf{V}) = Z\Big(\beta, \mu + \frac{1}{2} m\mathbf{V}^2, 0\Big) \,. \tag{2.5.51}$$

We may now make use of Eqs. (2.5.5), (2.5.6), and (2.5.7) to find that

$$\begin{aligned} \langle \mathbf{P} \rangle &= \frac{\partial}{(\partial \beta \mathbf{V})} \ln Z \\ &= m\mathbf{V} \frac{\partial}{\partial(\beta\mu)} \ln Z \\ &= m\mathbf{V} \langle N \rangle \,. \end{aligned} \tag{2.5.52}$$

The average total momentum of the moving system is the average total mass $m\langle N \rangle$ times the velocity $\mathbf{V}$. The chemical potential measures the energy increase when a particle is added to the system. We see that μ_{eff} includes the kinetic energy of adding a particle to the moving system. It should be noted that what has been done is to implement a Galilean transformation using the functional integral formulation. This is made explicit in the working of Problem 1.

2.6 Free gas

We turn now to compute the grand partition function for a free gas of Bosons or Fermions. We have just seen that there is no loss of generality in setting $\mathbf{V} = 0$. We first note that the orthogonal mode expansions (2.5.42) and (2.5.43) put the free particle (imaginary time) action in the form

$$\int_0^\beta d\tau \, \tilde{L} = \int_0^\beta d\tau \int (d^3\mathbf{r}) \, \psi^*(\mathbf{r}, \tau) \Big[\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu \Big] \psi(\mathbf{r}, \tau)$$

$$= \sum_\nu \int (d^3\mathbf{r}) \, \psi_\nu^* \beta \Big[-\frac{2\pi i \nu}{\beta} - \frac{\nabla^2}{2m} - \mu \Big] \psi_\nu \,. \tag{2.6.1}$$

Introducing an operator notation similar to that used previously in Chapter 1, Section 6, with an inner product involving $\sum_\nu$ and $\int (d^3\mathbf{r})$, gives

$$\int_0^\beta d\tau \tilde{L} = \psi^* G^{-1} \psi \,. \tag{2.6.2}$$

Hence the partition function for a free gas becomes

$$\begin{aligned} Z &= [\operatorname{Det} S]^{\pm 1} \int \prod_\nu [d\psi_\nu^* d\psi_\nu] \, e^{-\psi^* G^{-1} \psi} \\ &= [\operatorname{Det} S]^{\pm 1} \, [\operatorname{Det} G^{-1}]^{\mp 1} \,. \end{aligned} \tag{2.6.3}$$

The Gaussian integration in the Bose case gives a factor of Det^{-1} not $\operatorname{Det}^{-\frac{1}{2}}$ because it involves a pair of integration variables* (ψ^*, ψ). The Gaussian integration in the Fermi case gives a factor of Det^{+1} according to our previous discussion. Overall constant factors such as $(2\pi)^\infty$ have been omitted since they can be absorbed into $\operatorname{Det} S$. Writing the determinant in the trace-log form and calculating the trace in the inner product basis just described gives

$$\begin{aligned} \operatorname{Det} G^{-1} &= \exp\{ \operatorname{Tr} \ln G^{-1} \} \\ &= \exp\left\{ \operatorname{Tr} \ln \left[\frac{\partial}{\partial \tau} + \frac{\mathbf{p}^2}{2m} - \mu \right] \right\} \\ &= \exp\left\{ \sum_\nu \int (d^3\mathbf{r}) \langle \mathbf{r} | \ln \beta \left[\frac{-2\pi i \nu}{\beta} + \frac{\mathbf{p}^2}{2m} - \mu \right] | \mathbf{r} \rangle \right\} . \end{aligned} \tag{2.6.4}$$

Introducing the resolution of the identity in the momentum basis allows the abstract operator $\mathbf{p}^2$ inside the logarithm to be replaced by its eigenvalue — which to save a clutter of notation we represent by the same symbol — and we obtain the explicit form

$$\operatorname{Det} G^{-1} = \exp\left\{ \sum_\nu \int (d^3\mathbf{r}) \int (d^3\mathbf{p}) \langle \mathbf{r} | \mathbf{p} \rangle \ln \beta \left[\frac{-2\pi i \nu}{\beta} + \frac{\mathbf{p}^2}{2m} - \mu \right] \langle \mathbf{p} | \mathbf{r} \rangle \right\} . \tag{2.6.5}$$

Since

$$\langle \mathbf{r} | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{r} \rangle = |\langle \mathbf{r} | \mathbf{p} \rangle|^2 = \frac{1}{(2\pi)^3} \,, \tag{2.6.6}$$

* Alternatively, one can understand the appearance of $\operatorname{Det}^{-1} G^{-1}$ as the Jacobian of the transformation $G^{-1}\psi = \phi$, $\psi^* = \phi^*$.

the integrand is independent of $\mathbf{r}$, and the $\mathbf{r}$-integration gives simply

$$\int (d^3\mathbf{r})1 = V\,, \tag{2.6.7}$$

where V is the volume of the (large) box which holds the gas. Thus, defining

$$\mathcal{E} = \frac{\mathbf{p}^2}{2m} - \mu\,, \tag{2.6.8}$$

Eq. (2.6.5) becomes

$$\mathrm{Det}\, G^{-1} = \exp\left\{V \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \sum\nolimits_\nu \ln\left[-2\pi i\nu + \beta\mathcal{E}\right]\right\}. \tag{2.6.9}$$

To compute the sum over ν, we write

$$\sum_\nu \ln[-2\pi i\nu + \beta\mathcal{E}] = \{\ln\beta\mathcal{E}\} + \sum_{\nu>0} \ln\left[1 + \left(\frac{\beta\mathcal{E}}{2\pi\nu}\right)^2\right] + \sum_{\nu>0} \ln(2\pi\nu)^2\,, \tag{2.6.10}$$

where the curly brackets enclose the $\nu = 0$ terms that occurs only for the Bose case. The last ν-sum diverges, but it gives rise only to an overall constant that again can be absorbed into Det$S$. The remaining sum can be done for the Bose case, ν = integers, with the aid of the product representation

$$\sinh z = z \prod_{k=1}^{\infty} \left[1 + \left(\frac{z}{\pi k}\right)^2\right]. \tag{2.6.11}$$

Similarly, for the Fermi case, ν = odd-half-integers, we may use

$$\cosh z = \prod_{k=1}^{\infty} \left[1 + \left(\frac{z}{\pi(k-\frac{1}{2})}\right)^2\right]. \tag{2.6.12}$$

Writing

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \qquad \cosh z = \frac{1}{2}(e^z + e^{-z})\,, \tag{2.6.13}$$

and doing a little algebra one finds that (with the infinite but temperature independent terms discarded)

$$\sum\nolimits_\nu \ln[-2\pi i\nu + \beta\mathcal{E}] = \frac{1}{2}\beta\mathcal{E} + \ln(1 \mp e^{-\beta\mathcal{E}})\,, \tag{2.6.14}$$

where the upper $-$ sign is for Bosons, the lower $+$ sign for Fermions. As will be verified in the next paragraph, these are just the right signs to get the usual Bose and Fermi distributions. These signs follow directly from the periodic and anti-periodic boundary conditions on the functional integral.

The first term on the right-hand side of this equation, $\frac{1}{2}\beta\mathcal{E}$, corresponds to a "zero point energy" contribution. Our original operator Hamiltonian did not contain such terms since it involved the operator ordering $\psi^\dagger \ldots \psi$. Evidently, this term has arisen from our somewhat cavalier manipulations, and it should not appear if we had made a careful calculation such as that in Section 9 of Chapter 1. We shall simply discard it to get

$$\begin{aligned} Z &= [\mathrm{Det}\, SG^{-1}]^{\mp 1} \\ &= (\mathrm{const}) \exp\left\{\mp V \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \ln\left(1 \mp e^{-\beta\mathcal{E}}\right)\right\}. \end{aligned} \tag{2.6.15}$$

Here all our sins of computation have been amassed into the overall (const) factor out in front. As was mentioned before, this factor is determined by taking the chemical potential to be negative, and taking the low temperature limit, $\beta \to \infty$. In this case $\mathcal{E}$ is positive, the exponential in the logarithm in Eq. (2.6.15) vanishes, and so does the logarithm. Since only the vacuum state contributes in this limit, $Z \to 1$, and the constant prefactor is determined to be unity. Hence we arrive at the familiar formula for the grand partition function,

$$\begin{aligned} Z &= \mathrm{Tr}\, e^{-\beta(H-\mu N)} \\ &= \exp\left\{\mp gV \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \ln\left[1 \mp \exp\left\{-\beta\left(\frac{p^2}{2m} - \mu\right)\right\}\right]\right\}. \end{aligned} \tag{2.6.16}$$

Here we have generalized the result a bit by including a degeneracy factor g. The gas particles may have some internal degree of freedom such as spin. In this case, the trace in Eq. (2.6.4) includes a sum over g identical terms and produces an overall factor of g. For spin s, $g = 2s + 1$. Note that the chemical potential for a free Bose gas must not be positive, for otherwise the argument of the logarithm in Eq. (2.6.16) would be negative at small p.

As was described in the preceding section, the grand partition function gives all the thermodynamic properties of a system. For the present case of a free Bose or Fermi gas, the average particle number is given by

$$\begin{aligned} \langle N\rangle &= \frac{\partial}{\partial(\beta\mu)} \ln Z\Big|_{\beta \text{ fixed}} \\ &= gV \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \left[\exp\left\{\beta\left(\frac{p^2}{2m} - \mu\right)\right\} \mp 1\right]^{-1}. \end{aligned} \tag{2.6.17}$$

This can be used to determine the chemical potential μ in terms of the particle density $n = \langle N\rangle/V$ and the temperature $T = 1/k\beta$. The average

or "internal" energy is given by

$$\langle H\rangle = -\frac{\partial}{\partial\beta}\ln Z\Big|_{(\beta\mu)\text{ fixed}}$$
$$= gV\int\frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{p^2}{2m}\left[\exp\left\{\beta\left(\frac{p^2}{2m}-\mu\right)\right\}\mp 1\right]^{-1}. \quad (2.6.18)$$

The general formula for the pressure yields for the free gas

$$\mathcal{P} = \frac{1}{\beta V}\ln Z$$
$$= \mp\frac{g}{\beta}\int\frac{(d^3\mathbf{p})}{(2\pi)^3}\ln\left[1\mp\exp\left\{-\beta\left(\frac{p^2}{2m}-\mu\right)\right\}\right]. \quad (2.6.19)$$

Scaling the momentum by writing $\mathbf{p} = \mathbf{q}/\sqrt{\beta}$ exhibits the functional form

$$\ln Z = \left(V/\beta^{\frac{3}{2}}\right)F(\beta\mu). \quad (2.6.20)$$

Hence

$$-\frac{\partial}{\partial\beta}\ln Z\Big|_{(\beta\mu)\text{ fixed}} = \frac{3}{2\beta}\ln Z, \quad (2.6.21)$$

and so the energy may be expressed as

$$\langle H\rangle = \frac{3}{2}\mathcal{P}V. \quad (2.6.22)$$

This formula, which is familiar from classical, Boltzmann statistics is also valid for (non-relativistic) quantum gases of either Bose or Fermi statistics. To complete this list of general formulae, we recall that the entropy per unit volume is given by the derivative of the pressure with respect to the temperature at fixed chemical potential:

$$\frac{S}{V} = \frac{\partial\mathcal{P}}{\partial T}\Big|_{\mu\text{ fixed}}. \quad (2.6.23)$$

Using the formula above for the pressure of a free gas it is easy to compute this entropy explicitly.

2.7 Spontaneous symmetry breaking

A lump of iron at high temperatures is paramagnetic. An applied magnetic field $\mathbf{B}$ induces a magnetization $\mathbf{M}$ which vanishes when the magnetic field is turned off. However, if the temperature T is lowered below a critical temperature T_c (the Curie temperature) before the field is turned off, the magnetization $\mathbf{M}$ does not vanish. The iron has become

a ferromagnet.* From a naive point of view this is a very peculiar state of affairs. The Hamiltonian of the ferromagnet is rotationally invariant so one would think that the ground state should be spherically symmetric. Yet there is a preferred direction in the ground state, the direction along the magnetization $\mathbf{M}$, a direction along that of the original magnetizing field $\mathbf{B}$. But this is an arbitrary direction. There are infinitely many degenerate ground states of the ferromagnet labeled by the infinitely many directions of $\mathbf{M}$.

The ferromagnet is a familiar example of a general class of systems that exhibit spontaneous symmetry breakdown: Systems whose dynamics is invariant under some symmetry group but whose ground state does not share this symmetry. Such systems are important not only in condensed matter physics but also in elementary particle physics and even in cosmology. Most of the formal machinery has now been developed to describe a very interesting illustration of the spontaneous symmetry breakdown that is provided by liquid helium at low temperatures, superfluid liquid helium. This we shall do using a simplified model that describes a good deal of the essential physics of the problem. In addition to illustrating the character of spontaneous symmetry in a real and interesting physical system, the introduction of the notions of the effective action and effective potential in this work should motivate their further study in Chapter 6.

The electrons in helium atoms are tightly bound. The atoms are rather rigid and not easily polarized. Thus the long-distance attractive force between two helium atoms (which results from their polarizability) is weak and the short-distance force between two atoms is strongly repulsive. This character of the interatomic potential between two helium atoms is reflected in the fact that they do not bind to form a molecule He^2. If the circumstances are not too extreme (e.g. pressures not so high as to crush the atoms into their component nuclei and electrons), one can describe a system of helium atoms using only two-body forces. Since we will be interested in investigating the superfluid phase of helium whose character is revealed by its motion, it will be convenient to use the extended grand canonical ensemble with a velocity parameter, characterized by the "thermodynamic Hamiltonian"

$$\tilde{H} = H - \mu N - \mathbf{V}\cdot\mathbf{P}$$
$$= \int(d\mathbf{r})\,\psi^\dagger(\mathbf{r})\left[-\frac{\nabla^2}{2m}-\mu-\mathbf{V}\cdot\frac{1}{i}\nabla\right]\psi(\mathbf{r})$$
$$+\frac{1}{2}\int(d\mathbf{r})(d\mathbf{r}')\,\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}). \quad (2.7.1)$$

* To keep the discussion simple, the domain structure in a real ferromagnet is neglected.

This Hamiltonian is invariant against the constant, "global" phase transformations

$$\psi(\mathbf{r}) \to e^{i\theta}\psi(\mathbf{r}), \quad \psi^\dagger(\mathbf{r}) \to e^{-i\theta}\psi^\dagger(\mathbf{r}). \tag{2.7.2}$$

As we shall see, the phase symmetry (2.7.2) is broken in superfluid helium. In general, when spontaneous symmetry breaking can occur, in order to define a unique ground state, or in our case the equilibrium thermal ensemble, it is convenient to introduce some external perturbation that explicitly breaks the symmetry (like the external field $\mathbf{B}$ for the ferromagnet), calculate the response, and then take the limit where the perturbation vanishes. In our case, the simplest choice is to couple external sources to ψ and $\psi^\dagger$. These are idealized sources that destroy and create particles. Although it would suffice to take the sources to be constants, we shall at first be more general and consider sources $\eta(\mathbf{r},\tau)$, $\eta^*(\mathbf{r},\tau)$ that depend both on the spatial point $\mathbf{r}$ and on the imaginary time parameter τ that enters into the thermodynamic description. Thus we shall consider the functional integral for this Bosonic system

$$Z[\eta,\eta^*] = \int [d\psi][d\psi^*] \exp\left\{ -\int_0^\beta d\tau \tilde{L} + \int_0^\beta d\tau \int (d\mathbf{r}) \left[\psi^*(\mathbf{r},\tau)\eta(\mathbf{r},\tau) + \eta^*(\mathbf{r},\tau)\psi(\mathbf{r},\tau)\right] \right\}, \tag{2.7.3}$$

in which

$$\tilde{L} = \int (d\mathbf{r})\, \psi^*(\mathbf{r},\tau)\frac{\partial}{\partial\tau}\psi(\mathbf{r},\tau) + \tilde{H}, \tag{2.7.4}$$

where $\tilde{H}$ is the previous "thermodynamic Hamiltonian", but with the operators $\psi, \psi^\dagger$ replaced by the functional integration variables ψ, ψ^*.

As was discussed in Chapter 1, Section 3, an additional factor of an arbitrary functional of the fields within the functional integral produces the corresponding matrix element of the functional with the numerical fields replaced by time-ordered products of the operator fields. The source terms in Eq. (2.7.3) comprise such an additional factor. The functional integral in Eq. (2.7.3) has been continued to imaginary time (with an effective Hamiltonian $\tilde{H}$), and periodic boundary conditions on the fields are imposed to produce the trace. Hence

$$Z[\eta,\eta^*] = \mathrm{Tr}\, e^{-\beta\tilde{H}} T \exp\left\{ \int_0^\beta d\tau \int (d\mathbf{r}) [\psi^\dagger(\mathbf{r},\tau)\eta(\mathbf{r},\tau) + \eta^*(\mathbf{r},\tau)\psi(\mathbf{r},\tau)] \right\}, \tag{2.7.5}$$

in which

$$\psi(\mathbf{r},\tau) = e^{\tilde{H}\tau}\psi(\mathbf{r},0)e^{-\tilde{H}\tau}, \tag{2.7.6}$$

$$\psi^\dagger(\mathbf{r},\tau) = e^{\tilde{H}\tau}\psi^\dagger(\mathbf{r},0)e^{-\tilde{H}\tau}, \tag{2.7.7}$$

and T denotes a τ-ordering where operators with the larger values of τ are placed to the left. Note that although $\psi^\dagger(\mathbf{r},0)$ is the Hermitian adjoint of $\psi(\mathbf{r},0)$, this relationship does not extend to $\psi^\dagger(\mathbf{r},\tau)$ and $\psi(\mathbf{r},\tau)$ since the evolution in imaginary time is not unitary. Treating η and η^* as independent variables and expanding in powers of η and η^*, shows that $Z[\eta,\eta^*]$ is the generating functional for "thermal Green's functions". These are the thermal ensemble averages of "time"-ordered products of field operators. After the sources have been used to generate these thermal Green's functions, they must be restricted to be independent of the imaginary time variable τ with $\eta^*(\mathbf{r})$ the true complex conjugate of $\eta(\mathbf{r})$ so as to get an Hermitian Hamiltonian which includes the addition of[†] $\psi^\dagger\eta + \eta^*\psi$.

1 Effective potential

To describe possible spontaneous symmetry breaking, we need to develop some formalism. Although the functionals that we shall introduce have thermodynamic analogs with different notation, we shall use symbols that are now common in relativistic field theory except for various factors of i that are removed here since we deal with "imaginary time". First we introduce the effective action $W[\eta,\eta^*]$ which is the logarithm of the grand partition function in the presence of sources,[‡]

$$\exp\{W[\eta,\eta^*]\} = Z[\eta,\eta^*]. \tag{2.7.8}$$

Now if X is any operator functional of the field operators $\psi(\mathbf{r},\tau)$ and $\psi^\dagger(\mathbf{r},\tau)$,

$$\langle X\rangle = \frac{1}{Z[\eta,\eta^*]}\int [d\psi][d\psi^*]\, X \exp\left\{ -\int_0^\beta d\tau \tilde{L} + \int_0^\beta d\tau \int (d\mathbf{r})[\psi^*\eta + \eta^*\psi] \right\} \tag{2.7.9}$$

[†] With η and η^* independent of τ, it follows from the functional integral representation that partition function of Eq. (7.5) reduces to $\mathrm{Tr}\exp\{-\beta\tilde{H}_{\mathrm{tot}}\}$, where $\tilde{H}_{\mathrm{tot}}$ includes the constant sources. This, of course, is just an example of the relationship between the interaction and Heisenberg pictures, but for a time-independent perturbation. This relationship between the two pictures, which can be proven directly with operator techniques, was derived from the functional integral representation in Eq. (1.3.32).

[‡] A general discussion of the effective action is given in Chapter 6. We derive here only the basic properties needed for the present work.

defines an ensemble average or expectation value of X in the presence of the external sources $\eta(\mathbf{r},\tau)$ and $\eta^*(\mathbf{r},\tau)$. Here, of course, when we write X within the functional integral, the operators $\psi, \psi^\dagger$ are to be replaced by the functional integration variables ψ, ψ^*. Since the variational derivative of the functional integral with respect to $\eta(\mathbf{r},\tau)$ or $\eta^*(\mathbf{r},\tau)$ brings down a factor of $\psi^*(\mathbf{r},\tau)$ or $\psi(\mathbf{r},\tau)$ into the integrand, it is easy to see that

$$\frac{\delta W[\eta,\eta^*]}{\delta\eta(\mathbf{r},\tau)} = \langle\psi^\dagger(\mathbf{r},\tau)\rangle\,, \tag{2.7.10}$$

and

$$\frac{\delta W[\eta,\eta^*]}{\delta\eta^*(\mathbf{r},\tau)} = \langle\psi(\mathbf{r},\tau)\rangle\,. \tag{2.7.11}$$

Thus $W[\eta,\eta^*]$ is the generating functional for expectation values of field operators — which is the reason for its definition.

We have mentioned that the spontaneous symmetry breaking which we are investigating is connected to a phase symmetry. The phase symmetry is made explicit by changing functional integration variables from ψ, ψ^* to $\psi', \psi^{*\prime}$, where the two sets differ by a constant (global) phase rotation:

$$\psi(\mathbf{r},\tau) = e^{i\theta}\psi'(\mathbf{r},\tau)\,, \qquad \psi^*(\mathbf{r},\tau) = e^{-i\theta}\psi^{*\prime}(\mathbf{r},\tau)\,. \tag{2.7.12}$$

Clearly, this change has a unit Jacobian, and it leaves the functional integral unaltered except for the source terms. Hence

$$Z[\eta,\eta^*] = Z[e^{i\theta}\eta\,,\, e^{-i\theta}\eta^*]\,, \tag{2.7.13}$$

which symmetry also carries over to the effective action,

$$W[\eta,\eta^*] = W[e^{i\theta}\eta\,,\, e^{-i\theta}\eta^*]\,. \tag{2.7.14}$$

The sources η and η^* were introduced in order to select a unique equilibrium state — so as to set the "alignment" of the system just as the magnetic field does for the ferromagnet. These sources induce non-vanishing values of $\langle\psi\rangle$ and $\langle\psi^\dagger\rangle$. For a normal system, these field expectation values vanish when the sources are turned off, since non-vanishing field expectation values violate the phase symmetry (2.7.2). As we shall see, the field expectation values do not vanish when the system undergoes spontaneous symmetry breakdown, even though the sources have been removed. To exhibit this behavior, it is convenient to consider the field expectation values $\langle\psi\rangle$ and $\langle\psi^\dagger\rangle$ as the independent variables rather than the sources η and η^*. This is done by carrying out a functional Legendre

transformation which defines[§]

$$\Gamma = \int_0^\beta d\tau \int (d\mathbf{r}) \left\{\langle\psi^\dagger(\mathbf{r},\tau)\rangle\eta(\mathbf{r},\tau) + \eta^*(\mathbf{r},\tau)\langle\psi(\mathbf{r},\tau)\rangle\right\} - W[\eta,\eta^*]\,. \tag{2.7.15}$$

Consider the variation of this equation with respect to η and η^*. These variations induce changes $\delta\langle\psi^\dagger\rangle$, $\delta\langle\psi\rangle$, and so

$$\begin{aligned}\delta\Gamma = &\int_0^\beta d\tau \int (d\mathbf{r}) \left\{\delta\langle\psi^\dagger\rangle\eta + \eta^*\delta\langle\psi\rangle\right\} \\ &+ \int_0^\beta d\tau \int (d\mathbf{r}) \left\{\langle\psi^\dagger\rangle\delta\eta + \delta\eta^*\langle\psi\rangle - \frac{\delta W}{\delta\eta}\delta\eta - \frac{\delta W}{\delta\eta^*}\delta\eta^*\right\}.\end{aligned} \tag{2.7.16}$$

Since $W[\eta,\eta^*]$ is the generating functional for $\langle\psi\rangle$ and $\langle\psi^\dagger\rangle$ [Eqs. (2.7.10) and (2.7.11)], the terms in the curly brackets cancel. Thus, we may regard Γ as a functional of $\langle\psi\rangle$ and $\langle\psi^\dagger\rangle$ with the properties

$$\frac{\delta\Gamma[\langle\psi\rangle,\,\langle\psi^\dagger\rangle]}{\delta\langle\psi(\mathbf{r},\tau)\rangle} = \eta^*(\mathbf{r},\tau)\,, \tag{2.7.17}$$

$$\frac{\delta\Gamma[\langle\psi\rangle,\,\langle\psi^\dagger\rangle]}{\delta\langle\psi^\dagger(\mathbf{r},\tau)\rangle} = \eta(\mathbf{r},\tau)\,. \tag{2.7.18}$$

Note that if we make a phase variation $\delta\langle\psi\rangle = i\delta\theta\langle\psi\rangle$, $\delta\langle\psi^\dagger\rangle = -i\delta\theta\langle\psi^\dagger\rangle$, we find that

$$\begin{aligned}\delta_\theta\Gamma &= i\delta\theta\int\int[\eta^*\langle\psi\rangle - \langle\psi^\dagger\rangle\eta] \\ &= i\delta\theta\int\int\left[\eta^*\frac{\delta W}{\delta\eta^*} - \eta\frac{\delta W}{\delta\eta}\right] \\ &= \delta_\theta W = 0\,.\end{aligned} \tag{2.7.19}$$

Thus Γ also shares in the phase symmetry:

$$\Gamma[e^{i\theta}\langle\psi\rangle,\, e^{-i\theta}\langle\psi^\dagger\rangle] = \Gamma[\langle\psi\rangle,\,\langle\psi^\dagger\rangle]\,. \tag{2.7.20}$$

We now specialize to uniform sources that are constant in space and imaginary time,

$$\eta(\mathbf{r},\tau) \to \eta\,, \qquad \eta^*(\mathbf{r},\tau) \to \eta^*\,. \tag{2.7.21}$$

In this case the expectation values must also be constant, and we have,

[§] The properties of Γ are also discussed extensively in Chapter 6.

for example,

$$\delta_\eta W = \int_0^\beta d\tau \int (d\mathbf{r}) \frac{\delta W}{\delta \eta} \delta\eta = \beta V \langle \psi^\dagger \rangle \delta\eta \,, \tag{2.7.22}$$

or

$$\frac{\partial W}{\partial \eta} = \beta V \langle \psi^\dagger \rangle \,. \tag{2.7.23}$$

Hence W contains a factor of the spatial volume V, and so must Γ. We also remember the phase invariance to conclude that now

$$\Gamma = \beta V \mathcal{V}(\langle \psi^\dagger \rangle \langle \psi \rangle) \,, \tag{2.7.24}$$

where $\mathcal{V}$ is an ordinary function that is independent of the volume of the system. It is called the "effective potential". In the limit of uniform sources, Eqs. (2.7.17) and (2.7.18) reduce to the ordinary partial derivatives

$$\frac{\partial \mathcal{V}(|\langle \psi \rangle|^2)}{\partial \langle \psi \rangle} = \eta^* \,, \tag{2.7.25}$$

and

$$\frac{\partial \mathcal{V}(|\langle \psi \rangle|^2)}{\partial \langle \psi^\dagger \rangle} = \eta \,. \tag{2.7.26}$$

The sources η and η^* were introduced to make the equilibrium state unique, but once this is achieved they are taken to vanish (just as was the case for the ferromagnet). In the limit $\eta^* = 0 = \eta$,

$$\frac{\partial \mathcal{V}}{\partial \langle \psi \rangle} = 0 = \frac{\partial \mathcal{V}}{\partial \langle \psi^\dagger \rangle} \,. \tag{2.7.27}$$

Thus the correct thermodynamic state is determined as a stationary point of the effective potential $\mathcal{V}$. At a stationary point $|\langle \psi \rangle|$ is determined by the thermodynamic parameters β, μ, $\mathbf{V}$, and so $\mathcal{V}$ at such a point is only a function of these parameters. In view of the chain of definitions — (2.7.8) for W, (2.7.15) for Γ, and (2.7.24) for $\mathcal{V}$ — the grand canonical partition function is now expressed as

$$Z = e^{-\beta V \mathcal{V}} \,. \tag{2.7.28}$$

Accordingly, when the effective potential $\mathcal{V}$ is evaluated at a stationary point it reduces to the usual thermodynamic potential Ω [Eq. (2.5.8)] divided by the volume V. The thermodynamic ground state is given by the stationary point which is a global minimum — other stationary points may, in some circumstances, correspond to different metastable phases of higher energy.

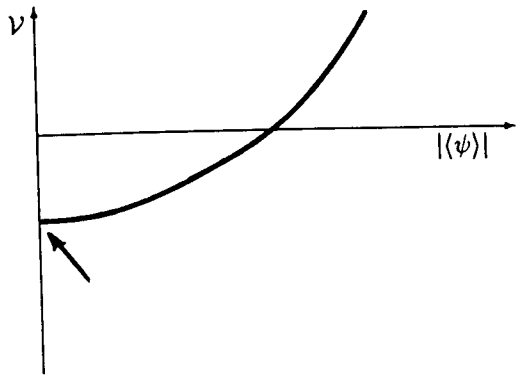


Fig. 2.1. Effective potential with no spontaneous symmetry breaking.

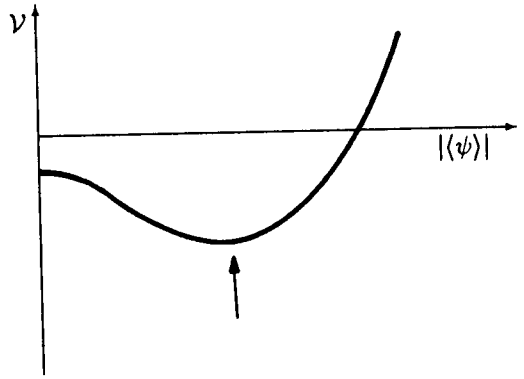


Fig. 2.2. Effective potential with spontaneous symmetry breaking.

2 *Symmetry broken spontaneously*

Although more complicated situations can arise, there are two cases which illustrate the cases of ordinary symmetry and spontaneously broken symmetry. First suppose that a plot of the effective potential versus $|\langle \psi \rangle|$ appears as shown in Fig. 2.1. In this case, the stationary conditions (2.7.27) require that $|\langle \psi \rangle| = 0$ as shown by the position of the arrow in the sketch. In this case, the field expectation value is completely determined, and there was really no need to introduce external sources — the system is already well-defined without them. On the other hand, suppose that the plot of the effective potential versus $|\langle \psi \rangle|$ appears as in Fig. 2.2. In this case, the stationary conditions (2.7.27) require that $|\langle \psi \rangle| \neq 0$ as

shown by the position of the arrow in the sketch.¶ The effective potential with such a shape gives rise to a spontaneously broken symmetry: The phase of $\langle\psi\rangle$ can be chosen arbitrarily. (This is the analog of the direction of the magnetization in a ferromagnet.) Changing the phase of $\langle\psi\rangle$ does not change the value of the effective potential. Extending this global phase rotation to a local phase rotation where the phase varies very slowly in space gives a very small alteration of the energy of the system brought about by the spatial gradient of the phase. In the limit where the wavelength of this spatial variation becomes infinite, there is no gap separating the energies of such excited states from that of the ground state. This behavior will be shown explicitly later on. We will find that there is a sound wave excitation in the system. Finally, we should note that the effective potential depends upon parameters such as the temperature and the chemical potential, and altering these parameters may change the symmetric situation shown in Fig. 2.1 into the spontaneously broken case shown in Fig. 2.2. This change corresponds to a phase transformation.

We have remarked that, in the case of spontaneously broken symmetry, the phase of the field expectation value $\langle\psi\rangle$ can be chosen arbitrarily. With non-vanishing sources this phase ambiguity is removed, and the system is well defined. Remembering that the sources must be true complex conjugates of each other when they play the role of defining the system with an Hermitian Hamiltonian, we set $\eta = e^{i\theta}|\eta|$ and $\eta^* = e^{-i\theta}|\eta|$ which locks the phase of the field expectation values, $\langle\psi\rangle = e^{i\theta}|\langle\psi\rangle|$ and $\langle\psi^\dagger\rangle = e^{-i\theta}|\langle\psi\rangle|$. This phase locking follows from Eqs. (2.7.25) and (2.7.26). The phase remains determined in the limit $|\eta| \to 0$ like the smile of the Cheshire cat. At this point, it should be remembered that in order to investigate the superfluid phase, we are working out a thermal ensemble with the velocity $\mathbf{V}$. But we need to allow for the possibility that the "condensate" described by $\langle\psi\rangle$ moves with a different velocity $\mathbf{V}_S$. This will exhibit the two-fluid behavior characteristic of superfluid helium. In view of the work at the end of Section 5, we are motivated to consider not the restriction to uniform sources as was done to provide a first illustration, but rather consider the limit to sources which are uniform in magnitude but with an oscillatory phase,

$$\eta(\mathbf{r},\tau) \to e^{im\mathbf{V}_S\cdot\mathbf{r}}\,\eta\,, \tag{2.7.29}$$

and

$$\eta^*(\mathbf{r},\tau) \to e^{-im\mathbf{V}_S\cdot\mathbf{r}}\,\eta^*\,. \tag{2.7.30}$$

As in the work leading to Eq. (2.5.48), the phase variation in the sources

¶ The stationary point at $|\langle\psi\rangle| = 0$ has higher energy and corresponds to the normal phase which has now become metastable.

is removed by transforming the field variables

$$\psi(\mathbf{r},\tau) \to e^{im\mathbf{V}_S\cdot\mathbf{r}}\,\psi(\mathbf{r},\tau)\,, \tag{2.7.31}$$

and

$$\psi^*(\mathbf{r},\tau) \to e^{-im\mathbf{V}_S\cdot\mathbf{r}}\psi^*(\mathbf{r},\tau)\,, \tag{2.7.32}$$

which puts the effective thermodynamic Hamiltonian in the form

$$\tilde{H} = \int(d\mathbf{r})\psi^*(\mathbf{r},\tau)\Big[-\frac{\nabla^2}{2m} - \Big(\mu - \frac{m}{2}\mathbf{V}_S\cdot(\mathbf{V}_S - 2\mathbf{V})\Big) - (\mathbf{V}-\mathbf{V}_S)\cdot\frac{1}{i}\nabla\Big]\psi(\mathbf{r},\tau) + H_2\,. \tag{2.7.33}$$

Thus, the effect of oscillatory sources is taken into account by using uniform fields but writing

$$\mathcal{V} = \mathcal{V}\Big(\beta\,,\,\mu - \frac{1}{2}m\mathbf{V}_S\cdot(\mathbf{V}_S - 2\mathbf{V})\,,\,\mathbf{V}-\mathbf{V}_S\,;\,|\langle\psi\rangle|^2\Big)\,, \tag{2.7.34}$$

where $\mathcal{V}\,(\beta\,,\,\mu\,,\,\mathbf{V}\,;\,|\langle\psi\rangle|^2)$ is the previous effective potential. Thus we may calculate the effective potential with $\mathbf{V}_S = 0$ and then make the substitutions shown in Eq. (2.7.34) to reinstate $\mathbf{V}_S$.

3 Loop expansion

We cannot simply make a power series expansion in the strength of the interaction potential since the zeroth order term, that which describes a free gas, does not give rise to spontaneous symmetry breaking and is thus not an appropriate starting point. Instead what we shall do is to expand about the classical approximation to the functional integral. As we shall show in Chapter 6, Section 4, this corresponds to a well-defined "loop expansion". To achieve the expansion, we introduce a parameter λ into the functional integral representation by writing

$$Z[\eta,\eta^*] = \int[d\psi][d\psi^*]\exp\left\{-\frac{1}{\lambda}\int_0^\beta d\tau\left[\tilde{L} - \int(d\mathbf{r})(\psi^*\eta + \eta^*\psi)\right]\right\}. \tag{2.7.35}$$

We shall expand (the logarithm of Z) in powers of λ and then set $\lambda = 1$. Here λ plays the role of $\hbar$ in the previous quantum mechanical case discussed in Chapter 1, Section 4, and the first term in this expansion is just the "stationary phase" or classical limit. In the first approximation

$$Z[\eta,\eta^*] = \exp\left\{-\frac{1}{\lambda}\int_0^\beta d\tau\left[\tilde{L}_0 - \int(d\mathbf{r})(\psi_0^*\eta + \eta^*\psi_0)\right]\right\}, \tag{2.7.36}$$

where $\tilde{L}_0$ is composed of the fields $\psi_0\,,\,\psi_0^*$ which make this expression stationary — the solutions of the classical field equations. In lowest order,

$\psi_0 = \langle\psi\rangle$ and $\psi_0^* = \langle\psi^\dagger\rangle$. The Legendre transformation to Γ just removes the source terms and with $\lambda = 1$ gives, in this first approximation,

$$\Gamma_0 = \int_0^\beta d\tau \tilde{L}_0 \,. \tag{2.7.37}$$

We are interested in the effective potential $\mathcal{V} = \Gamma/\beta V$ for uniform fields $\langle\psi\rangle, \langle\psi^\dagger\rangle$. Remembering the structure of $\tilde{L}$ which involves the Hamiltonian (2.7.1), we find that

$$\mathcal{V}_0 = \frac{1}{2}\tilde{V}(0)|\langle\psi\rangle|^4 - \mu_{\text{eff}}|\langle\psi\rangle|^2 \,. \tag{2.7.38}$$

Here $\tilde{V}(0)$ is the volume integral of the two body interaction potential,

$$\tilde{V}(0) = \int (d\mathbf{r}) V(\mathbf{r}) \,. \tag{2.7.39}$$

This notation is used because we shall shortly need to work with the Fourier transform for arbitrary momentum transfer,

$$\tilde{V}(\mathbf{p}) = \int (d\mathbf{r}) e^{i\mathbf{p}\cdot\mathbf{r}} V(\mathbf{r}) \,. \tag{2.7.40}$$

The stability of the spontaneously broken phase requires that the volume integral of the potential be positive, $\tilde{V}(0) > 0$, for otherwise $\mathcal{V}_0$ would dive to $-\infty$ with $|\langle\psi\rangle| \to \infty$. That is, the interatomic force, on the average, must be repulsive. This is in fact the case for two helium atoms. According to Eq. (2.7.34), the effect of the velocity parameter $\mathbf{V}_S$ is taken into account by writing

$$\mu_{\text{eff}} = \mu - \frac{1}{2} m \mathbf{V}_S \cdot (\mathbf{V}_S - 2\mathbf{V}) \,. \tag{2.7.41}$$

The effective potential $\mathcal{V}$ is sketched for various values of the effective chemical potential μ_{eff} in Fig. 2.3. There is a broken symmetry ground state determined by

$$\frac{\partial \mathcal{V}_0}{\partial |\langle\psi\rangle|} = 0 \,, \tag{2.7.42}$$

if μ_{eff} is positive. (Recall that the chemical potential must be negative for a free Bose gas.) The stationary point which gives the global minimum is

$$|\langle\psi\rangle|^2 = \frac{\mu_{\text{eff}}}{\tilde{V}(0)} \,. \tag{2.7.43}$$

The non-vanishing expectation value $\langle\psi\rangle$ is often referred to as the "condensate". It is an "order parameter" akin to the magnetization $\mathbf{M}$ in the

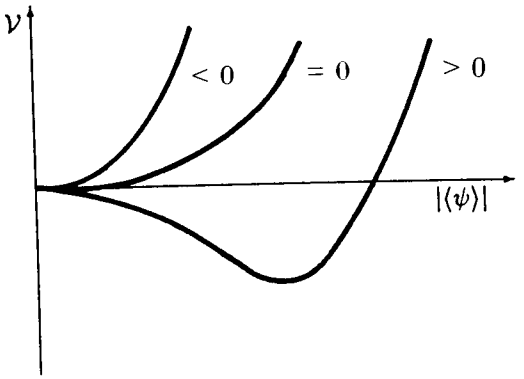


Fig. 2.3. Effective potential for different values of the chemical potential μ_{eff}.

ferromagnetic case, with $\langle\psi\rangle$ and η playing roles analogous[‖] to $\mathbf{M}$ and $\mathbf{B}$. At the stationary point, the effective potential reduces to the usual thermodynamic potential divided by the volume,

$$\frac{\Omega}{V} = \mathcal{V} = -\frac{\mu_{\text{eff}}^2}{2\tilde{V}(0)} \,. \tag{2.7.44}$$

In view of the general thermodynamic relations (2.5.8) and (2.5.6), the average number density in this approximation is just the square of the condensate amplitude,

$$\frac{\langle N\rangle}{V} = \frac{\mu_{\text{eff}}}{\tilde{V}(0)} = |\langle\psi\rangle|^2 \,. \tag{2.7.45}$$

Moreover, in view of Eq. (2.5.5), the average energy density is given by

$$\frac{\langle H\rangle}{V} = \frac{1}{2}\frac{\mu_{\text{eff}}^2}{\tilde{V}(0)} = \frac{1}{2}\left(\frac{\langle N\rangle}{V}\right)^2 \tilde{V}(0) \,. \tag{2.7.46}$$

These should be obvious results. The pressure is the negative of the thermodynamic potential Ω divided by the volume [Eq. (2.5.9)]. With the chemical potential μ_{eff} eliminated in favor of the average number density,

$$\mathcal{P} = \frac{1}{2}\left(\frac{\langle N\rangle}{V}\right)^2 \tilde{V}(0) = \frac{\langle H\rangle}{V} \,. \tag{2.7.47}$$

Since, in this lowest approximation, the thermodynamic potential is independent of the temperature, Eq. (2.5.12) informs us that the entropy

[‖] This is only a rough correspondence because $\langle\psi\rangle$ and η are not directly physically measurable quantities whereas $\mathbf{M}$ and $\mathbf{B}$ are.

vanishes,

$$S = 0\,. \tag{2.7.48}$$

This again is quite reasonable since in this approximation there is a perfectly ordered state. Finally, we recall that the average momentum is given by the velocity gradient (2.5.7). Hence it follows from Eqs. (2.7.38) and (2.7.41) that

$$\langle \mathbf{P} \rangle = m\mathbf{V}_S \left(\frac{\mu_{\text{eff}}}{\tilde{V}(0)} \right) V = m\mathbf{V}_S |\langle \psi \rangle|^2 V = m\mathbf{V}_S \langle N \rangle\,. \tag{2.7.49}$$

Thus the condensate moves with the velocity $\mathbf{V}_S$.

4 *First correction*

Thus far we have a rather unrealistic model which does not contain any temperature dependence. We must go on to the next level of approximation. This we do by writing out the functional integration variables as

$$\psi = \psi_0 + \sqrt{\lambda}\psi'\,, \qquad \psi^* = \psi_0^* + \sqrt{\lambda}\psi^{*\prime}\,, \tag{2.7.50}$$

expanding the exponent in powers of λ, and keeping the lowest correction. Since $\psi = \psi_0$, $\psi^* = \psi_0^*$ is a stationary point, this lowest correction entails the quadratic terms in the fluctuation variables ψ' and $\psi^{*\prime}$. Hence, setting $\lambda = 1$, and taking uniform sources η, η^*, we get

$$Z[\eta, \eta^*] = \exp\left\{ -\int_0^\beta d\tau \left[\tilde{L}_0 - \int (d\mathbf{r})(\psi_0^* \eta + \eta^* \psi_0) \right] \right\} Z_2[\psi_0, \psi_0^*]\,, \tag{2.7.51}$$

where

$$Z_2[\psi_0, \psi_0^*] = \int [d\psi'][d\psi'^*] \exp\left\{ -\int_0^\beta d\tau \left[\int (d\mathbf{r}) \psi^{*\prime} \frac{\partial}{\partial \tau} \psi' + H^{(2)} \right] \right\}, \tag{2.7.52}$$

with, in view of Eqs. (2.7.1) and (2.7.34),

$$\begin{aligned} H^{(2)} = & \int (d\mathbf{r})\, \psi^{*\prime}(\mathbf{r}, \tau) \left[-\frac{\nabla^2}{2m} - \bar{\mu}_{\text{eff}} - (\mathbf{V} - \mathbf{V}_S) \cdot \frac{1}{i} \nabla \right] \psi'(\mathbf{r}, \tau) \\ & + \frac{1}{2} \int (d\mathbf{r})(d\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \{ 2|\psi_0|^2 \psi^{*\prime}(\mathbf{r}, \tau) \psi'(\mathbf{r}', \tau) \\ & + \psi_0^{*2} \psi'(\mathbf{r}, \tau) \psi'(\mathbf{r}', \tau) + \psi_0^2 \psi^{*\prime}(\mathbf{r}, \tau) \psi^{*\prime}(\mathbf{r}', \tau) \}\,, \end{aligned} \tag{2.7.53}$$

in which

$$\bar{\mu}_{\text{eff}} = \mu_{\text{eff}} - \tilde{V}(0)|\psi_0|^2 \tag{2.7.54}$$

defines an altered effective chemical potential. Here (ψ_0, ψ_0^*) is the solution to the classical field equations driven by (η, η^*). These solutions (ψ_0, ψ_0^*) now differ from the expectation values $(\langle \psi \rangle, \langle \psi^\dagger \rangle)$ defined by Eqs. (2.7.10) and (2.7.11) since we are now including first order corrections in $W[\eta, \eta^*]$. However, the classical "thermodynamic action", the term in the exponential in Eq. (2.7.51), is stationary about (ψ_0, ψ_0^*), and thus it is not altered by first order corrections. Hence to our level of approximation we may replace (ψ_0, ψ_0^*) in this classical "action" by $(\langle \psi \rangle, \langle \psi^\dagger \rangle)$. Moreover, since $Z_2[\psi_0, \psi_0^*]$ is already a first order correction, we may also make this replacement here. Accordingly,

$$\Gamma[\langle \psi \rangle, \langle \psi^\dagger \rangle] = \Gamma_0[\langle \psi \rangle, \langle \psi^\dagger \rangle] - \ln Z_2[\langle \psi \rangle, \langle \psi^\dagger \rangle]\,. \tag{2.7.55}$$

To compute $Z_2[\langle \psi \rangle, \langle \psi^\dagger \rangle]$, we first express the functional integration variables in terms of Fourier transforms

$$\psi'(\mathbf{r}, \tau) = \int \frac{(d\mathbf{p})}{(2\pi)^3} \psi(\mathbf{p}, \tau) e^{i\mathbf{p}\cdot\mathbf{r}}\,, \tag{2.7.56}$$

$$\psi^{*\prime}(\mathbf{r}, \tau) = \int \frac{(d\mathbf{p})}{(2\pi)^3} \psi^*(\mathbf{p}, \tau) e^{-i\mathbf{p}\cdot\mathbf{r}}\,, \tag{2.7.57}$$

which gives

$$\begin{aligned} & H^{(2)} \\ & = \int \frac{(d\mathbf{p})}{(2\pi)^3} \left\{ \psi^*(\mathbf{p}, \tau) \psi(\mathbf{p}, \tau) \left[\frac{\mathbf{p}^2}{2m} - \bar{\mu}_{\text{eff}} - (\mathbf{V} - \mathbf{V}_S) \cdot \mathbf{p} + |\langle \psi \rangle|^2 \tilde{V}(\mathbf{p}) \right] \right. \\ & \quad \left. + \psi(-\mathbf{p}, \tau) \psi(\mathbf{p}, \tau) \frac{1}{2} \langle \psi^\dagger \rangle^2 \tilde{V}(\mathbf{p}) + \psi^*(\mathbf{p}, \tau) \psi^*(-\mathbf{p}, \tau) \frac{1}{2} \langle \psi \rangle^2 \tilde{V}(\mathbf{p}) \right\}. \end{aligned} \tag{2.7.58}$$

This is almost of the form of a free (quasi)particle Hamiltonian except that $\psi\psi, \psi^*\psi^*$ terms appear as well as the standard $\psi^*\psi$ structure. But it can be put into the standard structure. To do this, we first note that if $\langle \psi \rangle = |\langle \psi \rangle| e^{i\theta}$, then the constant phase rotation of the functional integration variables $\psi \to e^{i\theta}\psi, \psi^* \to e^{-i\theta}\psi^*$ replaces $\langle \psi \rangle^2$ and $\langle \psi^\dagger \rangle^2$ with $|\langle \psi \rangle|^2$. We perform this rotation and then make use of the transformation

$$\psi(\mathbf{p}, \tau) = u(\mathbf{p}) \tilde{\psi}(\mathbf{p}, \tau) - v(\mathbf{p}) \tilde{\psi}^*(-\mathbf{p}, \tau)\,, \tag{2.7.59}$$

$$\psi^*(\mathbf{p}, \tau) = u(\mathbf{p}) \tilde{\psi}^*(\mathbf{p}, \tau) - v(\mathbf{p}) \tilde{\psi}(-\mathbf{p}, \tau)\,, \tag{2.7.60}$$

where $u(\mathbf{p}) = u(-\mathbf{p})$ and $v(\mathbf{p}) = v(-\mathbf{p})$ are real functions. This change of functional integration variables entails a Jacobian which is the (infinite)

product of terms containing

$$\begin{vmatrix} \partial\psi(\mathbf{p})/\partial\tilde{\psi}(\mathbf{p}) & \partial\psi(\mathbf{p})/\partial\tilde{\psi}^*(-\mathbf{p}) \\ \partial\psi^*(-\mathbf{p})/\partial\tilde{\psi}(\mathbf{p}) & \partial\psi^*(-\mathbf{p})/\partial\tilde{\psi}^*(-\mathbf{p}) \end{vmatrix} = \begin{vmatrix} u(\mathbf{p}) & -v(\mathbf{p}) \\ -v(\mathbf{p}) & u(\mathbf{p}) \end{vmatrix} . \tag{2.7.61}$$

One chooses

$$u(\mathbf{p})^2 - v(\mathbf{p})^2 = 1 \tag{2.7.62}$$

to ensure that this Jacobian is unity. With this choice it is easy to check that

$$\int_0^\beta d\tau \int \frac{(d\mathbf{p})}{(2\pi)^3}\psi^*(\mathbf{p},\tau)\frac{\partial}{\partial\tau}\psi(\mathbf{p},\tau) = \int_0^\beta d\tau \int \frac{(d\mathbf{p})}{(2\pi)^3}\tilde{\psi}^*(\mathbf{p},\tau)\frac{\partial}{\partial\tau}\tilde{\psi}(\mathbf{p},\tau) , \tag{2.7.63}$$

since total derivatives with respect to τ give no contribution in view of the cyclic boundary condition on the fields. A similar calculation shows that

$$\int \frac{(d\mathbf{p})}{(2\pi)^3}\psi^*(\mathbf{p},\tau)\mathbf{p}\psi(\mathbf{p},\tau) = \int \frac{(d\mathbf{p})}{(2\pi)^3}\tilde{\psi}^*(\mathbf{p},\tau)\mathbf{p}\tilde{\psi}(\mathbf{p},\tau) . \tag{2.7.64}$$

To go on with the "diagonalization" of $H^{(2)}$, we see that the coefficient of the $\tilde{\psi}(-\mathbf{p})\tilde{\psi}(\mathbf{p})$ terms vanishes if

$$[u(\mathbf{p})^2 + v(\mathbf{p})^2]|\langle\psi\rangle|^2\tilde{V}(\mathbf{p}) = 2u(\mathbf{p})v(\mathbf{p})\left[\frac{\mathbf{p}^2}{2m} - \bar{\mu}_{\text{eff}} + |\langle\psi\rangle|^2\tilde{V}(\mathbf{p})\right] . \tag{2.7.65}$$

The same condition ensures that the coefficient of $\tilde{\psi}^*(-\mathbf{p})\tilde{\psi}^*(\mathbf{p})$ also vanishes. Equation (2.7.65) together with the constraint (2.7.62) is easily solved. To put the solution in a compact form, we first define

$$\mathcal{E}(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \bar{\mu}_{\text{eff}} , \tag{2.7.66}$$

and

$$E(\mathbf{p}) = \sqrt{\mathcal{E}(\mathbf{p})^2 + 2|\langle\psi\rangle|^2\tilde{V}(\mathbf{p})\mathcal{E}(\mathbf{p})} . \tag{2.7.67}$$

Then a little algebra shows that the solution is given by

$$u(\mathbf{p}) + v(\mathbf{p}) = \sqrt{\frac{E(\mathbf{p})}{\mathcal{E}(\mathbf{p})}} , \tag{2.7.68}$$

and

$$u(\mathbf{p}) - v(\mathbf{p}) = \sqrt{\frac{\mathcal{E}(\mathbf{p})}{E(\mathbf{p})}} . \tag{2.7.69}$$

A little more algebra is now needed to show that the transformations (2.7.59), (2.7.60) reduce Eq. (2.7.58) to

$$\begin{aligned} H^{(2)} = &\int \frac{(d\mathbf{p})}{(2\pi)^3}\tilde{\psi}^*(\mathbf{p},\tau)\left[-(\mathbf{V}-\mathbf{V}_S)\cdot\mathbf{p}\right]\tilde{\psi}(\mathbf{p},\tau) \\ &+ \int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{4\mathcal{E}(\mathbf{p})}\Big\{ [E(\mathbf{p})+\mathcal{E}(\mathbf{p})]^2\,\tilde{\psi}^*(\mathbf{p},\tau)\tilde{\psi}(\mathbf{p},\tau) \\ &\qquad - [E(\mathbf{p})-\mathcal{E}(\mathbf{p})]^2\,\tilde{\psi}(\mathbf{p},\tau)\tilde{\psi}^*(\mathbf{p},\tau)\Big\} . \end{aligned} \tag{2.7.70}$$

We have written this in such a fussy way because, as we have seen in the case of the free gas, the functional integral representation, calculated in a naive fashion, does not correctly reproduce zero point energy contributions — one must be careful in dealing with operator ordering problems. The functional integral that we started with is presumed to be defined such that it represents a Hamiltonian with all $\psi^\dagger$ operators placed to the left of all ψ operators. The Fourier transformation of the canonical commutation relation gives

$$[\psi(\mathbf{p}),\, \psi^\dagger(\mathbf{p}')] = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}') , \tag{2.7.71}$$

with the other commutators vanishing. The condition (2.7.62) ensures that the operator transformation corresponding to Eqs. (2.7.59) and (2.7.60) is canonical, giving

$$[\tilde{\psi}(\mathbf{p}),\, \tilde{\psi}^\dagger(\mathbf{p}')] = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}') , \tag{2.7.72}$$

with the other commutators of the new fields vanishing. We have written $H^{(2)}$ with the ordering that would have been obtained if it had entailed the field operators, $\psi \to \psi$, $\psi^* \to \psi^\dagger$. Since the functional integral must be carefully defined to represent this operator ordering, we should replace the numerical $H^{(2)}$ above with the corresponding operator, order the $\psi, \psi^\dagger$ operators, and then identify the result as that to be produced by the functional integration over the new $\tilde{\psi}, \tilde{\psi}^*$ variables. Since

$$(2\pi)^3\delta(0) = \int (d^3\mathbf{r}) = V , \tag{2.7.73}$$

we must interpret $H^{(2)}$ as corresponding to the operator

$$H^{(2)} = V\mathcal{E}_0 + \int \frac{(d\mathbf{p})}{(2\pi)^3}\tilde{\psi}^\dagger(\mathbf{p},\tau)\left[E(\mathbf{p}) - (\mathbf{V}-\mathbf{V}_S)\cdot\mathbf{p}\right]\tilde{\psi}(\mathbf{p},\tau) , \tag{2.7.74}$$

where the zero point energy is given by

$$\mathcal{E}_0 = -\int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{4\mathcal{E}(\mathbf{p})}\left[E(\mathbf{p})-\mathcal{E}(\mathbf{p})\right]^2 . \tag{2.7.75}$$

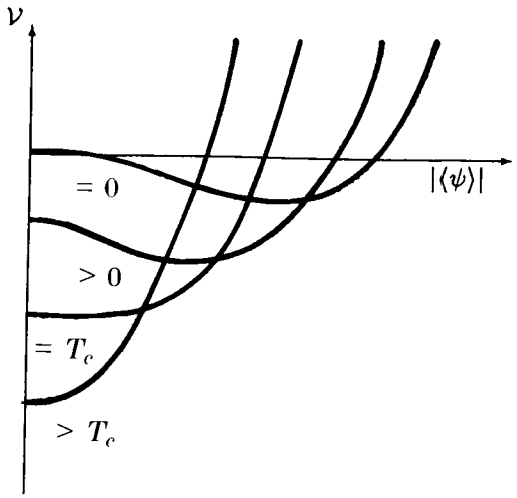


Fig. 2.4. Effective potential at various temperatures T.

We have found that, in addition to a zero point energy contribution, $H^{(2)}$ describes a set of free quasi-particles with the energy-momentum relation given by $E(\mathbf{p})$. Therefore, referring to our earlier work on a free Bose gas, we see that

$$\ln Z_2[\langle\psi\rangle, \langle\psi^\dagger\rangle] = -\beta V \mathcal{E}_0 - V \int \frac{(d\mathbf{p})}{(2\pi)^3} \ln\left[1 - \exp\left\{-\beta\left[E(\mathbf{p}) - (\mathbf{V} - \mathbf{V}_S)\cdot\mathbf{p}\right]\right\}\right] . \quad (2.7.76)$$

Accordingly, remembering Eq. (2.7.55), the effective potential including the first-order correction now appears as

$$\mathcal{V}(|\langle\psi\rangle|^2) = \frac{1}{2}\tilde{V}(0)|\langle\psi\rangle|^4 - \mu_{\text{eff}}|\langle\psi\rangle|^2 + \mathcal{E}_0 + \frac{1}{\beta}\int \frac{(d\mathbf{p})}{(2\pi)^3} \ln\left[1 - \exp\left\{-\beta\left[E(\mathbf{p}) - (\mathbf{V} - \mathbf{V}_S)\cdot\mathbf{p}\right]\right\}\right] . \quad (2.7.77)$$

Without investigating the functional form of Eq. (2.7.77) in detail, we expect that as the temperature is raised there is a sequence of effective potentials whose qualitative form is shown in the sketch in Fig. 2.4. We expect that as the temperature is raised the magnitude of the condensate $|\langle\psi\rangle|$ decreases as shown by the positions marked by the arrows. At the temperature $T = T_c$ the condensate has disappeared and at higher temperatures there is an unbroken, normal system. It should emphasized that although the one-loop approximation correctly describes this qualitative character, it does not provide an accurate, quantitative description of the

behavior of the transition in liquid helium.

The thermodynamic relations which follow from the first approximation (2.7.77) to the effective potential $\mathcal{V}$ are obtained from the evaluation of $\mathcal{V}$ at its stationary point. Since the computation is valid only to first order, in the first-order correction terms we may use the lowest-order value for the stationary point condensate (2.7.43) which we repeat here:

$$|\langle\psi\rangle|^2 = \frac{\mu_{\text{eff}}}{\tilde{V}(0)} . \quad (2.7.78)$$

With this choice, the altered chemical potential defined in Eq. (2.7.54) vanishes, $\bar{\mu}_{\text{eff}} = 0$, and Eqs. (2.7.66) and (2.7.67) reduce to

$$\mathcal{E}(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} , \quad (2.7.79)$$

and

$$E(\mathbf{p}) = \sqrt{\mathcal{E}(\mathbf{p})^2 + 2\mu_{\text{eff}}\left(\tilde{V}(\mathbf{p})/\tilde{V}(0)\right)\mathcal{E}(\mathbf{p})} . \quad (2.7.80)$$

The first two terms in the first-order approximation (2.7.77) comprise the zeroth-order effective potential $\mathcal{V}_0(|\langle\psi\rangle|)$. Since this function is stationary against small changes in $|\langle\psi\rangle|$ about the the zeroth-order stationary point (2.7.78), the first-order correction to $|\langle\psi\rangle|$ does not contribute to $\mathcal{V}_0$, and to first order it may be evaluated by using the zeroth-order result (2.7.78). Therefore, including all the terms of first order, the evaluation of Eq. (2.7.77) at the stationary point yields the thermodynamic potential

$$\frac{\Omega}{V} = \mathcal{V} = -\frac{1}{2}\frac{\mu_{\text{eff}}^2}{\tilde{V}(0)} + \mathcal{E}_0 + \frac{1}{\beta}\int \frac{(d\mathbf{p})}{(2\pi)^3} \ln\left[1 - \exp\left\{-\beta\left[E(\mathbf{p}) - (\mathbf{V} - \mathbf{V}_S)\cdot\mathbf{p}\right]\right\}\right] . \quad (2.7.81)$$

The effective chemical potential was defined in Eq. (2.7.41) which we repeat here:

$$\mu_{\text{eff}} = \mu - \frac{1}{2}m\mathbf{V}_S\cdot(\mathbf{V}_S - 2\mathbf{V}) . \quad (2.7.82)$$

The energies $\mathcal{E}(\mathbf{p})$ and $E(\mathbf{p})$ are now given by Eqs. (2.7.79) and (2.7.80). The zero-point energy $\mathcal{E}_0$ is given by Eq. (2.7.75) using these forms of the energies.

In this first approximation, the number density appears as

$$\frac{\langle N\rangle}{V} = \frac{\mu_{\text{eff}}}{\tilde{V}(0)} - \frac{\partial\mathcal{E}_0}{\partial\mu} - \int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{\partial E(\mathbf{p})}{\partial\mu}\frac{1}{\exp\left\{\beta\left[E(\mathbf{p}) - (\mathbf{V} - \mathbf{V}_S)\cdot\mathbf{p}\right]\right\} - 1} .$$

(2.7.83)

The quasi-particle contribution given by the momentum integral above vanishes in the zero temperature, $\beta \to \infty$, limit since it involves an exponentially damped term. But even in the zero-temperature limit, the particle number density departs from the square of the zeroth-order condensate wave function $|\langle\psi\rangle|^2$ given by the first term in Eq. (2.7.83) because of the $\partial\mathcal{E}_0/\partial\mu$ term which accounts (approximately) for the effect of the particle–particle interaction. As the temperature is increased from zero, the quasi-particle excitations also contribute to the density. Since at low temperature β is large, only small values of $\mathbf{p}$ contribute in the temperature-dependent corrections. In this case we may make the replacement

$$E(\mathbf{p}) \to |\mathbf{p}|\sqrt{\mu_{\text{eff}}/m}\,. \tag{2.7.84}$$

Thus the low-temperature excitations are phonons, sound waves with a speed

$$C = \sqrt{\mu_{\text{eff}}/m}\,. \tag{2.7.85}$$

To further elucidate the distinction between the condensate and quasi-particle contributions, we use the general thermodynamic relation (2.5.7) to calculate the average total momentum of the system from the approximate thermodynamic potential (2.7.81). Remembering the definition of the effective chemical potential (2.7.82), the required velocity gradient is given by $m\mathbf{V}_S$ times $\partial/\partial\mu$ plus the gradient of the explicit velocity dependence in appearing in Eq. (2.7.81). The chemical potential derivative produces the total average number of particles, and so

$$\langle\mathbf{P}\rangle = m\langle N\rangle\mathbf{V}_S + V\int\frac{(d^3\mathbf{p})}{(2\pi)^3}\,\mathbf{p}\,\frac{1}{\exp\{\beta[E(\mathbf{p}) - (\mathbf{V}-\mathbf{V}_S)\cdot\mathbf{p}]\} - 1}\,. \tag{2.7.86}$$

With $\mathbf{V} = \mathbf{V}_S$, the quasi-particle contribution that appears here vanishes since it involves the integration of $\mathbf{p}$ weighted by a rotationally invariant function of $\mathbf{p}$. In this case the whole fluid moves together with velocity $\mathbf{V}_S$. However, if $\mathbf{V}$ departs from $\mathbf{V}_S$, the quasi-particles give a separate contribution to the total momentum of the fluid — this is the two-fluid behavior that is characteristic of liquid helium II.

The superfluid state becomes a normal fluid when the velocity of the quasi-particle excitations relative to the fluid as a whole, $\mathbf{V}-\mathbf{V}_S$, becomes large. This is exhibited, in a heuristic fashion, by a breakdown in our model. The exponential term within the logarithm in the integral in the thermodynamic potential (2.7.81) must be less than unity to keep this logarithm real. Thus the inequality

$$E(\mathbf{p}) - (\mathbf{V}-\mathbf{V}_S)\cdot\mathbf{p} > 0 \tag{2.7.87}$$

must hold for all momenta $\mathbf{p}$. This implies that there is a maximum relative velocity given by

$$|\mathbf{V}-\mathbf{V}_S| < [\,E(\mathbf{p})/|\mathbf{p}|\,]_{\text{min}}\,, \tag{2.7.88}$$

where the expression on right-hand side is evaluated at its absolute minimum. When the relative velocity exceeds this maximum value, the one-loop thermal correction diverges, signaling the breakup of the superfluid state by thermal fluctuations. A bound on the maximum relative velocity is provided by the low-momentum limit (2.7.84) of the quasi-particle energy: The relative velocity cannot exceed the sound velocity,

$$|\mathbf{V}-\mathbf{V}_S| < C\,. \tag{2.7.89}$$

Problems

> But be ye doers of the word,
> and not hearers only,
> deceiving your own selves.
>
> James, i, 22

1. Galilean Invariance

Non-relativistic, many-particle systems are not invariant under Lorentz transformations but, in the absence of external forces, they are invariant under the transformations of Galilean relativity. The purpose of this problem is to explore the consequences of that invariance.

Let us first consider the special case of spatial rotations. Under an infinitesimal rotation of the coordinate frame, an n-particle state transforms according to*

$$\langle \mathbf{r}_1, \cdots, \mathbf{r}_n; t| \to \langle \mathbf{r}_1 + \delta\omega \times \mathbf{r}_1, \cdots, \mathbf{r}_n + \delta\omega \times \mathbf{r}_n; t| = \langle \mathbf{r}_1, \cdots, \mathbf{r}_n; t|U_t(\delta\omega)\,,$$

with the adjoint giving

$$|\mathbf{r}_1, \cdots, \mathbf{r}_n; t\rangle \to U_t(\delta\omega)^\dagger\, |\mathbf{r}_1, \cdots, \mathbf{r}_n; t\rangle\,.$$

Since the rotation preserves the equal-time transformation function (it is a product of rotationally invariant δ-functions), the operator $U_t(\delta\omega)$ must be unitary

$$U_t(\delta\omega)U_t(\delta\omega)^\dagger = 1\,.$$

The time-dependent transformation function

$$\langle \mathbf{r}_1, \cdots, \mathbf{r}_n; t_2|\mathbf{r}_1, \cdots, \mathbf{r}_n; t_1\rangle$$

viewed in the rotated frame is given by

$$\langle \mathbf{r}_1, \cdots, \mathbf{r}_n; t_2|U_{t_2}(\delta\omega)U_{t_1}(\delta\omega)^\dagger|\mathbf{r}_1, \cdots, \mathbf{r}_n; t_1\rangle\,.$$

The rotational invariance of the dynamics requires that the Hamiltonian obeys

$$U_t H U_t^\dagger = H\,,$$

* That these conventions do indeed define a (passive) transformation of the coordinate frame follows from considering the behavior of a single-particle wave function $\langle \mathbf{r}|S\rangle$ which is transformed to $\langle \mathbf{r} + \delta\mathbf{r}|S\rangle$. Thus if the original wave function is peaked at $\mathbf{r} = \bar{\mathbf{r}}$, the transformed wave function is peaked when $\mathbf{r} + \delta\mathbf{r} = \bar{\mathbf{r}}$, or at $\mathbf{r} = \mathbf{r} - \delta\mathbf{r}$, which corresponds to a transformation of the coordinates by the vector $\delta\mathbf{r}$.

or since U_t is unitary

$$U_t H = H U_t\,.$$

Hence

$$i\frac{\partial}{\partial t}\, U_t = [U_t, H] = 0\,,$$

and U_t is independent of the time t. Hence

$$\langle \mathbf{r}_1, \cdots, \mathbf{r}_n; t_2|U(\delta\omega)U(\delta\omega)^\dagger\, |\mathbf{r}_1, \cdots, \mathbf{r}_n; t_1\rangle = \langle \mathbf{r}_1, \cdots, \mathbf{r}_n; t_2|\mathbf{r}_1, \cdots, \mathbf{r}_n; t_1\rangle\,,$$

and so the transition amplitude viewed in the new frame is identical to the transition amplitude viewed in the old frame.

Although we have been discussing a fixed number of particles, the rotation operator $U(\delta\omega)$ can be defined for an arbitrary number of particles. We simply take the vacuum to be rotationally invariant

$$\langle 0|U(\delta\omega) = \langle 0|\,,$$

and require that the field operator, which creates the coordinate eigenbras when acting to the left on the vacuum state, obeys

$$\psi(\mathbf{r} + \delta\omega \times \mathbf{r}) = U(\delta\omega)^{-1}\psi(\mathbf{r})U(\delta\omega)\,.$$

Now

$$\psi(\mathbf{r} + \delta\omega \times \mathbf{r}) = [1 + i\delta\omega \cdot \mathbf{L}]\psi(\mathbf{r})\,,$$

in which

$$\mathbf{L} = \frac{1}{i}\, \mathbf{r} \times \nabla\,.$$

[There would be a spin matrix $\mathbf{S}$ added to $\mathbf{L}$ if the field described particles with spin.] The operator $\mathbf{J}$ is the generator of infinitesimal rotations defined by

$$U(\delta\omega) = 1 + i\delta\omega \cdot \mathbf{J}\,.$$

Hence, comparing the results above,

$$\mathbf{L}\psi = [\psi, \mathbf{J}]\,.$$

a) Prove that if an operator X commutes with both $\psi(\mathbf{r})$ and $\psi^\dagger(\mathbf{r})$, then this operator is a multiple of the unit operator,

$$X = \langle 0|X|0\rangle 1\,.$$

Hint: Consider the matrix element

$$\langle \mathbf{r}_1 \cdots \mathbf{r}_m|X|\mathbf{r}'_1 \cdots \mathbf{r}'_n\rangle\,.$$

b) The group property of rotations is expressed by

$$[L_k, L_l] = i\,\epsilon_{klm}\,L_m\,.$$

Use this and $\mathbf{L}\psi = [\psi, \mathbf{J}]$ to deduce that

$$[J_k, J_l] = i\,\epsilon_{klm}\,J_m\,.$$

[Hint: Remember the Jacobi identity.]

c) Write a short essay (as was done in the introduction to this problem) to explain why spatial translations are expressed by

$$\frac{1}{i}\,\nabla\psi = [\psi, \mathbf{P}]\,.$$

Compute

$$[P_k, P_l] = ?$$

d) Compute

$$[J_k, P_l] = ?$$

e) Express J_k and P_l in terms of volume integrals of field bilinears $\psi^\dagger \cdots \psi$.

f) We already know that time translations are generated by

$$i\,\frac{\partial}{\partial t}\,\psi = [\psi, H]\,.$$

Assuming that the system is translationally and rotationally invariant gives

$$[H, P_k] = 0 = [H, J_l]\,.$$

There remains the determination of the properties of the "Boost" operators G_k that correspond to viewing the system in moving coordinates, $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \delta\mathbf{v}t$. To determine these properties, first consider the sequence of geometrical transformations

$$\begin{pmatrix} \mathbf{r} \to \mathbf{r}' &=& \mathbf{r} + \delta\mathbf{v}t \\ t \to t' &=& t \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{r}' \to \mathbf{r}'' &=& \mathbf{r}' \\ t' \to t'' &=& t' + \delta t \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{r}'' \to \mathbf{r}''' &=& \mathbf{r}'' - \delta\mathbf{v}t'' \\ t' \to t''' &=& t'' \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{r}''' \to \mathbf{r}'''' &=& \mathbf{r}''' \\ t''' \to t'''' &=& t''' - \delta t \end{pmatrix}.$$

On the one hand these correspond to the operators

$$[1 + i\delta v_k G_k]\,, \qquad [1 - i\delta t H]\,,$$

and so forth. On the other hand, one can trivially compute the net result of this sequence of geometrical operations to conclude that $t'''' = t$ while $\mathbf{r}'''' = \mathbf{r} - \delta\mathbf{v}\delta t$. Show that the comparison yields the commutation relation

$$[H, G_l] = iP_l\,.$$

This appears to conflict with the time-independence of the Galilean boost G_l. Not to worry, just write

$$G_k = \bar{G}_k + tP_k\,,$$

where $\bar{G}_k$ has no explicit time dependence. Then

$$\frac{d}{dt}\,G_k = -i[G_k, H] + P_k = 0\,.$$

g) According to the geometrical transformation one would expect that

$$[1 - i\delta\mathbf{v}\cdot G]\psi(\mathbf{r}, t)\,[1 + i\delta\mathbf{v}\cdot G] = \psi(\mathbf{r} + \delta\mathbf{v}t,\, t)\,.$$

This, however, is *wrong*. It is wrong because something has been hidden: Quantum mechanics gives, in general, only a ray representation of an abstract group, not a true representation. There can be extra phase factors because the overall phase of a quantum state has no physical meaning. It turns out after a close examination that any potential phase factors associated with $\mathbf{J}$ and $\mathbf{P}$ and be removed by a re-definition of these generators. However, there is an intrinsic phase associated with G_k that cannot be so removed. Thus the correct formula is

$$[1 - i\delta\mathbf{v}\cdot\mathbf{G}]\psi(\mathbf{r}, t)\,[1 + i\delta\mathbf{v}\cdot\mathbf{G}] = e^{i\phi}\,\psi(\mathbf{r} + \delta\mathbf{v}t,\, t)\,.$$

To determine the phase ϕ, take the case of a free particle where

$$H = \int (dr)\psi^\dagger \left(\frac{-\nabla^2}{2m}\right)\psi\,,$$

and use the correctly transformed ψ given above to compute $[H, G_k]$ and compare with the previous, geometrical result for this commutator.

h) Now compute $[P_k, [G_l, \psi]]$ and $[G_l, [P_k, \psi]]$ to infer

$$[G_l, P_k] = ?$$

Note that for a true representation of the group this commutator would vanish.

i) Express

$$G_k = \int (d\mathbf{r})\psi^\dagger \cdots \psi\,.$$

j) Now prove an interesting theorem:

There is no particle production
in a Galilean invariant theory.

To prove this theorem, suppose that there are several particle types with associated annihilation operators $\psi_a(\mathbf{r}, t)$ with the subscript $a = 1, 2, \ldots$

labeling the different particle types which have different masses m_a. Derive the formula for X_{kl} defined by

$$X_{kl} = i[G_k, P_l]\,,$$

where G_k and P_l are boosts and translations for the complete system. Since

$$\frac{d}{dt}\,G_k = 0 = \frac{d}{dt}\,P_k\,,$$

we must have

$$\frac{d}{dt}\,X_{kl} = 0\,.$$

Thus X_{kl} is unchanged by any scattering or reaction process. Use this fact to prove the theorem.

The theorem can be easily understood. Particle production does occur of course in a Lorentz invariant theory according to the famous formula $E = mc^2$. But Galilean invariance corresponds to the $c \to \infty$ limit of the Lorentz invariant theory, and in the non-relativistic theory particle production cannot occur since with $c \to \infty$ this would require an infinite amount of energy.

k) Consider at time $t = 0$ a finite Galilean transformation

$$U(\mathbf{u}) = e^{i\mathbf{u}\cdot\mathbf{G}}\,.$$

Vary $\mathbf{u}$ to obtain differential equations and then solve these differential equations to compute

$$U(\mathbf{u})^{-1} H U(\mathbf{u}) = ?\,,$$

$$U(\mathbf{u})^{-1} N U(\mathbf{u}) = ?\,,$$

and

$$U(\mathbf{u})^{-1} \mathbf{P} U(\mathbf{u}) = ?\,.$$

Hence show that a value of $\mathbf{u}$ can be chosen to give

$$U(\mathbf{u})^{-1} \exp\{-\beta[H - \mu N - \mathbf{V}\cdot\mathbf{P}]\} U(\mathbf{u}) = \exp\{-\beta[H - (\mu + \frac{1}{2}m\mathbf{V}^2)N]\}\,.$$

The cyclic symmetry of the trace now implies the result Eq. (2.5.51) that was derived in the text using functional integral methods.

2. Harmonic Oscillator Partition Function

Calculate the partition function

$$Z = \mathrm{Tr}\; e^{-\beta H}$$

for the simple harmonic oscillator, $H = \omega a^\dagger a$, in three ways:

a) The easy way. Compute the trace in the number basis $a^\dagger a|n\rangle = |n\rangle n$ and do the sum over n.

b) The not too hard way. Use the result for Problem 4 of Chapter 1 to obtain $\langle z^*, -i\beta|z\rangle$. Integrate this over z^* and z with the proper weight to compute the trace.

c) The hard way. Compute the partition function following the methods used in the text to calculate the partition function of a non-interacting quantum gas.

3. Thermal Green's Functions and the Partition Function. Let us denote the thermal average by

$$\langle X\rangle = \frac{\mathrm{Tr}\; e^{-\beta(H-\mu N)}X}{\mathrm{Tr}\; e^{-\beta(H-\mu N)}}\,.$$

Here H is the Hamiltonian and N the number operator. Throughout this problem we shall consider only a single species of non-interacting particles with

$$H = \int (d\mathbf{r})\; \psi^\dagger \left(\frac{-\nabla^2}{2m}\right)\psi\,.$$

a) Let $\psi(\mathbf{r}, t)$ be a Bose annihilation operator. Compute

$$i\,\frac{\partial}{\partial t}\langle\, [\psi(\mathbf{r}, t),\, \psi^\dagger(\mathbf{r}', t')]\,\rangle = ?$$

and solve the resulting differential equation in terms of a Fourier transform $\int (d^3\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\cdots$ since the initial $[\psi(\mathbf{r},\, t), \psi^\dagger(\mathbf{r}',\, t)]$ is known.

b) Let $\psi(\mathbf{r}, t)$ be a Fermi annihilation operator. Compute

$$\langle\{\psi(\mathbf{r}, t),\, \psi^\dagger(\mathbf{r}', t')\}\rangle$$

in a similar way.

c) Suppose that

$$\langle A(t)B(t')\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; g^{(+)}(\omega)\; e^{-i\omega(t-t')}\,,$$

$$\langle B(t')A(t)\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; g^{(-)}(\omega)\; e^{-i\omega(t-t')}\,,$$

and

$$\langle [A(t),\, B(t')]\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; c(\omega)\; e^{-i\omega(t-t')}\,.$$

Prove that for $\mu = 0$,

$$g^{(\pm)}(\omega) = \pm \frac{c(\omega)}{1 - e^{\mp\beta\omega}} .$$

d) Use this result, generalized to a non-vanishing chemical potential μ, to compute $\langle \psi^\dagger(\mathbf{r}, t)\, \psi(\mathbf{r}', t') \rangle$ for Bose fields.

e) Use a similar result to compute $\langle \psi^\dagger(\mathbf{r}, t)\, \psi(\mathbf{r}', t') \rangle$ for Fermi fields.

f) Compute $\langle H \rangle$ for both Bose and Fermi fields.

4. Thermal Correlation Functions

As discussed in Section 7, the functional integral representation of the partition function may be extended to produce thermodynamic averages of products of field operators at various imaginary time values τ which lie in the interval $(0, \beta)$. The "time" development is governed by the thermodynamic Hamiltonian $\tilde{H} = H - \mu N$. Field operators at a common τ-value may be used to construct arbitrary operators $A(\tau)$, $B(\tau), \cdots$. For example, $A(\tau)$ may be the density operator $\rho(\mathbf{r}, \tau) = \psi^\dagger(\mathbf{r}, \tau)\psi(\mathbf{r}, \tau)$, or the field operator $\psi(\mathbf{r}, \tau)$ itself. (The spatial coordinate labels $\mathbf{r} \cdots$ are left implicit in the definition of the operators $A(\tau) \cdots$.) These thermodynamic averages of $A(\tau_1)B(\tau_2) \cdots$ are *time ordered* in the imaginary times $\tau_1, \tau_2, \cdots$,

$$\langle TA(\tau_1)B(\tau_2) \cdots \rangle = [\mathrm{Tr}\ e^{-\beta\tilde{H}}]^{-1} \mathrm{Tr}\ e^{-\beta\tilde{H}} TA(\tau_1)B(\tau_2) \cdots .$$

The time-ordering contains the appropriate $\pm$ signs if an operator is Fermionic — an operator containing an odd number of Fermion fields. The time-ordered product $TA(\tau_1)B(\tau_2) \cdots$ for Fermi fields is defined to be the product $A(\tau_1)B(\tau_2) \cdots$ if $\tau_1 > \tau_2 > \cdots$ and then the ordinary time-ordered sequence for other time orderings but with a minus sign if an odd permutation of Fermionic operators is required to return the product to the original order.

a) The thermodynamic averages define functions which are periodic (or anti-periodic) in their τ-variables in the interval $(0, \beta)$. The function at $\tau_a = 0$ (where τ_a is any of the "time" variables with the other "time" variables having arbitrary values) is identical to function at $\tau_a = \beta$ except for an overall minus sign if the operator is Fermionic. Use the cyclic symmetry of the trace to prove this.

Since the functions are periodic in the imaginary time variables, they may be expressed in terms of Fourier series. The purpose of the remainder of this problem is to describe the continuation from imaginary time back to real time variables. In the general case this can be rather complicated and so here we shall restrict the discussion to a two-point function.

b) Show that the two-point function is a function of the imaginary time difference,

$$\langle TA(\tau)B(\tau') \rangle = G(\tau - \tau') .$$

Thus the Fourier representation has the form

$$G(\tau) = \frac{1}{\beta} \sum\nolimits_\nu e^{-i\omega_\nu \tau} g(i\omega_\nu) ,$$

in which $\omega_\nu = 2\pi\nu/\beta$, and ν ranges over integer or half-odd-integer values.

c) Let us further restrict the discussion to Bosonic operators $A(\tau)$, $B(\tau')$. (The Fermionic case just involves various changes of sign.) Use the results of Problem 3, part c) continued to $t \to -i\tau$ to express $G(\tau)$ in terms of an integral representation involving the commutator weight $c(\omega)$. Write down the two representations for the two intervals $0 \le \tau < \beta$ and $-\beta < \tau \le 0$ and explain why the two representations are valid in these intervals. Show that these representations yield a periodic function in that if $\tau = \tau' - \beta$, $0 < \tau' < \beta$, then $G(\tau) = G(\tau')$. Show that

$$g(i\omega_\nu) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{c(\omega')}{\omega' - i\omega_\nu} .$$

d) Suppose that, at a remote time in the past, the system is initially described by a density operator $e^{-\beta\tilde{H}}/Z$. Suppose further that the system is then weakly perturbed by the Hamiltonian addition

$$H_I(t) = B(t)\phi(t) .$$

[Here ϕ could be a field $\phi(\mathbf{r}, t)$ coupled to a field $B(\mathbf{r}, t)$, in which case an additional volume integration $\int(d\mathbf{r})$ is implied.] Finally, suppose that the average value of some operator A is measured at time t. Prove that the first-order perturbation alters this expectation value by

$$\delta\langle A(t) \rangle = \int_{-\infty}^{\infty} dt' R(t - t')\phi(t') ,$$

where the response function is the thermal average of the retarded commutator,

$$R(t - t') = -i\theta(t - t')\langle [A(t), B(t')] \rangle .$$

With our Fourier transform convention,

$$R(t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} r(\omega)\, e^{-i\omega(t - t')} .$$

Prove that

$$r(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{c(\omega')}{\omega - \omega' + i\epsilon} .$$

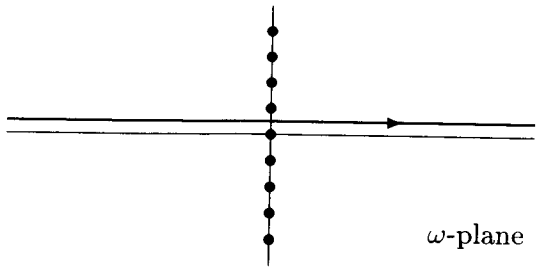


Fig. 2.5. The function $f(\omega)$ evaluated at discrete points on the imaginary axis yields the Fourier coefficients of the imaginary time Green's function. The same analytic function, evaluated along a line just above the real axis, is the Fourier transform of the real time, retarded response function.

Comparing this result with the representation given in part b) above for $g(i\omega_\nu)$ gives an important lesson to be learned from this problem. The Fourier coefficients $g(i\omega_\nu)$ of the periodic function in imaginary time are just the evaluations of an analytic function

$$f(z) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{c(\omega')}{z - \omega'}$$

at the points $z = i\omega_\nu = 2\pi i\nu/\beta$ along the imaginary axis. The evaluation of this same analytic function just above the real axis gives the Fourier transform of the retarded response function $r(\omega)$. This situation is illustrated in Fig. 2.5.

5. Time Reversal Invariance

Correlation functions of field operators in imaginary time were introduced in Section 7 and also discussed in the previous Problem 4. These correlation functions have the functional integral representation†

$$\begin{aligned}\langle T\psi(\mathbf{r}_1,\tau_1)\cdots\psi(\mathbf{r}_n,\tau_n)\psi^\dagger(\mathbf{r}'_n,\tau'_1)\cdots\psi^\dagger(\mathbf{r}'_1,\tau'_1)\rangle \\ = \frac{1}{Z}\int[d\psi][d\psi^*]\,\psi(\mathbf{r}_1,\tau_1)\cdots\psi(\mathbf{r}_n,\tau_n)\psi^*(\mathbf{r}'_n,\tau'_n)\cdots\psi^*(\mathbf{r}'_1,\tau'_1) \\ \exp\left\{-\int_0^\beta d\tau\left[\int(d\mathbf{r})\psi^*(\mathbf{r},\tau)\frac{\partial}{\partial\tau}\psi(\mathbf{r},\tau)+\tilde{H}\right]\right\}.\end{aligned}$$

† In the case of Fermions, both the sources η, η^* and the field variables in the functional integral ψ, ψ^* are anti-commutative, Grassmann variables. Sources multiplying fields $\eta^*\psi$, $\psi^*\eta$ are commutative and the functional integral produces ordinary time-ordered products of the corresponding field operators $\eta^*\psi$, $\psi^\dagger\eta$. Placing all the anti-commutative η^*, η in some prescribed order outside the matrix element to define the correlation function, as in writing $\eta^*\cdots\eta^*\langle\ \ \rangle\eta\cdots\eta$, produces the Fermionic time-ordered product described in Problem 4.

To start the discussion of time-reversal invariance, let us first consider the simplest case of a single Bose field. In this case the (thermodynamic) Hamiltonian $\tilde{H}$ is a real function of the ψ, ψ^* - that is, $\tilde{H}$ contains no factors that involve the imaginary unit‡ i. Using the prescription, discussed in Chapter 1, Section 8, that ψ and ψ^* are independent integration variables, they are not altered by complex conjugation, and so the correlation functions in imaginary time are real functions,

$$\begin{aligned}\langle T\psi(\mathbf{r}_1,\tau_1)\cdots\psi(\mathbf{r}_n,\tau_n)\psi^\dagger(\mathbf{r}'_n,\tau'_n)\cdots\psi^\dagger(\mathbf{r}'_1,\tau'_1)\rangle^* \\ = \langle T\psi(\mathbf{r}_1,\tau_1)\cdots\psi(\mathbf{r}_n,\tau_n)\psi^\dagger(\mathbf{r}'_n,\tau'_n)\cdots\psi^\dagger(\mathbf{r}'_1,\tau'_1)\rangle\,.\end{aligned}$$

Although this prescription facilitates the proof, it is not necessary. Once could instead write $\psi = \phi_1 + i\phi_2$, $\psi^* = \phi_1 - i\phi_2$ and integrate over the real variables ϕ_1, ϕ_2. Using these real variables, the first term in the action in the functional integral is now purely imaginary,

$$\psi^*\frac{\partial}{\partial\tau}\psi = 2i\phi_1\frac{\partial}{\partial\tau}\phi_2\,,$$

since a total derivation makes no contribution in view of the periodicity of the fields. To compensate the minus sign brought about by the complex conjugation of this term, we change the (dummy) field variables by making the replacements $\phi_1 \to \phi_1$, $\phi_2 \to -\phi_2$. This gives precisely the previous result in which the ψ, ψ^* were considered to be independent variables.

In general, the Bose field may have several components ψ_a, ψ_a^* corresponding to spin or other degrees of freedom. The situation here is essentially the same as in the single-component case except that a further transformation of field variables $\psi_a \to U_{ab}\psi_b$, $\psi_a^* \to U_{ab}^*\psi_b^*$ or, in matrix form $\psi \to U\psi$, $\psi^* \to \psi^*U^\dagger$, may be needed to keep $\tilde{H}$ invariant after it has been complex conjugated. The invariance of $\psi^*\frac{\partial}{\partial\tau}\psi$ requires that the matrix U be unitary, $U^\dagger U = 1$. Thus, in the general Bose case one has

$$\begin{aligned}\langle T\psi(\mathbf{r}_1,\tau_1)\cdots\psi(\mathbf{r}_n,\tau_n)\psi^\dagger(\mathbf{r}'_1,\tau'_1)\rangle^* \\ = \langle TU\psi(\mathbf{r}_1,\tau_1)\cdots U\psi(\mathbf{r}_n,\tau_n)\psi^\dagger(\mathbf{r}'_n,\tau'_n)U^\dagger\cdots\psi^\dagger(\mathbf{r}'_1,\tau'_1)U^\dagger\rangle\,.\end{aligned}$$

Using the rules for integration over Grassmann variables, it is not difficult to check that the same result also holds for Fermions. The simplest case of a two-component Fermi field which describes spin $\frac{1}{2}$ particles illustrates the need of the additional U-matrix transformation. In this case, there can be a spin–spin (or spin–orbit) interaction involving the spin density $\psi^*\vec{\sigma}\psi$. Here $\vec{\sigma}$ are the Pauli spin matrices with σ_1, and σ_3 real while

‡ This excludes the coupling of the system to an external magnetic field through the vector potential interaction $\psi^*(\frac{1}{i}\nabla - eA)^2\psi$ which does involve a factor of i in the cross term. But this coupling does not give a time-reversal invariant dynamics.

σ_2 is purely imaginary. Since $\sigma_2^2 = 1$, $\sigma_2^\dagger = \sigma_2$, and σ_2 anticommutes with σ_1, σ_3, $\vec{\sigma}^* = -\sigma_2^\dagger \vec{\sigma} \sigma$. Thus complex conjugation followed by the transformation $\psi \to \sigma_2 \psi$, $\psi^* \to \psi^* \sigma_2^\dagger$ sends the spin density into its negative, which is appropriate for time reversal. The spin–spin (or spin–orbit) interaction involves two minus signs and is invariant.

The complex conjugation result obtained from the functional integral representation, compared with the direct operator construction where the complex conjugate of the trace of an operator is the trace of the Hermitian adjoint of that operator, yields the statement of time-reversal invariance. Explain why

$$[\psi(\mathbf{r}, \tau)]^\dagger = \psi^\dagger(\mathbf{r}, -\tau)$$

and

$$[\psi^\dagger(\mathbf{r}, \tau)]^\dagger = \psi(\mathbf{r}, -\tau)\,.$$

Use these results to prove that

$$\langle T\psi(\mathbf{r}_1, \tau_1) \cdots \psi(\mathbf{r}_n, \tau_n) \psi^\dagger(\mathbf{r}'_n \tau'_n) \cdots \psi^\dagger(\mathbf{r}'_1, \tau'_1) \rangle^* = \langle T\psi(\mathbf{r}'_1, -\tau'_1) \cdots \psi(\mathbf{r}'_n, -\tau'_n) \psi^\dagger(\mathbf{r}_n, -\tau_n) \cdots \psi(\mathbf{r}_1, -\tau_1) \rangle\,.$$

Combine this with the previous result to write down the statement of time-reversal invariance. Note that this final result may easily be continued to real time. The discussion and explanation given above has been rather laconic. Give further explanation and details to make it clear.

3
Elementary scalar field theory

This chapter introduces the simplest example of a relativistic field theory, that provided by a scalar (spin zero) field*. First we consider the free, noninteracting theory. After making contact with the non-relativistic, particle creation and destruction fields of the previous chapter, the relativistic, free-particle propagation functions are described. The interacting theory is then examined by computing the two-particle scattering cross section in lowest order. Using the general structure of the scattering amplitude, which is illustrated by the simple scalar theory, we derive a formula for the cross section that is easily generalized to any two-particle scattering process. This we do with some rigor by using a wave-packet technique. This discussion is an example of what will often be done in subsequent chapters: The scalar field theory will be employed to motivate and illustrate many features of relativistic quantum field theory that hold in general. We use the scalar field so as to make these features clear and not encumbered with complicated notation. This chapter also contains the calculation of the scalar-scalar scattering cross section to second order. This requires regularization — which we do by the method of dimensional continuation — and renormalization. The renormalization is done with a method that naturally leads to the idea of the running coupling constant which is of great importance in modern elementary particle theory.

3.1 Free theory, non-relativistic limit

We turn now to discuss the quantum mechanics of a real (Hermitian) scalar field $\phi(x)$, with $x^\mu = (t, \mathbf{r})$ labeling the points of space-time. To obtain the quantum theory, we first treat $\phi(x)$ as a classical field so as

* The basic concepts of special relativity and the relativistic notation that we employ are reviewed in the problems at the end of this chapter.

to identify canonically conjugate variables. This is done by using the stationary action principle to describe the dynamics. Thus we begin with a Lagrange function (Lagrange density) $\mathcal{L}$, form the action

$$S = \int (d^4x)\mathcal{L}\,, \tag{3.1.1}$$

and demand that it be stationary,

$$\delta S = 0\,, \tag{3.1.2}$$

to infer the equations of motion. The simplest Lorentz invariant Lagrange function is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\,. \tag{3.1.3}$$

The minus signs of the two terms are dictated by the requirement that the energy – the Hamiltonian — be a positive-definite form,[†] as will soon be clear. The first factor of $\frac{1}{2}$ is conventional — the magnitude of $\phi(x)$ can always be scaled by some constant to make this a factor of $\frac{1}{2}$. With this convention fixed, the parameter m^2 is uniquely determined. Since ∂_μ has the dimensions L^{-1} so does m; with our system of natural units, m has the dimensions of mass. With a partial integration (assuming $\delta\phi$ vanishes on the space-time boundary), the variation of the action becomes

$$\delta S = -\int (d^4x)\,\delta\phi(-\partial^2 + m^2)\phi\,, \tag{3.1.4}$$

and thus the principle of stationary action asserts that

$$(-\partial^2 + m^2)\phi(x) = 0\,. \tag{3.1.5}$$

This is the Klein-Gordon equation.

To write down the quantum theory, one must first identify the field Π which is canonically conjugate to ϕ,

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi\,, \tag{3.1.6}$$

and write the Lagrange function in the standard first order form

$$\mathcal{L} = \Pi\partial_0\phi - \mathcal{H}\,. \tag{3.1.7}$$

Here $\mathcal{H}$ is the Hamiltonian density which is to be written as a function of the conjugate variables (ϕ, Π). A little algebra gives

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\,, \tag{3.1.8}$$

[†] The Hamiltonian must be bounded from below to have a stable ground state. With the addition of some constant, this requires that the Hamiltonian be positive definite.

which indeed is a positive function. To show how the first-order formulation works, we note that requiring the new action

$$S = \int (d^4x)\,\{\Pi\partial_0\phi - \mathcal{H}\} \tag{3.1.9}$$

to be stationary against variations of Π,

$$\delta_\Pi S = 0\,, \tag{3.1.10}$$

we recover

$$\Pi = \partial_0\phi\,, \tag{3.1.11}$$

while the stationary action principle for ϕ,

$$\delta_\phi S = 0\,, \tag{3.1.12}$$

yields

$$-\partial_0\Pi = -\nabla^2\phi + m^2\phi\,. \tag{3.1.13}$$

These two equations combine to produce the previous Klein-Gordon equation. Although this first order form is not manifestly Lorentz invariant, it is a necessary intermediate step in obtaining the quantum theory. The theory will later be put back together in a relativistic form.

The classical theory is quantized by assuming the equal-time commutation relation

$$[\phi(\mathbf{r}, t), \Pi(\mathbf{r}', t)] = i\delta^{(3)}(\mathbf{r} - \mathbf{r}')\,, \tag{3.1.14}$$

with

$$[\phi(\mathbf{r}, t), \phi(\mathbf{r}', t)] = 0 = [\Pi(\mathbf{r}, t), \Pi(\mathbf{r}', t)]\,. \tag{3.1.15}$$

Here $\phi(\mathbf{r}, t)$ is the field generalization of a discrete set of $q_a(t)$ variables in which the index a now becomes the continuous coordinate label $\mathbf{r}$ and $\Pi(\mathbf{r}, t)$ similarly generalizes the canonically conjugate set $p_a(t)$. The commutation relations for the fields are simply the generalizations of the commutators of ordinary quantum mechanics extended to an infinite set of q_a and p_a. The quantum Hamiltonian appears as

$$H = \int (d\mathbf{r})\mathcal{H}\,, \tag{3.1.16}$$

where the Hamiltonian density $\mathcal{H}$ is given by Eq. (3.1.8), but with the classical fields replaced by their quantum operator counterparts. The time development of the quantum fields is governed by the Heisenberg equations of motion

$$\partial_0\phi = i[H, \phi] \tag{3.1.17}$$

and

$$\partial_0 \Pi = i[H, \Pi]\,. \tag{3.1.18}$$

Since the Hamiltonian commutes with itself, it is time-independent and it may be evaluated using fields evaluated at any time. Using the explicit form of the Hamiltonian and the canonical equal-time commutation relations (3.1.14) and (3.1.15), it is a simple matter to verify that the Heisenberg equations of motion (3.1.17) and (3.1.18) yield the same equations (3.1.11) and (3.1.13) obeyed by the classical fields. Combining these equations shows that the quantum field operator $\phi(x)$ obeys the Klein-Gordon equation (3.1.5).

To produce the generating functional Z for the quantum field theory, we follow our previous quantum mechanical development and adjoin external forces or field sources ρ and σ that are coupled to ϕ and Π, respectively. The quantum mechanical first-order form of the functional integral is generalized to

$$Z[\rho,\,\sigma] = \int [d\phi][d\Pi] \exp\left\{ i \int_{-\infty}^{\infty} dt \int (d\mathbf{r})[\mathcal{L} + \phi(x)\rho(x) + \Pi(x)\sigma(x)] \right\}. \tag{3.1.19}$$

We recall from our previous discussion of quantum mechanical systems that extending the time interval $-\infty < t < +\infty$ makes the functional integral represent the ground state-ground state transformation function or probability amplitude.‡ For reasons that will soon be clear, this ground state will be called the "vacuum state", and it will be denoted by $|0\rangle$. Since the functional integral of any quantity, weighted by exponential of the action, produces the time-ordered expectation value of that quantity,

$$Z[\rho,\,\sigma] = \langle 0 + \infty | 0 - \infty \rangle^{(\rho,\sigma)} = \langle 0 | T e^{i \int (dx)[\phi\rho + \Pi\sigma]} | 0 \rangle\,. \tag{3.1.20}$$

In the last line we have implicitly indicated that the vacuum state unperturbed by sources is time independent — it is the state of lowest energy and, with a suitable normalization, this energy vanishes, $H|0\rangle = 0$, and so $|0\rangle$ is time independent. The functional integration measure is assumed to be normalized so that

$$\int [d\phi][d\Pi] e^{iS} = \langle 0 | 0 \rangle = 1\,. \tag{3.1.21}$$

What we have done is to write down the simplest relativistically invariant theory of a wave equation. We identified the canonically conjugate momentum to ϕ, the field Π. We then noted that ϕ and Π appear as an infinite set of q's and p's and assumed that the quantum theory of the scalar field theory was obtained by requiring that the ϕ and Π obey a commutation relation that is the infinite dimensional generalization of a finite set of q_a, p_b commutation relations. Having made this assumption, the functional integral representation follows without any further assumptions — it is simply the infinite dimension generalization of the functional integral for a finite set of q_a and p_a. One cannot, of course, deduce new and extended physical laws from old and restricted laws, and so we had to simply postulate the commutation relations. Moreover, we do not yet know what kind of system the ϕ field represents! To see what relativistic system ϕ should describe, we take the non-relativistic limit. Thus although we cannot logically deduce new physics from old physics, we can find out what old physics follows from a limit of the new physics — a limit into the old physics — to check that the new physics is reasonable and to motivate what the new physics should describe. So we consider the non-relativistic limit of the relativistic scalar field theory. To do this, we first anticipate that the relativistic fields which have been introduced have a rapid time dependence $\exp\{\mp iEt\}$, with $E \approx m$ in the non-relativistic limit. The rapid phase oscillation is removed by changing variables from $(\phi,\,\Pi)$, to

$$\psi(\mathbf{r},\,t) = e^{imt}\left[\sqrt{\frac{m}{2}}\phi(\mathbf{r},\,t) + \frac{i}{\sqrt{2m}}\Pi(\mathbf{r},\,t)\right], \tag{3.1.22}$$

$$\psi^\dagger(\mathbf{r},\,t) = e^{-imt}\left[\sqrt{\frac{m}{2}}\phi(\mathbf{r},\,t) - \frac{i}{\sqrt{2m}}\Pi(\mathbf{r},\,t)\right]. \tag{3.1.23}$$

This is akin to the transformation $a = (q + ip)/\sqrt{2}$, $a^\dagger = (q - ip)/\sqrt{2}$ that was used in the treatment of the simple harmonic oscillator. It is essentially a canonical transformation, and the equal-time commutation relations for $(\phi,\,\Pi)$ imply that

$$[\psi(\mathbf{r},\,t),\,\psi^\dagger(\mathbf{r}',\,t)] = \delta^{(3)}(\mathbf{r} - \mathbf{r}')\,, \tag{3.1.24}$$

and

$$[\psi(\mathbf{r},\,t),\,\psi(\mathbf{r}',\,t)] = 0 = [\psi^\dagger(\mathbf{r},\,t),\,\psi^\dagger(\mathbf{r}',\,t)]\,. \tag{3.1.25}$$

These are just the commutation relations of the creation and annihilation operators for a many-body Bose system. However, one must not rush to the conclusion that the new relativistic system is identical to the old non-relativistic theory. To investigate the correspondence, we note that the transformation is easily inverted to give

$$\phi(\mathbf{r},\,t) = \frac{1}{\sqrt{2m}}\left[\psi(\mathbf{r},\,t)e^{-imt} + \psi^\dagger(\mathbf{r},\,t)e^{imt}\right], \tag{3.1.26}$$

‡ This was explained in the discussion about Eq. (1.5.51) in Chapter 1 where the limit $t_2 \to -i\infty$, $t_1 \to +i\infty$ was taken. Such an absolute convergence into the ground state is obtained in the $T \to \infty$ limit of $t_2 - t_1 = Te^{-i\theta}$ for any θ in the open interval between 0 and π. The real limit $t_2 - t_1 \to \infty$ also produces the ground state expectation value if it is understood that averages over intervals Δt_1, Δt_2 in t_1, t_2 are also made with $\Delta t_{1,2} \to \infty$ as well as $t_2 - t_1 \to \infty$.

and

$$\Pi(\mathbf{r}, t) = -i\sqrt{\frac{m}{2}}\left[\psi(\mathbf{r}, t)e^{-imt} - \psi^\dagger(\mathbf{r}, t)e^{imt}\right]. \tag{3.1.27}$$

Substituting this into the first order form of the Lagrangian yields

$$\mathcal{L} = \frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi\right) - \frac{1}{2m}(\nabla\psi^*)\cdot(\nabla\psi) + \cdots e^{\pm 2imt}. \tag{3.1.28}$$

Here we have replaced the field operators by the classical fields which appear in the functional integral. We replace the sources (ρ, σ) by appropriate linear combinations (f, g) that couple to (ψ^*, ψ). We assume that these new sources are used to create non-relativistic particles and thus have a time variation $\exp\{\pm i\mathcal{E}t\}$ which varies slowly in comparison with the rapid phase variation $e^{\pm 2imt}$. We then expect that the numerically important paths in the functional integral involve fields $\psi(\mathbf{r}, t)$, $\psi^*(\mathbf{r}, t)$ that are also slowly varying in comparison with $e^{\pm 2imt}$. In this case we should be able to omit the $\cdots e^{\pm 2imt}$ terms that have not been written out explicitly in $\mathcal{L}$, for the time integration in the action makes these terms negligible owing to their rapid phase variation. Thus, in the non-relativistic limit, we should obtain the generating functional

$$Z = \int [d\psi][d\psi^*] \exp\left\{i\int dt \int (d\mathbf{r})\left[\psi^* i\frac{\partial\psi}{\partial t} - \psi^*\left(-\frac{\nabla^2}{2m}\right)\psi + \psi^* f + \psi g\right]\right\}, \tag{3.1.29}$$

where partial integrations have been made in the action to simplify the expression. In this limit,

$$Z = \langle 0|Te^{i\int[\psi^\dagger f + \psi g]}|0\rangle \tag{3.1.30}$$

is precisely the generating functional for the free, many-body, non-relativistic Bose theory described in the previous chapter.

3.2 Free field propagators

We return now to the relativistic theory. We omit the source σ which is not needed for our purposes, and turn to compute

$$Z[\rho] = \int [d\phi][d\Pi] e^{i\int[\mathcal{L} + \phi\rho]} = \left\langle Te^{i\int\phi\rho}\right\rangle. \tag{3.2.1}$$

Here the simplified notation is adopted in which the vacuum expectation value is denoted by angular brackets,

$$\langle 0|\cdots|0\rangle = \langle\cdots\rangle, \tag{3.2.2}$$

a convention which often will be used. Just as was done previously in the quantum mechanical case, one can pass from the first order formulation to the second order form by integrating out the conjugate field variable Π:

$$\begin{aligned}\int [d\Pi] e^{i\int\mathcal{L}} &= \int [d\Pi] e^{i\int\left[-\frac{1}{2}(\Pi - \partial_0\phi)^2 + \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right]} \\ &= (\text{const.}) e^{i\int\left[\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right]}.\end{aligned} \tag{3.2.3}$$

Here, just as in the discussion of the quantum mechanical case in Chapter 1, Section 2, we complete a square, change the integration variable to $\Pi' = \Pi - \partial_0\phi$ (a translation which gives a unit Jacobian), and do a Gaussian (functional) integral which produces an overall constant. This constant is absorbed into a new measure for $[d\phi]$. Thus we have now returned to the original Lorentz-invariant Lagrange function

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2, \tag{3.2.4}$$

with

$$Z[\rho] = \left\langle Te^{i\int\phi\rho}\right\rangle = \int [d\phi] e^{i\int[\mathcal{L} + \phi\rho]}. \tag{3.2.5}$$

There should be no confusion with the notation in which now $\mathcal{L}$ is in the Lorentz-invariant form. The measure is normalized such that, in the absence of the source $\rho(x)$, $Z[0] = 1$.

The ϕ-functional integral is again of Gaussian form, and it can be done by completing the square. To do this it is convenient to introduce an operator notation so that, on integrating by parts, we may write

$$\int \mathcal{L} = -\frac{1}{2}\phi[-\partial^2 + m^2]\phi. \tag{3.2.6}$$

Here, on the right-hand side of the equation, an operator notation is employed in which the product $\phi\cdots\phi$ is a scalar product in the coordinate space that includes the integration over x. The inverse operator Δ_+ is defined by

$$(-\partial^2 + m^2)\Delta_+ = 1. \tag{3.2.7}$$

In ordinary notation this reads

$$(-\partial^2 + m^2)\Delta_+(x - x') = \delta(x - x'), \tag{3.2.8}$$

which identifies the propagator $\Delta_+(x - x')$ as the scalar field Green's function. The boundary conditions which are necessary to complete the definition of $\Delta_+(x - x')$ will be spelled out later. Using this operator notation, the functional integral in Eq. (3.2.5),

$$Z[\rho] = \int [d\phi] \exp\left\{-i\left[\frac{1}{2}\phi(-\partial^2 + m^2)\phi - \phi\rho\right]\right\}, \tag{3.2.9}$$

is computed as follows:

$$Z[\rho] = \int [d\phi] \exp\left\{ \frac{-i}{2} \left[(\phi - \rho\Delta_+)(-\partial^2 + m^2)(\phi - \Delta_+\rho) - \rho\Delta_+\rho \right] \right\}$$
$$= \exp\left\{ \frac{i}{2} \rho\Delta_+\rho \right\}, \tag{3.2.10}$$

since the change of integration variables to $\phi' = \phi - \Delta_+\rho$ produces a Gaussian integral which, in conjunction with the normalization of the measure, must yield unity, the value of $Z[0]$. To be sure that our shorthand notation is understood, we write out the result in detail:

$$Z[\rho] = \left\langle T \exp\left\{ i \int (d^4x)\, \phi(x)\rho(x) \right\} \right\rangle$$
$$= \exp\left\{ \frac{i}{2} \int (d^4x)(d^4x')\, \rho(x)\Delta_+(x - x')\rho(x') \right\}. \tag{3.2.11}$$

Let us give an alternative derivation of this result so as to tone our computational muscle. The integral of a total derivative vanishes:

$$0 = \int [d\phi] \frac{1}{i} \frac{\delta}{\delta\phi(x)} e^{i\int[\mathcal{L}+\phi\rho]}. \tag{3.2.12}$$

On taking the functional derivative one obtains

$$\int [d\phi]\, e^{i\int[\mathcal{L}+\phi\rho]} \left\{ (-\partial^2 + m^2)\phi(x) - \rho(x) \right\} = 0. \tag{3.2.13}$$

The functional integral with the action including the source term produces the vacuum–to–vacuum transformation function driven by the source. An additional factor of the integration variable $\phi(x)$ produces the insertion of the operator $\phi(x)$ in this matrix element. Equation (3.2.13) shows that the source-driven Klein-Gordon equation is obeyed by the $\phi(x)$ operator. Since the functional derivative of the generating functional with respect to the source $\rho(x)$ brings down an insertion of the field $\phi(x)$, Eq. (3.2.13) may be written as

$$(-\partial^2 + m^2) \frac{1}{i} \frac{\delta}{\delta\rho(x)} Z[\rho] = \rho(x) Z[\rho], \tag{3.2.14}$$

which has the solution

$$Z[\rho] = e^{\frac{i}{2}\rho\Delta_+\rho}. \tag{3.2.15}$$

This is just the previous result.

Taking two functional derivatives with respect to the source and then setting the source to zero produces the two-point function,

$$\frac{\delta}{\delta\rho(x)} \frac{1}{i} \frac{\delta}{\delta\rho(x')} \left\langle T e^{i\int\phi\rho} \right\rangle \Big|_{\rho=0} = \langle iT\phi(x)\phi(x') \rangle. \tag{3.2.16}$$

Using the result (3.2.11) one finds that

$$\langle iT\phi(x)\phi(x') \rangle = \Delta_+(x - x'). \tag{3.2.17}$$

Thus the Green's function or propagator is the vacuum expectation value of the time-ordered product (times i) of two scalar field operators. To examine the structure of the higher order correlations, let us abbreviate the coordinate labels $x_1^\mu, x_2^\mu, \ldots$, by $1, 2, \ldots$. Then, for example, using Eq. (3.2.11) the four-point function is given by

$$-\frac{\delta}{\delta\rho(1)} \frac{\delta}{\delta\rho(2)} \frac{\delta}{\delta\rho(3)} \frac{\delta}{\delta\rho(4)} \left\langle T e^{i\int\phi\rho} \right\rangle \Big|_{\rho=0}$$
$$= \langle -T\phi(1)\phi(2)\phi(3)\phi(4) \rangle$$
$$= \Delta_+(1-2)\Delta_+(3-4) + \Delta_+(1-3)\Delta_+(2-4)$$
$$+ \Delta_+(1-4)\Delta_+(2-3). \tag{3.2.18}$$

Note that this result can be expressed by saying that the four-point function is the sum over all possible bilinear contractions as illustrated in

$$\langle -T\,\phi(1)\phi(2)\phi(3)\phi(4) \rangle = \langle \overbrace{i\phi(1)\phi(2)}^{\Delta_+(1-2)}\ \underbrace{i\phi(3)\phi(4)}_{\Delta_+(3-4)} \rangle$$
$$+ \langle \overbrace{i\phi(1)\,\underbrace{i\phi(2)\phi(3)}_{\Delta_+(2-3)}\,\phi(4)}^{\Delta_+(1-4)} \rangle + \langle \overbrace{i\phi(1)\,\underbrace{i\phi(2)\phi(4)}_{\Delta_+(2-4)}\,\phi(3)}^{\Delta_+(1-3)} \rangle. \tag{3.2.19}$$

It should be clear that a similar "contraction rule" holds for the general n-point function. The contraction rule is often referred to as "Wick's Theorem". Thus the whole of the free scalar field theory is determined by the propagator or Green's function $\Delta_+(x - x')$.

Before proceeding to the construction of the Green's function it is worthwhile giving another derivation of the equation which the Green's function obeys. In view of Eq. (3.2.13), in the absence of a source, the field operator $\phi(x)$ satisfies the equation of motion

$$(-\partial^2 + m^2)\phi(x) = 0. \tag{3.2.20}$$

The Green's function equation (3.2.8) follows from its operator construction (3.2.17) and this operator equation of motion together with the equal-time commutation relation

$$[\phi(\mathbf{r}, t),\, \partial_0\phi(\mathbf{r}', t)] = i\delta(\mathbf{r} - \mathbf{r}'). \tag{3.2.21}$$

The double time derivative of the discontinuous time-ordering of the quantum field operators produces $\delta(t - t')$ times this equal-time commutator

and results in the inhomogeneous $\delta^4(x-x')$ term on the right-hand side of the Green's function equation (3.2.8). This is easy to prove by integrating Eq. (3.2.8) over a small time interval which contains the time t', integrating over t from $t'-\epsilon$ to $t'+\epsilon$.

The Green's function equation can be solved by Fourier transformation. In order to do this, we first note that the functional integral representation with its oscillating exponential structure is made to converge absolutely if we add a small imaginary constant $i\epsilon$, $\epsilon > 0$, as shown:

$$Z[\rho] = \int [d\phi] e^{-\frac{i}{2}\phi[-\partial^2+m^2-i\epsilon]\phi+i\phi\rho} . \tag{3.2.22}$$

A well-defined theory is obtained if the mass parameter m^2 is continued to a complex value, $m^2 \to m^2 - i\epsilon$, with $\epsilon \to 0^+$ defining the proper analytic continuation. With this in mind, the Fourier transform solution of the Green's function equation is given by

$$\Delta_+(x-x') = \int \frac{(d^4p)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2+m^2-i\epsilon} , \tag{3.2.23}$$

for this formula gives

$$(-\partial^2+m^2)\Delta_+(x-x') = \int \frac{(d^4p)}{(2\pi)^4} e^{ip(x-x')} = \delta(x-x') . \tag{3.2.24}$$

We should note that the four-dimensional integration in Eq. (3.2.23) defines a Lorentz invariant quantity as must be the case.

The Green's function can be expressed in terms of a Bessel function. (See Problem 12.) Here we shall put the Green's function in a form that elucidates the $i\epsilon$ prescription and leads to its connection with the boundary conditions that also define this function. We write

$$E(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2} , \tag{3.2.25}$$

and note that the dp^0 part of the (d^4p) integration in Eq. (3.2.23) involves

$$I(t-t') = \int \frac{dp^0}{2\pi} \frac{e^{-ip^0(t-t')}}{E(\mathbf{p})^2 - p^{0\,2} - i\epsilon} . \tag{3.2.26}$$

The $i\epsilon$ prescription is tantamount to defining the p^0 integration along the contour shown in Fig. 3.1. If $t-t' > 0$, the integration contour may be closed in the lower half p^0 plane. In this case the pole at $p^0 = E(\mathbf{p})$ is encircled in a clockwise (negative) sense. On the other hand, if $t-t' < 0$, the integration may be closed in the upper half p^0 plane and the pole at $p^0 = -E(\mathbf{p})$ is encircled in counter-clockwise (positive) sense. The result of either case may be written as

$$I(t-t') = \frac{i}{2E(\mathbf{p})} e^{-iE(\mathbf{p})|t-t'|} . \tag{3.2.27}$$

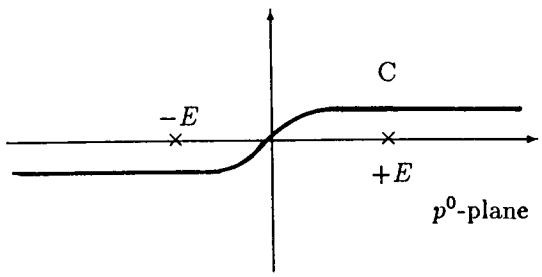


Fig. 3.1. Integration contour for the Green's function. It provides positive frequency boundary conditions.

Therefore*

$$\Delta_+(x-x') = i \int \frac{(d\mathbf{p})}{(2\pi)^3} \frac{1}{2E(\mathbf{p})} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-iE(\mathbf{p})|t-t'|} . \tag{3.2.28}$$

Since $\Delta_+(x-x')$ is a Lorentz invariant so must be the integral in this equation. Indeed, as discussed in Problem 4, the measure $(d\mathbf{p})/E(\mathbf{p})$ is a Lorentz invariant.

The form (3.2.28) finally makes contact with the positive frequency boundary conditions that must be obeyed by the Green's function. To see this, we note that for $t > t'$ the Heisenberg equation of motion gives

$$\langle T\phi(x)\phi(x')\rangle = \langle 0|e^{iHt}\phi(\mathbf{r})e^{-iH(t-t')}\phi(\mathbf{r}')e^{-iHt'}|0\rangle . \tag{3.2.29}$$

We are assuming that the energy is normalized such that the vacuum has none,

$$H|0\rangle = 0 . \tag{3.2.30}$$

We make use of a complete set of energy eigenstates,

$$1 = \sum_n |n\rangle\langle n| , \tag{3.2.31}$$

with

$$H|n\rangle = |n\rangle E_n . \tag{3.2.32}$$

Since the vacuum is the lowest energy state, the other states must have positive energies,

$$n > 0 \quad : \quad E_n > 0 . \tag{3.2.33}$$

* Note that this Green's function is essentially a sum of the simple harmonic oscillator Green's functions (1.5.52) for each momentum mode. Moreover, the field-theory generating functional (3.2.10) is a generalization of the simple oscillator result (1.5.51).

Introducing the resolution of the identity (3.2.31) into the matrix element (3.2.29), we find that for $t > t'$,

$$\langle T\phi(x)\phi(x')\rangle = \sum_n \langle 0|\phi(\mathbf{r})|n\rangle e^{-iE_n(t-t')}\langle n|\phi(\mathbf{r}')|0\rangle\,, \tag{3.2.34}$$

which contains only the positive frequencies exhibited in Eq. (3.2.28). Similarly, for $t < t'$ one finds that

$$\langle T\phi(x)\phi(x')\rangle = \sum_n \langle 0|\phi(\mathbf{r}')|n\rangle e^{-iE_n(t'-t)}\langle n|\phi(\mathbf{r})|0\rangle\,, \tag{3.2.35}$$

which also has only the positive frequencies exhibited in $\Delta_+(x-x')$. These positive frequency boundary conditions, which follow from the time-ordering in the vacuum matrix element, are equivalent to the $m^2 \to m^2 - i\epsilon$ prescription. That is, this prescription is equivalent to routing the p^0 in such a way that the positive frequency boundary conditions are obtained. Conversely, imposing the positive frequency boundary conditions requires that the p^0 contour be routed about the two poles that appear with $p^2 + m^2 = 0$ in a manner which is equivalent to the $m^2 \to m^2 - i\epsilon$ prescription.

We have seen that in the non-relativistic limit the scalar field ϕ is related to creation and annihilation operators that can be used to form states of one or more particles out of the vacuum state. To extend this to the relativistic case we shall use a general method to infer such a construction from the field correlation functions. This is the first case of our later repeated use of the simplest scalar field example to illustrate general methods. The general method that we shall use can be easily extended to the case of interacting fields. It is worthwhile noting that most texts straightaway decompose ϕ in terms of particle creation and annihilation operators. We believe that this is a bad practice because not only does it not generalize to the case of interacting fields, but there are even situations in which a relativistic field has nothing to do with the creation of single particle states from the vacuum. Thus this standard treatment inculcates the wrong mind set. What we shall do instead is to examine the functions that have been constructed and use general principles to infer their physical consequences, rather than trying to impose this physics in an ad-hoc way that could turn out to be wrong in other circumstances.

Having mounted this sermon, let us turn to the details. Relativistic invariance requires that the energy operator H be the time-component of a four-vector $P^\mu = (H, \mathbf{P})$ that comprises the momentum $\mathbf{P}$ as well as the energy H. Moreover, relativistic invariance also requires that the Heisenberg equation of motion is just the time-component of

$$[\phi(x),\, P^\mu] = \frac{1}{i}\partial^\mu\phi(x)\,. \tag{3.2.36}$$

Thus, just as the energy operator $H = P^0$ is also the generator of translations in time, the momentum operator $\mathbf{P}$ is also the generator of translations in space.† Since such geometrical transformations commute,

$$[P^\mu,\, P^\nu] = 0\,, \tag{3.2.37}$$

the differential equation (3.2.36) can be integrated to give

$$\phi(x) = e^{-iP^\mu x_\mu}\phi(0)e^{iP^\mu x_\mu}\,. \tag{3.2.38}$$

This is the generalization of the familiar result for the time dependence of Heisenberg operators. The vacuum state is translation invariant and thus a null eigenstate of the spatial momentum operators, $\mathbf{P}|0\rangle = 0$. Relativistic invariance requires that it also be a state of zero energy, and so

$$P^\mu|0\rangle = 0\,. \tag{3.2.39}$$

Since the P^μ commute, they can be simultaneously diagonalized — a complete set of states,

$$1 = \sum_n |n\rangle\langle n|\,, \tag{3.2.40}$$

exist with

$$P^\mu|n\rangle = |n\rangle P_n^\mu\,. \tag{3.2.41}$$

Hence

$$\begin{aligned}\langle\phi(x)\phi(x')\rangle &= \langle 0|e^{-iPx}\phi(0)e^{iP(x-x')}\phi(0)e^{iPx'}|0\rangle \\ &= \sum_n e^{iP_n(x-x')}\langle 0|\phi(0)|n\rangle\langle n|\phi(0)|0\rangle\,.\end{aligned} \tag{3.2.42}$$

This result, in comparison with the previous explicit formula (3.2.28), analytically continued to all values of $t - t'$,

$$\langle\phi(x)\phi(x')\rangle = \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-iE(\mathbf{p})(t-t')}\,, \tag{3.2.43}$$

tells us the action of $\phi(x)$. Since the Fourier transform representation is unique, the values of the intermediate state four-momenta must equal those of the four-vector $(E(\mathbf{p}), \mathbf{p})$. Since $E(\mathbf{p})^2 - \mathbf{p}^2 = m^2$, we conclude that

$$-(P_n^\mu)^2 = m^2\,. \tag{3.2.44}$$

Therefore $\langle 0|\phi|n\rangle = 0$ unless $-P_n^2 = m^2$. The scalar field ϕ connects the vacuum state $|0\rangle$ only to states with the mass m — only to *single*

† These statements are verified in Problem 13 where P^μ is constructed from the stress-energy tensor $T^{\mu\nu}$ which in turn is formed from the basic field variables.

particle states.[‡] We label a single particle state of mass m by its (three) momentum, $|p\rangle$, with

$$P^\mu|p\rangle = |p\rangle p^\mu\,, \tag{3.2.45}$$

in which

$$p^0 = E(\mathbf{p})\,. \tag{3.2.46}$$

The familiar proof that states with different eigenvalues of the Hermitian operator are necessarily orthogonal can be applied here. One evaluates $\langle p|P^\mu|p'\rangle$ by letting P^μ act both to the left and the right to conclude that

$$(p^\mu - p'^\mu)\langle p|p'\rangle = 0\,. \tag{3.2.47}$$

Hence $\langle p|p'\rangle$ must be a multiple of the three-dimensional $\delta^{(3)}(\mathbf{p}-\mathbf{p}')$ since $p^0 = E(\mathbf{p})$ is not an independent variable. We shall adopt the normalization convention

$$\langle p|p'\rangle = (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,. \tag{3.2.48}$$

With this convention, the resolution of the identity reads

$$1 = |0\rangle\langle 0| + \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})}|p\rangle\langle p| + \sum\{\text{multiparticle states}\}\,, \tag{3.2.49}$$

since this satisfies

$$1|p'\rangle = |p'\rangle\,. \tag{3.2.50}$$

We can now see the reason for the choice of the $E(\mathbf{p})$ factor in the normalization convention (3.2.48) — it puts the single-particle contribution to the resolution of the identity in an Lorentz invariant form. As we shall soon see, the other conventional factors are put in Eq. (3.2.48) so as to get simple wave functions. Using this resolution of the identity in the previous sum over intermediate states in Eq. (3.2.42) gives

$$\langle\phi(x)\phi(x')\rangle = \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})}e^{ip(x-x')}\langle 0|\phi(0)|p\rangle\langle p|\phi(0)|0\rangle\,, \tag{3.2.51}$$

which is to be compared to the direct evaluation of the Green's function

$$\langle\phi(x)\phi(x')\rangle = \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})}e^{ip(x-x')}\,. \tag{3.2.52}$$

Hence, remembering that $\phi^\dagger(x) = \phi(x)$ is an Hermitian operator, we must have

$$|\langle 0|\phi(0)|p\rangle|^2 = 1\,, \tag{3.2.53}$$

[‡] The "mass" of a multiparticle state is just the total energy in the center-of-mass frame, and this "mass" is a continuously varying quantity not a discrete number. So ϕ cannot couple $|0\rangle$ to multiparticle states.

and with a standard phase convention

$$\langle 0|\phi(0)|p\rangle = 1 = \langle p|\phi(0)|0\rangle\,. \tag{3.2.54}$$

Accordingly, the *wave functions* associated with the scalar field are given by

$$\langle 0|\phi(x)|p\rangle = \langle 0|e^{-iPx}\phi(0)e^{iPx}|p\rangle = e^{ipx}\,, \tag{3.2.55}$$

and

$$\langle p|\phi(x)|0\rangle = \langle 0|\phi(x)|p\rangle^* = e^{-ipx}\,. \tag{3.2.56}$$

It has now been shown that the field $\phi(x)$ acting upon the vacuum state produces only single particle states of mass m. To make this construction explicit, it is convenient to define

$$\phi^{(\pm)}(p) = \pm i\int(d\mathbf{r})\,e^{\mp ipx}(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\phi(x)\,, \tag{3.2.57}$$

in which $p^0 = E(\mathbf{p})$. Since

$$(-\partial^2 + m^2)\phi(x) = 0\,, \tag{3.2.58}$$

and

$$(-\partial^2 + m^2)e^{\mp ipx} = 0\,, \tag{3.2.59}$$

we find that

$$\begin{aligned}\frac{\partial}{\partial t}\phi^{(\pm)}(p) &= \pm i\int(d\mathbf{r})\,e^{\mp ipx}(\overrightarrow{\partial}_0^{\,2} - \overleftarrow{\partial}_0^{\,2})\phi(x)\\ &= \pm i\int(d\mathbf{r})\,\nabla\cdot\left[e^{\mp ipx}(\overrightarrow{\nabla} - \overleftarrow{\nabla})\phi(x)\right]\\ &= 0\,.\end{aligned} \tag{3.2.60}$$

Thus the $\phi^{(\pm)}(p)$ are time independent. One can also show that the $\phi^{(\pm)}(p)$ are also Lorentz invariant.[§] Using the previous form of the wave function,

$$\begin{aligned}\langle 0|\phi^{(\pm)}(p)|p'\rangle &= \pm i\int(d\mathbf{r})e^{\mp ipx}(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\phi(x)|p'\rangle\\ &= \pm[E(\mathbf{p}')\pm E(\mathbf{p})](2\pi)^3\delta(\mathbf{p}\mp\mathbf{p}')\,.\end{aligned} \tag{3.2.61}$$

Hence

$$\langle 0|\phi^{(+)}(p)|p'\rangle = \langle p|p'\rangle\,, \tag{3.2.62}$$

[§] These features follow from the fact that the $\phi^{(\pm)}(p)$ are given by the spatial integral of the time component $f_0(x)$ of a current $f_\mu(x) = e^{\mp ipx}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\phi(x)$ which is conserved, $\partial_\mu f^\mu = 0$. This is made clear in Problem 10 and in the discussion following it.

while

$$\langle 0|\phi^{(-)}(p)|p'\rangle = 0\,. \tag{3.2.63}$$

Since $\phi(x)$ connects only single particle states with the vacuum, these results imply that

$$\langle 0|\phi^{(+)}(p) = \langle p|\,, \qquad \langle 0|\phi^{(-)}(p) = 0\,. \tag{3.2.64}$$

Since

$$\phi^{(-)}(p) = \phi^{(+)}(p)^\dagger\,, \tag{3.2.65}$$

the Hermitian adjoint of Eqs. (3.2.64) give

$$\phi^{(-)}(p)|0\rangle = |p\rangle\,, \qquad \phi^{(+)}(p)|0\rangle = 0\,. \tag{3.2.66}$$

The $\phi^{(+)}(p)$ and $\phi^{(-)}(p)$ are the relativistic generalizations of the non-relativistic annihilation and creation operators, or rather the Fourier transforms of these operators.

We have just found that

$$\langle p|p'\rangle = \langle 0|\phi^{(+)}(p)\phi^{(-)}(p')|0\rangle = (2\pi)^3 2E(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')\,. \tag{3.2.67}$$

To create multiparticle states, we recall that the vacuum expectation values of several field operators factorize into products of two-point functions according to the "contraction rule" that was described above. Thus, for example, we expect that the state of two free particles is given by the construction

$$|p_1p_2\rangle = \phi^{(-)}(p_1)\phi^{(-)}(p_2)|0\rangle\,, \tag{3.2.68}$$

which is analogous to that of the non-relativistic many particle theory. Indeed, using the contraction rule it follows that

$$\begin{aligned}\langle p_1'p_2'|p_1p_2\rangle &= \langle 0|\phi^{(+)}(p_1')\phi^{(+)}(p_2')\phi^{(-)}(p_1)\phi^{(-)}(p_2)|0\rangle\\ &= \langle\phi^{(+)}(p_1')\phi^{(+)}(p_2')\rangle\langle\phi^{(-)}(p_1)\phi^{(-)}(p_2)\rangle\\ &\quad +\langle\phi^{(+)}(p_1')\phi^{(-)}(p_1)\rangle\langle\phi^{(+)}(p_2')\phi^{(-)}(p_2)\rangle\\ &\quad +\langle\phi^{(+)}(p_1')\phi^{(-)}(p_2)\rangle\langle\phi^{(+)}(p_2')\phi^{(-)}(p_1)\rangle\\ &= 0 + \langle p_1'|p_1\rangle\langle p_2'|p_2\rangle + \langle p_1'|p_2\rangle\langle p_2'|p_1\rangle\,.\end{aligned} \tag{3.2.69}$$

This is precisely the form of the two-particle transformation function of free, identical Bosons and verifies our assumption. The structure of this result is indicated in Fig. 3.2.

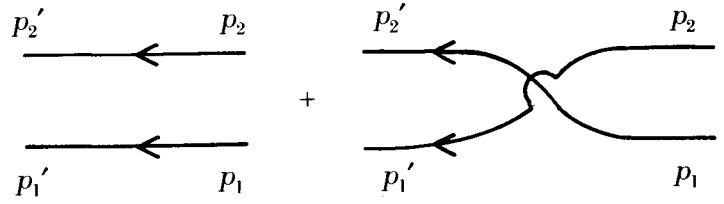


Fig. 3.2. Free particle propagation for Bosons includes exchange to maintain Bose statistics.

3.3 Self-interacting theory

Let us enlarge our world from a trivial non-interacting scalar field system to a theory which contains a self-interaction:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\,. \tag{3.3.1}$$

To generate matrix elements of field operators, we shall also couple the field ϕ to a source ρ, but once ρ has served its purpose to construct these matrix elements it is set to zero, and so we do not indicate its presence explicitly here. There is no need to introduce a cubic interaction involving ϕ^3 since performing a constant translation of the field variable $\phi \to \phi + \text{const.}$ in the Lagrange function (3.3.1) that contains the quartic interaction ϕ^4 augmented by such a term removes this cubic term and just redefines the parameter m and the implicit source term. This translation has no effect since the field ϕ is just a (dummy) functional integration variable. As will be seen in detail in our later development, the Lagrange function (3.3.1) is, in a well-defined sense, unique. Other terms involving higher derivatives and/or higher powers of the field cannot occur if the theory is to be renormalizable. The presence of such terms would give rise to infinities in scattering amplitudes whose removal would entail the introduction of yet more undetermined parameters. As one progresses to higher and higher orders of the perturbation series, more and more such undetermined constants must be introduced. Thus the theory is described by only two parameters m and λ or else by an infinite number*

* We should be quick to note, however, that unrenormalizable, "effective Lagrangians" with higher powers of the derivatives and/or the fields are often useful for the limited description of low-energy phenomena. The low-energy limit of more complete, renormalizable theories can be shown to be described by such effective Lagrangians. For example, the electroweak theory including the $W^\pm$, Z^0, photon, Higgs, and Fermi fields is renormalizable. At low energies, the short-ranged but non-local interaction brought about by the exchange of the $W^\pm$ and Z^0 particles is equivalent to a local interaction of four Fermi fields without

of parameters!

It is important to understand the dimensions of the quantities that enter into the theory. The action

$$S = \int (d^4x)\mathcal{L} \tag{3.3.2}$$

appears in the exponential in the functional integral and so it must be dimensionless. Let us use the symbols L to denote the unit of length and M to denote a unit of mass. With our natural system of units in which $\hbar = 1$, $M = L^{-1}$. The Lagrange density must carry the dimensions L^{-4} to keep the action dimensionless,

$$[\mathcal{L}]_d = L^{-4} = M^4 . \tag{3.3.3}$$

Since the gradient ∂_μ has the dimensions of mass, the $(\partial_\mu\phi)^2$ term in the Lagrange function now tells us that the scalar field carries the dimension of mass,

$$[\phi]_d = M . \tag{3.3.4}$$

Finally, since ϕ^4 has the same dimensions as $\mathcal{L}$, the coupling constant λ must be dimensionless,

$$[\lambda]_d = 0 . \tag{3.3.5}$$

We can now indicate very roughly why other parameters cannot be introduced into the Lagrange function if the theory is to be renormalizable. The sketchy remarks that are about to be made are meant only to give the flavor of what happens — a full discussion will appear later. Suppose, for example, that a coupling of the form $\kappa\phi^6$ is introduced. The parameter κ has the dimensions $1/M^2$. In second order perturbation theory this interaction will contribute to the two particle scattering process that initially was driven by the $\lambda\phi^4$ interaction, where λ is dimensionless. Crudely speaking, the second-order correction which results when both interactions act once alters the original interaction, with λ being replaced by $\lambda[1 + 0(\kappa)]$. To make this dimensionally correct, we must write it as $\lambda \to \lambda[1 + 0(E^2\kappa)]$, where E is the energy of the scattering process.† The introduction of this new interaction gives a scattering amplitude that diverges very badly as the energy increases. It is not only

the $W^\pm$ and Z^0 fields. This interaction has the weak coupling constant G_F bearing the dimension (mass)$^{-2}$, and it by itself yields a non-renormalizable theory. In general, non-renormalizable, effective Lagrangians can certainly be used if one can show that they result from the low-energy limit of a more complete, renormalizable theory. They may also be useful even in the absence of such a larger theory if physical arguments can be made to show that the non-renormalizable couplings involve a mass scale which is much larger that the energy region to be described by the effective Lagrangian.

† The particle mass can be neglected in the high-energy limit which we consider, and so it cannot be used to make up the dimensions.

that the scattering amplitude increases rapidly with energy — going to higher orders involves essentially integrating the square of this amplitude over all intermediate states which contain infinite energies — and highly divergent integrals appear.

Setting $\lambda \to 0$ produces the previous free theory. Thus it is natural to expect to be able to do perturbation theory and expand in powers of the coupling parameter λ. This is indeed possible in our quantum theory. But it is not so in the classical theory. To see this, we redefine the field amplitude by setting

$$\phi = \lambda^{-1/2}\phi' \tag{3.3.6}$$

to obtain

$$\mathcal{L} = -\frac{1}{\lambda}\left[\frac{1}{2}(\partial_\mu\phi')^2 + \frac{1}{2}m^2\phi'^2 + \frac{1}{4!}\phi'^4\right] . \tag{3.3.7}$$

In classical physics the overall scale of the Lagrange function has no significance, and so the strength of the coupling parameter λ is irrelevant (so long as it is non-vanishing). Hence in classical physics it makes no sense to try to solve the theory by expanding in powers of λ. On the other hand, in quantum physics the scale of the Lagrange function does have meaning because the theory is defined by a functional integral with the exponential of the action as its weight. If we would use the new form with the field ϕ' we can expand in powers of λ: The leading term is just the stationary phase approximation‡ discussed in the previous chapters, and it involves the classical solution $\phi' = 0$. As we have seen, to get the higher order corrections in a systematic way, it was convenient to scale the field by $\lambda^{1/2}$. Doing so here just takes us back to our original form, which is the form that we shall use. Thus we write

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{4!}\lambda\phi^4 , \tag{3.3.8}$$

where $\mathcal{L}_0$ is the free field Lagrange function, and expand the generating functional

$$Z[\rho] = \int [d\phi] \exp\left\{i\int (d^4x)\left[\mathcal{L}_0 - \frac{\lambda}{4!}\phi^4 + \phi\rho\right]\right\} \tag{3.3.9}$$

in powers of λ.

The basic, elementary process brought about by the $\lambda\phi^4$ interaction is the scattering of two particles. Two particles that are initially at a remote time in the past with momenta p_1, p_2 scatter and emerge with momenta p_1', p_2' in the distant future as shown in Fig. 3.3. We shall now

‡ It is evident from Eq. (3.7) that the coupling λ can be taken to play the role of Planck's constant $\hbar$.

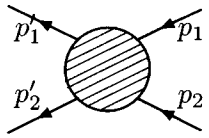


Fig. 3.3. Graphical representation of the elastic scattering amplitude.

compute this process in lowest order, deferring the discussion of higher order corrections to subsequent sections. To first order, the generating functional gives

$$\left\langle T e^{i\int \phi\rho} \right\rangle = \int [d\phi] e^{i\int [\mathcal{L}_0 + \rho\phi]} \left\{ 1 - \frac{i\lambda}{4!} \int \phi^4 \right\} . \tag{3.3.10}$$

On the one hand, the source $\rho(x)$ can be used to pull down field operators $\phi(x)$ at the initial and final times so as to create the initial state $|p_1,\, p_2-\rangle$ and the final state $\langle p_1',\, p_2'+|$. On the other hand, the functional integral with the free field action just produces the free particle theory. Hence, to first order, the probability amplitude for two particles prepared with momenta p_1 and p_2 in the remote past ($t \to -\infty$ which is indicated by $-$) to be found with the momenta p_1' and p_2' in the remote future ($t \to +\infty$ which is indicated by $+$) is given by

$$\langle p_1',\, p_2'+|p_1,\, p_2-\rangle = \langle p_1',\, p_2'| \left\{ 1 - \frac{i\lambda}{4!} \int (d^4x)\, \phi^4(x) \right\} |p_1,\, p_2\rangle^{(0)} , \tag{3.3.11}$$

in which the matrix element on the right-hand side is computed in the free theory with a vanishing source ($\rho = 0$) as is indicated by the (0) superscript. Here

$$\begin{aligned} &\int (d^4x) \langle p_1',\, p_2'|\phi^4(x)|p_1,\, p_2\rangle^{(0)} \\ &= \int (d^4x) \langle p_1',\, p_2'|e^{-iPx}\phi^4(0)e^{iPx}|p_1,\, p_2\rangle^{(0)} \\ &= \int (d^4x) e^{-i(p_1'+p_2'-p_1-p_2)x} \langle p_1',\, p_2'|\phi^4(0)|p_1,\, p_2\rangle^{(0)} \\ &= (2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2) \langle p_1',\, p_2'|\phi^4(0)|p_1,\, p_2\rangle^{(0)} , \end{aligned} \tag{3.3.12}$$

while, as was seen at the end of the previous section,

$$\langle p_1',\, p_2'|p_1,\, p_2\rangle^{(0)} = \langle p_1'|p_1\rangle\langle p_2'|p_2\rangle + \langle p_1'|p_2\rangle\langle p_2'|p_1\rangle . \tag{3.3.13}$$

Accordingly, the scattering amplitude has the general structure

$$\begin{aligned} \langle p_1', p_2'+|p_1,\, p_2-\rangle &= \langle p_1'|p_1\rangle\langle p_2'|p_2\rangle + \langle p_1'|p_2\rangle\langle p_2'|p_1\rangle \\ &- \frac{i\lambda}{4!}(2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2) \langle p_1',\, p_2'|\phi^4(0)|p_1,\, p_2\rangle^{(0)} . \end{aligned} \tag{3.3.14}$$

This decomposition is illustrated in Fig. 3.4. The disconnected pairs of

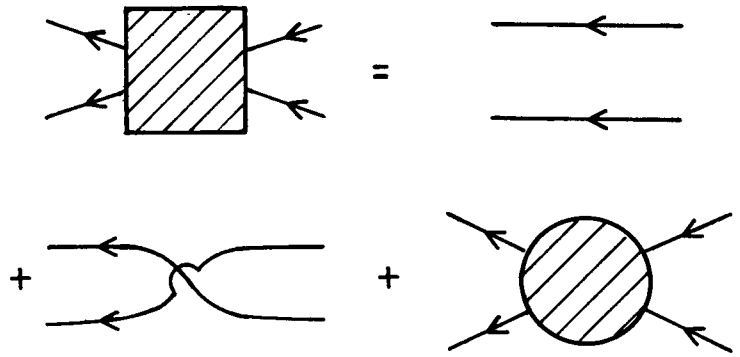


Fig. 3.4. The transformation function between times in the remote past and far future is the sum of free propagation terms plus the true scattering contribution.

lines correspond to the straight-through propagation without scattering with the additional twisted pair needed to account for the Bose statistics. The shaded circle denotes the term where scattering does occur.

The initial and final states are constructed by the $\phi^{(\pm)}(p)$ operators defined in Eq. (3.2.57). Thus

$$\begin{aligned} &\langle p_1',\, p_2'|\phi^4(0)|p_1,\, p_2\rangle^{(0)} \\ &= \langle 0|\phi^{(+)}(p_1')\phi^{(+)}(p_2')\phi(0)\phi(0)\phi(0)\phi(0)\phi^{(-)}(p_1)\phi^{(-)}(p_2)|0\rangle^{(0)} . \end{aligned} \tag{3.3.15}$$

This matrix element is easily calculated using the contraction rule. The $\phi^{(+)}(p_1')$ operator may be contracted with any of the four $\phi(0)$, giving a factor of 4 and a factor of $\langle 0|\phi^{(+)}(p_1')\phi(0)|0\rangle = \langle p_1'|\phi(0)|0\rangle = 1$. The $\phi^{(+)}(p_2')$ may then be contracted with any of the remaining three $\phi(0)$ fields, giving an additional factor of 3. The operator $\phi^{(-)}(p_1)$ may then be contracted with any of the remaining two $\phi(0)$ fields, giving a final factor of 2. There remains $\langle 0|\phi(0)\phi^{(-)}(p_2)|0\rangle = 1$. Hence

$$\langle p_1',\, p_2'|\phi^4(0)|p_1,\, p_2\rangle^{(0)} = 4! . \tag{3.3.16}$$

The alert reader will have noticed that terms have been omitted terms which involve the contraction of one $\phi(0)$ with another $\phi(0)$. These correspond to renormalization of the straight-through propagation and do not contribute to the scattering. Such corrections will be discussed later. Neglecting these terms, we have found that to first order

$$\begin{aligned} \langle p_1',\, p_2'+|p_1,\, p_2-\rangle &= \langle p_1'|p_1\rangle\langle p_2'|p_2\rangle + \langle p_1'|p_2\rangle\langle p_2'|p_1\rangle \\ &- i\lambda(2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2) , \end{aligned} \tag{3.3.17}$$

where, to remind ourselves of the normalization, we record

$$\langle p_1'|p_1\rangle = 2E_1(2\pi)^3\delta(\mathbf{p}_1' - \mathbf{p}_1)\,. \tag{3.3.18}$$

3.4 Scattering cross section

The structure of the result that has just been obtained for the scalar theory holds for any two-particle scattering process. Namely, in general,

$$\begin{aligned}\langle p_1',\, p_2'+|p_1,\, p_2-\rangle &= \langle p_1'|p_1\rangle\langle p_2'|p_2\rangle \pm \langle p_1'|p_2\rangle\langle p_2'|p_1\rangle \\ &\quad -i(2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)\,T(p_1',\, p_2';\; p_1,\, p_2)\,,\end{aligned} \tag{3.4.1}$$

where the $\pm$ accounts for the Bose or Fermi nature of the particles and the momentum label p also implicitly represents other degrees of freedom such as spin components or particle type labels. The first two terms describe the straight-through, unscattered propagation of one particle past the other while $T(p_1',\, p_2';\, p_1,\, p_2)$ is the amplitude for scattering. For the scalar theory in lowest order, $T = \lambda$ is simply the coupling constant. The scattering amplitude is accompanied by the $\delta^{(4)}(p_2' + p_1' - p_1 - p_2)$ factor so as to conserve the overall energy and momentum. In the general case, this factor arises from the translation invariance of the theory just as it did in the previous calculation for the scalar theory.

We turn now to obtain the general formula which relates the scattering amplitude to the scattering cross section. To do this clearly and with some rigor, we work with normalized, "wavepacket" initial states that represent the propagation of a reasonably localized, single particle. Thus we consider*

$$|f\rangle = \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2p^0} f(\mathbf{p})|p\rangle\,, \tag{3.4.2}$$

with the normalization

$$\langle f|f\rangle = 1\,. \tag{3.4.3}$$

Here, to save writing, we use $p^0 = E(\mathbf{p})$ as will often be done when it is clear that the energy is constrained to "be on the mass shell". In view of the conventional normalization (3.3.18) for $\langle p'|p\rangle$, the unit normalization (3.4.3) requires that

$$\int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2p^0}|f(\mathbf{p})|^2 = 1\,. \tag{3.4.4}$$

* Although we now explicitly work with the scalar theory, it should be clear that our results are easily extended to more general cases.

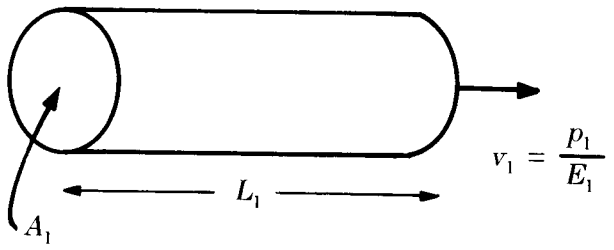


Fig. 3.5. Wave packet describing the initial beam particle.

The wave function for the normalized state is given by

$$\begin{aligned}\psi(x) &= \langle 0|\phi(x)|f\rangle \\ &= \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2p^0} f(\mathbf{p})\langle 0|\phi(x)|p\rangle \\ &= \int \frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2p^0} f(\mathbf{p})e^{ipx}\,.\end{aligned} \tag{3.4.5}$$

It is easy to check that

$$\int (d\mathbf{r})\,\psi^*(x)i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\psi(x) = 1\,. \tag{3.4.6}$$

Two wave packets are needed, one to describe the particle which is considered to be a "beam" particle, another to describe a "target" particle. To describe a beam particle, we take a weight $f_1(\mathbf{p})$ which is sharply peaked at $\mathbf{p} = \mathbf{p}_1$ and which gives a wave function modulus $|\psi_1(x)|$ that, except for small edge effects, is uniformly distributed over a moving cylinder of cross sectional area A_1 and length L_1. This packet is depicted in Fig. 3.5. Since the wavepacket is sharply peaked in momentum, its energy is close to $E_1 = E(\mathbf{p}_1)$, and the normalization condition (3.4.6) may be approximated by

$$2E_1\int (d\mathbf{r})|\psi_1|^2 \simeq 1\,. \tag{3.4.7}$$

Thus, neglecting small edge effects, the spatial wave function of the beam particle wave packet has the absolute values

$$|\psi_1|^2 \simeq \begin{cases} (2E_1A_1L_1)^{-1} & , \text{ inside the cylinder}\,, \\ 0 & , \text{ outside the cylinder}\,. \end{cases} \tag{3.4.8}$$

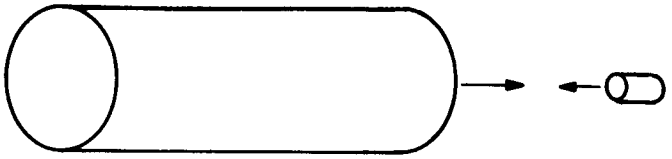


Fig. 3.6. The initial scattering configuration. It is convenient to take a much larger wave packet for the beam particle than the target particle as shown.

To describe the target particle, we take a weight $f_2(\mathbf{p})$ sharply peaked at $\mathbf{p} = \mathbf{p}_2$ with $\mathbf{p}_2$ antiparallel to $\mathbf{p}_1$, $-\mathbf{p}_2 \,||\, \mathbf{p}_1$. The normalization of this wave packet requires that

$$2E_2 \int (d\mathbf{r}) |\psi_2|^2 \simeq 1 \,. \tag{3.4.9}$$

It is convenient to assume that $\psi_2(x)$ is centered along the axis of the $\psi_1(x)$ cylinder, with the spatial extent of $\psi_2(x)$ much smaller than this cylinder. A sketch of the initial configuration is shown in Fig. 3.6.

With the final momenta p_1' and p_2' lying outside of the range of the straight-through propagation, only scattering occurs. Hence, in view of Eq.'s (3.4.1) and (3.4.2), the transformation function appears as

$$\langle p_1', p_2' + | f_1, f_2 - \rangle = -i \int \frac{(d\mathbf{q}_1)}{(2\pi)^3} \frac{f_1(\mathbf{q}_1)}{2q_1^0} \int \frac{(d\mathbf{q}_2)}{(2\pi)^3} \frac{f_2(\mathbf{q}_2)}{2q_2^0}$$
$$(2\pi)^4 \delta^{(4)}(p_1' + p_2' - q_1 - q_2) T(p_1', p_2'; q_1, q_2) \,. \tag{3.4.10}$$

With our normalization conventions, the resolution of the identity in the two-particle subspace is given by

$$1_{2-\text{part}} = \frac{1}{2} \int \frac{(d\mathbf{p}_1')}{(2\pi)^3} \frac{1}{2p_1'^0} \int \frac{(d\mathbf{p}_2')}{(2\pi)^3} \frac{1}{2p_2'^0} |p_1', p_2'\rangle \langle p_1', p_2'| \,. \tag{3.4.11}$$

Hence the probability $\Delta\mathcal{P}$ for scattering into some momentum interval Δ is given by

$$\Delta\mathcal{P} = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3} \frac{1}{2p_1'^0} \int \frac{(d\mathbf{p}_2')}{(2\pi)^3} \frac{1}{2p_2'^0} |\langle p_1', p_2' + | f_1, f_2 - \rangle|^2 \,. \tag{3.4.12}$$

There is no factor of $\frac{1}{2}$ here — the "identical particle factor" since $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are assumed to lie in different regions of momentum space. The factor of $\frac{1}{2}$ reappears in the calculation of the total cross section which is equivalent to summing over all regions of $\mathbf{p}_2'$ not already covered

by $\mathbf{p}_1'$. In computing the square of the amplitude (3.4.10) we encounter $\delta^{(4)}(p_1' + p_2' - q_1 - q_2)\delta^{(4)}(p_1' + p_2' - \bar{q}_1 - \bar{q}_2)$ where q_1 and q_2 are the previous integration variables for the wave packet weights that come into the amplitude and $\bar{q}_1$ and $\bar{q}_2$ are the similar variables for its complex conjugate. This product of δ-functions may be written as

$$\begin{aligned} &\delta^{(4)}(p_1' + p_2' - q_1 - q_2)\delta^{(4)}(p_1' + p_2' - \bar{q}_1 - \bar{q}_2) \\ &\quad = \delta^{(4)}(p_1' + p_2' - q_1 - q_2)\delta^{(4)}(q_1 + q_2 - \bar{q}_1 - \bar{q}_2) \\ &\quad \simeq \delta^{(4)}(p_1' + p_2' - p_1 - p_2)\delta^{(4)}(q_1 + q_2 - \bar{q}_1 - \bar{q}_2) \,, \end{aligned} \tag{3.4.13}$$

with the final approximate equality valid since $q_1 \simeq p_1$, $q_2 \simeq p_2$. Hence

$$\Delta\mathcal{P} = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3} \frac{1}{2p_1'^0} \int \frac{(d\mathbf{p}_2')}{(2\pi)^3} \frac{1}{2p_2'^0} (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) |T|^2 S \,, \tag{3.4.14}$$

in which

$$S = \int \frac{(d\mathbf{q}_1)}{(2\pi)^3} \frac{f_1(\mathbf{q}_1)}{2q_1^0} \int \frac{(d\mathbf{q}_2)}{(2\pi)^3} \frac{f_2(\mathbf{q}_2)}{2q_2^0} \int \frac{(d\bar{\mathbf{q}}_1)}{(2\pi)^3} \frac{f_1(\bar{\mathbf{q}}_1)^*}{2\bar{q}_1^0}$$
$$\int \frac{(d\bar{\mathbf{q}}_2)}{(2\pi)^3} \frac{f_2(\bar{\mathbf{q}}_2)^*}{2\bar{q}_2^0} (2\pi)^4 \delta^{(4)}(q_1 + q_2 - \bar{q}_1 - \bar{q}_2) \,. \tag{3.4.15}$$

On introducing

$$(2\pi)^4 \delta^{(4)}(q_1 + q_2 - \bar{q}_1 - \bar{q}_2) = \int (d^4x) e^{i(q_1 + q_2 - \bar{q}_1 - \bar{q}_2)x} \,, \tag{3.4.16}$$

this factor may be written simply as

$$S = \int (d^4x) |\psi_1(x)|^2 |\psi_2(x)|^2 \,. \tag{3.4.17}$$

The two wave packets shown in Fig. 3.6 move with the relative velocity

$$v_{12} = \left| \frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_2}{E_2} \right| \,. \tag{3.4.18}$$

The $dx^0 = dt$ integration in Eq.(3.4.17) extends only over the interval L_1/v_{12} during which the small $\psi_2(x)$ wave packet is inside the large $\psi_1(x)$ wave packet that extends over the length L_1. Otherwise the product $|\psi_1(x)|^2 |\psi_2(x)|^2$ vanishes. Therefore, in view of Eqs. (3.4.8) and (3.4.9),

$$S = \frac{L_1}{v_{12}} |\psi_1(\text{inside})|^2 \int (d\mathbf{r}) |\psi_2(x)|^2 = \frac{L_1}{v_{12}} \frac{1}{2E_1 A_1 L_1} \frac{1}{2E_2} \,, \tag{3.4.19}$$

and the scattering probability appears as

$$\Delta\mathcal{P} = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3} \frac{1}{2p_1'^0} \int \frac{(d\mathbf{p}_2')}{(2\pi)^3} \frac{1}{2p_2'^0} (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2)$$
$$\times |T(p_1', p_2'; p_1, p_2)|^2 \frac{1}{4E_1 E_2 v_{12}} \frac{1}{A_1} \,. \tag{3.4.20}$$

To interpret this result, we note that if the target displayed an effective scattering area $\Delta\sigma$, then the probability that the particle in the beam of area A_1 would scatter is given by

$$\mathcal{P} = \Delta\sigma/A_1 \,. \tag{3.4.21}$$

Hence, in view of Eq. (3.4.20), the partial scattering cross section $\Delta\sigma$ into the final momentum interval Δ is given by

$$\Delta\sigma = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3}\frac{1}{2p_1'^0}\int \frac{(d\mathbf{p}_2')}{(2\pi)^3}\frac{1}{2p_2'^0}(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)|T|^2\frac{1}{4E_1E_2v_{12}}. \tag{3.4.22}$$

Several remarks about this result should be made. First of all, a particular configuration of a very large "beam" and a very small "target" was chosen so as to make the calculation easy. Clearly nothing is changed if the roles of beam and target particles are interchanged. Moreover, the two wave packets could be replaced by packets of arbitrary shape by appropriately patching together many wave packets. Finally, various approximations have been made such as neglecting the small surface regions where the wave packets decay away or in approximating the wave packet momentum $\mathbf{q}$ by its mean value $\mathbf{p}$. These approximations entail neglecting quantities that are essentially the ratios of microscopic, "quantum-mechanical" sizes to macroscopic, "laboratory" sizes. They are quantities that are so small as to be unmeasurable. But they are necessarily present in a realistic description of an actual experimental scattering measurement — of which we have only given a rather idealized account. One should appreciate that we have given — in outline form at least — a rigorous connection of the scattering cross section with the scattering amplitude T, not just "mathematically rigorous", but "physically rigorous" as well.

Since the scattering cross section corresponds to a transverse area that is unchanged by Lorentz boosts along the beam direction, it may be put in a manifestly Lorentz invariant form. This is the form of all of the ingredients of formula (3.4.22) save for the last factor of $E_1E_2v_{12}$. To see how this factor may be written in invariant form, we keep the discussion general so that p_1 corresponds to a particle of mass m_1, p_2 to a particle of mass m_2, and note that with $\mathbf{p}_1$ and $\mathbf{p}_2$ collinear

$$\begin{aligned}(p_1p_2)^2 - m_1^2m_2^2 &= (p_1^0p_2^0 - \mathbf{p}_1\cdot\mathbf{p}_2)^2 - m_1^2m_2^2\\ &= (\mathbf{p}_1^2+m_1^2)(\mathbf{p}_2^2+m_2^2) - 2E_1E_2\mathbf{p}_1\cdot\mathbf{p}_2 + \mathbf{p}_1^2\mathbf{p}_2^2 - m_1^2m_2^2\\ &= E_1^2\mathbf{p}_2^2 + E_2^2\mathbf{p}_1^2 - 2E_1E_2\mathbf{p}_1\cdot\mathbf{p}_2\\ &= E_1^2E_2^2\left|\frac{\mathbf{p}_1}{E_1}-\frac{\mathbf{p}_2}{E_2}\right|^2 .\end{aligned} \tag{3.4.23}$$

Hence

$$E_1E_2v_{12} = \sqrt{(p_1p_2)^2 - m_1^2m_2^2} \tag{3.4.24}$$

is indeed a simple Lorentz invariant. To examine the kinematics further, we note that the squared center-of-mass energy s is given by

$$s = -(p_1+p_2)^2 \tag{3.4.25}$$

as is obvious by its evaluation in the center of mass where $\mathbf{p}_1+\mathbf{p}_2 = 0$. So

$$s = m_1^2 + m_2^2 - 2p_1p_2 \tag{3.4.26}$$

and

$$4E_1E_2v_{12} = 2\sqrt{[s-(m_1-m_2)^2][s-(m_1+m_2)^2]}\,. \tag{3.4.27}$$

Note that this vanishes, as it must, at the reaction threshold $s = (m_1+m_2)^2$.

Some further discussion of relativistic kinematics is useful. Since the total energy and momentum is conserved in the scattering process,

$$s = -(p_1+p_2)^2 = -(p_1'+p_2')^2\,. \tag{3.4.28}$$

In addition to this total energy variable, it is convenient to define the invariant, four-momentum-transfer-squared variable

$$t = -(p_1'-p_1)^2 = -(p_2'-p_2)^2\,. \tag{3.4.29}$$

Equally well, one could use the momentum transferred between the other particle pairs

$$u = -(p_2'-p_1)^2 = -(p_1'-p_2)^2\,. \tag{3.4.30}$$

A little algebra shows that

$$s+t+u = m_1^2+m_2^2+m_1'^2+m_2'^2\,, \tag{3.4.31}$$

where a generalization has been made with $-p_1'^2 = m_1'^2$, $-p_2'^2 = m_2'^2$ to cover also the case of inelastic scattering where the final particles may differ from the initial particles. These variables are displayed in Fig. 3.7.

The differential cross section appears as an integration over the 6-dimensional momentum space $(d\mathbf{p}_1')(d\mathbf{p}_2')$ constrained by the 4-dimensional δ-function of energy-momentum conservation. Thus there are two independent scattering variables which are conventionally taken to be the solid angle element into which one of the particles is scattered. Neglecting spin with polarized beams, the scattering process is invariant under rotations about the beam axis, and the differential cross section depends only on a scattering angle. Rather than calculating such a differential

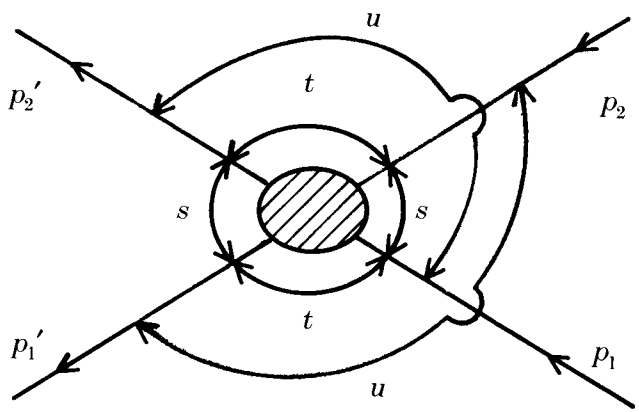


Fig. 3.7. Scattering variables.

cross section, which is not a Lorentz invariant, it is advantageous to compute the invariant partial cross section with respect to an increment of momentum transfer t. This can be done by inserting

$$1 = \int dt\, \delta(t + (p_1' - p_1)^2) \tag{3.4.32}$$

into the cross section formula (3.4.22), integrating over all final momenta and then identifying the resulting integrand for the t-integration as $d\sigma/dt$. Such a method obviates the computation of a Jacobian in transforming to the t variable. Thus

$$\begin{aligned}\frac{d\sigma}{dt} &= \int \frac{(d\mathbf{p}'_1)}{(2\pi)^3}\frac{1}{2p_1'^0}\int \frac{(d\mathbf{p}'_2)}{(2\pi)^3}\frac{1}{2p_2'^0}\delta(t+(p_1'-p_1)^2)\\ &\qquad \frac{(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)}{2\sqrt{[s-(m_1-m_2)^2]\,[s-(m_1+m_2)^2]}}\,|T|^2\\ &= \frac{\Phi(t,s)}{2\sqrt{[s-(m_1-m_2)^2]\,[s-(m_1+m_2)^2]}}\,|T|^2\,, \end{aligned} \tag{3.4.33}$$

which defines a relativistically invariant phase-space volume $\Phi(t,s)$. Since Φ is an invariant, it may be evaluated in the center-of-mass frame† where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. Using the spatial momentum δ-function, the resulting

† This evaluation is similar to that performed in Problem 5.

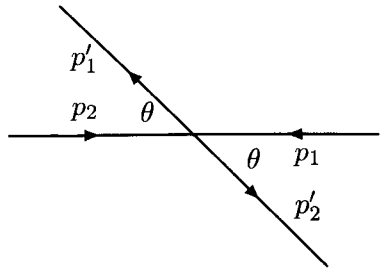


Fig. 3.8. Center-of-mass scattering kinematics.

integration over $\mathbf{p}_2'$ is trivial. Integrating over‡ $d\cos\theta$ and dp_1' then gives

$$\begin{aligned}\Phi &= \frac{1}{(2\pi)^2}\int d\Omega_1' \int_0^\infty p_1'^2 dp_1' \,\frac{1}{4E_1'(\mathbf{p}_1')E_2'(\mathbf{p}_1')}\delta(E_1'(\mathbf{p}_1')+E_2'(\mathbf{p}_1')-\sqrt{s})\\ &\qquad \delta(t-m_1'^2-m_1^2+2E_1'(\mathbf{p}_1')E_1(\mathbf{p}_1)-2p_1'p_1\cos\theta)\\ &= \frac{1}{16\pi}\,\frac{1}{p\sqrt{s}}\,, \end{aligned} \tag{3.4.34}$$

where p is the magnitude of the initial center-of-mass momentum defined by

$$E_1(\mathbf{p}) + E_2(\mathbf{p}) = \sqrt{s}\,. \tag{3.4.35}$$

In obtaining this result, we have used

$$\int dx\, \delta(f(x)) = \frac{1}{|f'(x_0)|}\,, \tag{3.4.36}$$

where the prime denotes the derivative and where x_0 is the root $f(x_0) = 0$. Squaring Eq. (3.4.35), rearranging and squaring again yields

$$p^2 = \frac{s^2 + (m_1^2 - m_2^2)^2}{4s} - \frac{m_1^2 + m_2^2}{2}\,. \tag{3.4.37}$$

The result simplifies for the equal mass elastic scattering case that motivated our work. In this case

$$\frac{d\sigma}{dt} = \frac{1}{16\pi}\,\frac{1}{s(s-4m^2)}\,|T|^2\,. \tag{3.4.38}$$

Working in the center-of-mass frame, the scattering is described by the scattering angle θ shown in Fig. 3.8. In terms of this angle,

$$-t = 2p^2(1-\cos\theta) = \frac{1}{2}(s-4m^2)(1-\cos\theta)\,, \tag{3.4.39}$$

‡ The condition that the $\cos\theta$ integration pass through the root of the delta function — that its argument be positive at $\cos\theta = -1$ and negative at $\cos\theta = +1$ — constrains the values of momentum transfer t which can occur in the scattering process.

and

$$dt = (s - 4m^2)\frac{d\Omega}{4\pi}\,. \tag{3.4.40}$$

Thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{4s}\left|\frac{T}{4\pi}\right|^2\,. \tag{3.4.41}$$

With identical particles, the total cross section is given by

$$\sigma = \frac{1}{2}\int d\Omega\,\frac{d\sigma}{d\Omega}\,. \tag{3.4.42}$$

In this case the amplitude T is invariant under the interchange $p_1' \leftrightarrow p_2'$ and the differential cross sections for angles θ and $\pi - \theta$ are identical. Integrating over the region $\pi/2 < \theta < \pi$ just repeats that over $0 < \theta < \pi/2$ and double counts which requires the factor of $\frac{1}{2}$ in the total cross section formula. This is just the factor $\frac{1}{2}$ that appears in the 2-particle resolution of the identity (3.4.11).

The general development was motivated by the first-order scalar particle scattering amplitude

$$T = \lambda\,. \tag{3.4.43}$$

We now see that for this process

$$\frac{d\sigma}{d\Omega} = \frac{1}{4s}\left(\frac{\lambda}{4\pi}\right)^2\,, \tag{3.4.44}$$

and

$$\sigma = \frac{\lambda^2}{32\pi}\frac{1}{s}\,. \tag{3.4.45}$$

A physically rigorous derivation of the formula that relates the scattering cross section to the scattering amplitude has been given. It is worthwhile concluding this section by giving another heuristic derivation that has the virtue of being so simple that one can use it as a mnemonic device to get the factors right. Moreover, similar methods can be used to quickly obtain formulas for other processes such as the calculation of decay rates. This is how the mnemonic method goes: Normalized single particle states are obtained if the system is put in a box so that the momentum eigenvalues become discrete and momentum integrals are replaced by sums. This procedure replaces the Dirac $\delta(\mathbf{p}' - \mathbf{p})$ with the Kronecker $\delta_{\mathbf{p}',\mathbf{p}}$, the two being related by

$$(d\mathbf{p}')\delta(\mathbf{p}' - \mathbf{p}) = \delta_{\mathbf{p}',\mathbf{p}}\,. \tag{3.4.46}$$

Since in the continuum we use the normalization

$$\langle p'|p\rangle = (2\pi)^3 2p^0\delta(\mathbf{p}' - \mathbf{p})\,, \tag{3.4.47}$$

the single particle states $|1_{\mathbf{p}}\rangle$ in the box are given by

$$|1_{\mathbf{p}}\rangle = |p\rangle\sqrt{\frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{2p^0}}\,, \tag{3.4.48}$$

for now

$$\langle 1_{\mathbf{p}'}|1_{\mathbf{p}}\rangle = \delta_{\mathbf{p}',\mathbf{p}}\,. \tag{3.4.49}$$

Hence the partial transition probability $\Delta\mathcal{P}$ for the scattering of two initial single particle states into some final momentum interval Δ is given by

$$\Delta\mathcal{P} = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3}\,\frac{1}{2p_1'^0}\,\frac{(d\mathbf{p}_2')}{(2\pi)^3}\,\frac{1}{2p_2'^0} \left|\langle p_1', p_2' + |p_1, p_2-\rangle\sqrt{\frac{(d\mathbf{p}_1)}{(2\pi)^3}\,\frac{1}{2p_1^0}}\sqrt{\frac{(d\mathbf{p}_2)}{(2\pi)^3}\,\frac{1}{2p_2^0}}\,\right|^2\,. \tag{3.4.50}$$

Here, neglecting forward scattering,

$$\langle p_1', p_2' + |p_1, p_2-\rangle = -i(2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)\,T\,, \tag{3.4.51}$$

and so we encounter

$$\delta^{(4)}(\Delta p)^2 = \delta^{(4)}(\Delta p)\delta^{(4)}(0)\,. \tag{3.4.52}$$

To interpret the $\delta^{(4)}(0)$ factor, we note that the partial scattering cross section $\Delta\sigma$ may be defined by

$$\Delta\sigma = \frac{\text{partial transition probability } = \Delta\mathcal{P}}{(\text{total time})(\text{incident flux})} = \frac{\text{rate}}{\text{flux}}\,, \tag{3.4.53}$$

with

$$(2\pi)\delta^{(1)}(0) = \int dt = (\text{total time})\,, \tag{3.4.54}$$

while

$$(2\pi)^3\delta^{(3)}(0)\,\frac{(d\mathbf{p})}{(2\pi)^3} = 1\,. \tag{3.4.55}$$

Thus Eq. (3.4.50) gives

$$\Delta\sigma = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3}\,\frac{1}{2p_1'^0}\,\frac{(d\mathbf{p}_2')}{(2\pi)^3}\,\frac{1}{2p_2'^0} (2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)\,|T|^2\,\frac{(d\mathbf{p})}{(2\pi)^3}\,\frac{1}{4p_1^0p_2^0(\text{flux})}\,. \tag{3.4.56}$$

Let us now consider $\mathbf{p}_2$ to represent the beam particle. Since

$$(2\pi)^3\delta^{(3)}(0) = \int(d\mathbf{r}) \tag{3.4.57}$$

is the volume of the box that contains the single particle states, the normalization is such that there is one particle in this volume, and in view of Eq. (3.4.55), the particle density is $(d\mathbf{p})/(2\pi)^3$. Hence the incident flux of the beam of momentum $\mathbf{p}_2$ is given by

$$\begin{aligned}\text{flux} &= (\text{relative velocity})\ (\text{particle density in beam}) \\ &= v_{12}\frac{(d\mathbf{p}_2)}{(2\pi)^3}\,.\end{aligned} \tag{3.4.58}$$

Therefore

$$\begin{aligned}\Delta\sigma = \int_\Delta \frac{(d\mathbf{p}_1')}{(2\pi)^3}\,\frac{1}{2p_1'^0}\,\frac{(d\mathbf{p}_2')}{(2\pi)^3}\,\frac{1}{2p_2'^0} \\ (2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)\,|T|^2\frac{1}{4p_1^0p_2^0v_{12}}\,,\end{aligned} \tag{3.4.59}$$

which is precisely the previous result that was carefully derived with the wave-packet technique.

3.5 Second-order scattering, renormalization

Including the second-order term in the interaction gives

$$\begin{aligned}\langle p_1', p_2' + |p_1, p_2-\rangle = \langle p_1', p_2'|\Big\{1 - \frac{i\lambda}{4!}\int(d^4x)\phi^4(x) \\ +\frac{1}{2}\left(-\frac{i\lambda}{4!}\right)^2\int(d^4x)(d^4y)T(\phi^4(x)\phi^4(y))\Big\}|p_1, p_2\rangle^{(0)}\,.\end{aligned} \tag{3.5.1}$$

Previously we saw that, for the first-order term, energy-momentum conservation was obtained by expressing

$$\phi^4(x) = e^{-iPx}\phi^4(0)e^{iPx}\,, \tag{3.5.2}$$

writing the operators P in terms of the eigenvalues of the states upon which they act, and then doing the x-integration. This procedure uses the time-space translational invariance rather directly to obtain the energy-momentum conservation. The same procedure may be employed for the second-order term. Namely, we may write

$$y = z + x \tag{3.5.3}$$

and integrate over z rather than y. Writing

$$T\left(\phi^4(x)\phi^4(z+x)\right) = e^{-iPx}T\left(\phi^4(0)\phi^4(z)\right)e^{iPx} \tag{3.5.4}$$

then gives us the same space-time integral that we had before, the integral which produces the δ-function of overall energy-momentum conservation.

Hence, to second order,

$$T = \langle p_1', p_2'|\left\{\frac{\lambda}{4!}\phi^4(0) - \frac{i}{2}\left(\frac{\lambda}{4!}\right)^2\int(d^4z)T\left(\phi^4(0)\phi^4(z)\right)\right\}|p_1, p_2\rangle^{(0)}\,. \tag{3.5.5}$$

Previously we used the contraction method to compute

$$\langle p_1', p_2'|\phi^4(0)|p_1, p_2\rangle^{(0)} = 4!\,. \tag{3.5.6}$$

This method is easily extended to compute the second-order term as well. One first creates, for example, the final state particle with momentum p_2' by the action of the field $\phi^{(+)}(p_2')$ and then removes this field by performing all possible contractions. These contractions produce vacuum values that are just various wave functions, with the result

$$\begin{aligned}\langle p_1', p_2'|T\phi^4(0)\phi^4(z)|p_1, p_2\rangle^{(0)} &= \langle p_1'|\phi^{(+)}(p_2')T\phi^4(0)\phi^4(z)|p_1, p_2\rangle^{(0)} \\ &= 4\langle p_2'|\phi(0)|0\rangle\langle p_1'|T\phi^3(0)\phi^4(z)|p_1, p_2\rangle^{(0)} \\ &\quad +4\langle p_2'|\phi(z)|0\rangle\langle p_1'|T\phi^4(0)\phi^3(z)|p_1, p_2\rangle^{(0)} \\ &= 4\langle p_1'|T\phi^3(0)\phi^4(z)|p_1, p_2\rangle^{(0)} \\ &\quad +4e^{-ip_2'z}\langle p_1'|T\phi^4(0)\phi^3(z)|p_1, p_2\rangle^{(0)}\,.\end{aligned} \tag{3.5.7}$$

The remaining particle in the final state may now be created by the action of $\phi^{(+)}(p_1')$ to the left on the vacuum, and the contraction process can be repeated. The particles in the initial state can then be dealt with in a similar way. One finds

$$\begin{aligned}&\langle p_1', p_2'|T\phi^4(0)\phi^4(z)|p_1, p_2\rangle^{(0)} \\ &\quad = 4\cdot 3\langle 0|T\phi^2(0)\phi^4(z)|p_1, p_2\rangle^{(0)} + 4\cdot 4e^{-ip_1'z}\langle 0|T\phi^3(0)\phi^3(z)|p_1, p_2\rangle^{(0)} \\ &\qquad +4\cdot 4e^{-ip_2'z}\langle 0|T\phi^3(0)\phi^3(z)|p_1, p_2\rangle^{(0)} \\ &\qquad\quad +4\cdot 3e^{-i(p_1'+p_2')z}\langle 0|T\phi^4(0)\phi^2(z)|p_1, p_2\rangle^{(0)} \\ &\quad = \quad \cdots\cdots\cdots \\ &\quad = (4\cdot 3)^2\langle 0|T\phi^2(0)\phi^2(z)|0\rangle\left\{e^{i(p_1+p_2)z} + e^{-ip_1'z}\left[e^{ip_1z} + e^{ip_2z}\right]\right. \\ &\qquad \left. +e^{-ip_2'z}\left[e^{ip_1z} + e^{ip_2z}\right] + e^{-i(p_1'+p_2')z}\right\}\,.\end{aligned} \tag{3.5.8}$$

It should be noted that again various contractions which do not contribute to the second-order scattering process have been omitted. These entail disconnected vacuum graphs and self-energy insertions whose effects will be discussed in the next chapter. We also omit contractions of this character in writing

$$\langle 0|T\phi^2(0)\phi^2(z)|0\rangle = -2\langle iT\phi(0)\phi(z)\rangle^2 = -2\Delta_+(z)^2\,. \tag{3.5.9}$$

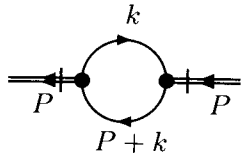


Fig. 3.9. The scattering amplitude in second order corresponds to a loop graph.

Now $\Delta_+(z)$ is unchanged by the reflection $z \to -z$ and the overall energy-momentum conservation gives $p_1' + p_2' = p_1 + p_2$. Thus, for example, the terms above involving $e^{i(p_1+p_2)z}$ and $e^{-i(p_1'+p_2')z}$ give identical contributions, and to second order we secure

$$T = \lambda + \frac{i}{2}\lambda^2 \int (d^4z)\, \Delta_+(z)^2 \left\{ e^{i(p_1+p_2)z} + e^{i(p_1-p_1')z} + e^{i(p_1-p_2')z} \right\}. \tag{3.5.10}$$

The integrals which appear here are Fourier transforms of a single relativistic scalar function $\Delta_+(x)^2$ with respect to three different momentum variables. Since the Fourier transform of a relativistic scalar is itself a scalar function, the integrals may be written as a sum of a single function but with three different arguments which can be expressed in terms of the relativistic scalars $s = -(p_1+p_2)^2$, $t = -(p_1-p_1')^2$, and $u = -(p_1-p_2')^2$. Thus

$$T = \lambda + \frac{1}{2}\lambda^2[F(s) + F(t) + F(u)]. \tag{3.5.11}$$

Introducing the Fourier representation

$$\Delta_+(z) = \int \frac{(d^4k)}{(2\pi)^4} \frac{e^{ikz}}{k^2 + m^2 - i\epsilon} \tag{3.5.12}$$

for each of the two $\Delta_+(z)$ functions that enter into

$$F(-P^2) = i \int (d^4z)\, e^{iPz} \Delta_+(z)^2, \tag{3.5.13}$$

and performing the z-integration to get $(2\pi)^4\delta^{(4)}(k + k' + P)$, gives

$$F(-P^2) = i \int \frac{(d^4k)}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(k+P)^2 + m^2 - i\epsilon}. \tag{3.5.14}$$

This integral can be represented by the Feynman graph shown in Fig. 3.9. The internal lines correspond to the propagator denominators in the integral. The external lines are slashed because they have no such denominator factor but simply are used to indicate the four-momentum that flows from the outside into the loop. The dots • denote vertices at which the lines meet and at which four-momentum is conserved as is shown by the labels on the graph. The scattering amplitude is represented by the sum of graphs shown:

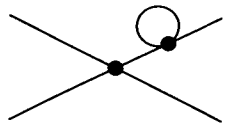
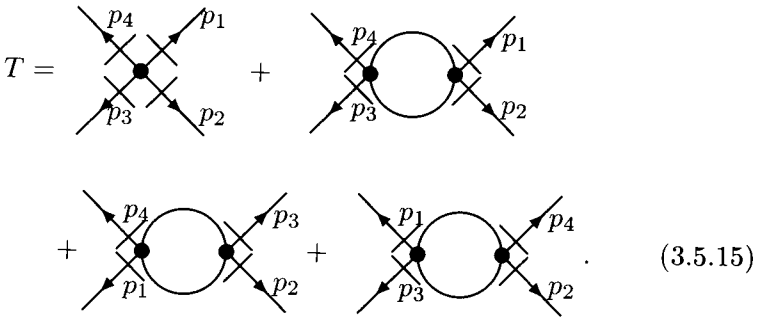


Fig. 3.10. A self-energy correction which may be neglected.



(3.5.15)

Incidentally Fig. 3.10 is one of the several graphs which were omitted in the computation of the scattering amplitude T. This graph represents a self-energy correction to the particle with momentum p_1. Such graphs are removed by the renormalization procedure that will be discussed at length in subsequent chapters. Please, reader, we cannot explain everything all at once, and so you must take it on faith for a while that the terms that have been omitted are irrelevant!

The integral defining $F(-P^2)$ does not, in fact, exist. It is logarithmically divergent in the ultraviolet since

$$F(-P^2) \sim \int^{\Lambda} k^3 dk \left(\frac{1}{k^2}\right)^2 \sim \ln \Lambda. \tag{3.5.16}$$

As we shall see, such a divergence, which is characteristic of relativistic quantum field theory, does not mean that the theory is wrong, but rather merely that it may not describe the world at infinite energies or, equivalently, at infinitesimally small distances. These divergencies are removed by the renormalization program which we are now starting to describe. Moreover, once the renormalizations have been performed, not only is the resulting theory finite, but it is also insensitive to high-energy processes. One should realize that this structure of the theory is an advantage: It has infinities arising from the high-energy, high-momentum parts — the short-time, short-distance parts of the theory. This reflects our basic ignorance of nature at infinite energies or infinitesimal distances. Indeed,

if the theory were finite it should be *the* theory describing everything which is inexcusable hubris on the part of the theorist in claiming to understand everything about nature. On the other hand, although our quantum field theories have infinities reflecting that we do not understand the asymptotic limit, they are absorbed into the renormalization of a few basic parameters — as we shall explain — with a unique and well-defined theory emerging which is valid at sub-asymptotic energies.* The resulting theory is completely insensitive to our lack of knowledge of the ultimate high-energy limit, and this is precisely as it should be.†

To perform the renormalization, the theory must first be made mathematically well-defined by some regularization process. One could, for example, cut off the momentum integrals at some large value. This procedure, however, would spoil Lorentz invariance, and this invariance would then be difficult to regain after the renormalization when the cutoff is to be removed. A regularization procedure should be employed that respects and maintains as many of the basic properties of the theory as is possible. Of course, it cannot respect every basic property of the theory, for then we would have a physically acceptable theory that is perfectly finite, a theory that we have just argued is an act of hubris. The regularization method that we shall employ throughout this text is dimensional regularization. It maintains all the basic properties of the theory at the expense of continuing it to a space-time world of n dimensions with n treated as a continuous (even complex) parameter. In the course of our work we shall see explicitly how this can be done.

For the case at hand we have simply a Fourier transform in n-dimensions rather than 4-dimensions, and so with dimensional regularization the function $F(-P^2)$ becomes

$$F(-P^2) = i \int \frac{(d^n k)}{(2\pi)^n} \frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(k+P)^2 + m^2 - i\epsilon} . \tag{3.5.17}$$

(It is easy to check that the denominators in the Fourier transform remain unchanged as the propagators are continued to n-dimensions.) The integral defines an analytic function of P^2. Hence it may be evaluated for space-like P^μ, $P^2 > 0$, and then analytically continued if the result is needed for time-like P^μ. Since the integral is a Lorentz-invariant function of P^2, P^μ may be taken to be a purely spatial vector with $P^0 = 0$. In this case, the denominators in the integral appear in the form (positive constant) $- k^{0\,2} - i\epsilon$. The k^0 integration may be continued

* Asymptotically free theories defined in Chapter 4, Section 4.3 remain valid even at arbitrarily high energies.

† As mentioned in a footnote at the beginning of Section 3, even non-renormalizable theories which do not have this complete insensitivity to corrections coming from high-energy processes may be useful.

into the complex plane so long as no poles are crossed. In particular, we may continue

$$k^0 \to k^0 = e^{i\theta} k_4 \tag{3.5.18}$$

with k_4 real, so long as $0 \le \theta < \pi$, since in this region the denominators never vanish in view of the $i\epsilon$ addition, $\epsilon > 0$. The choice

$$k^0 \to k^0 = +ik_4 \tag{3.5.19}$$

is preferred because this corresponds to transforming to Euclidean space-time.‡ Therefore

$$F(-P^2) = - \int \frac{(d_E^n k)}{(2\pi)^n} \frac{1}{k^2 + m^2} \frac{1}{(k+P)^2 + m^2} , \tag{3.5.20}$$

where now the n-vectors have Euclidean scalar products with, for example, $k^2 = k_4^2 + k_1^2 + \ldots$.

The method which will be employed to evaluate the integral illustrates a standard technique that is extraordinarily useful in field theory. One exponentiates the denominators using

$$\frac{1}{D} = \int_0^\infty ds\, e^{-sD} , \tag{3.5.21}$$

and interchanges integrals (which is permissible since we have continued to n-dimensions so that the integrals converge absolutely) to obtain

$$F(-P^2) = - \int_0^\infty ds \int_0^\infty dt e^{-(s+t)m^2} \int \frac{(d_E^n k)}{(2\pi)^n} \exp\left\{ - \left[(s+t)k^2 + 2kPt + P^2 t \right] \right\} . \tag{3.5.22}$$

One then completes the square by writing

$$(s+t)k^2 + 2kPt + P^2 t = (s+t)\left(k + \frac{Pt}{s+t} \right)^2 + \frac{st}{s+t} P^2 , \tag{3.5.23}$$

changes the integration variable to

$$\bar{k} = k + \frac{Pt}{s+t} , \tag{3.5.24}$$

and does the n-fold Gaussian $\bar{k}$ integrals to get

$$F(-P^2) = - \frac{1}{(4\pi)^{n/2}} \int_0^\infty ds \int_0^\infty dt \frac{1}{(s+t)^{n/2}} \exp\left\{ - \left[(s+t)m^2 + \left(\frac{st}{s+t} \right) P^2 \right] \right\} . \tag{3.5.25}$$

‡ In the dimensional continuation only the number of spatial variables is changed.

The final step is to change variables by inserting

$$1 = \int_0^\infty dS\, \delta(S - s - t) \tag{3.5.26}$$

into the integrand and by writing

$$s = \alpha S, \quad t = \beta S\,, \tag{3.5.27}$$

to secure

$$F(-P^2) = -\frac{1}{(4\pi)^{n/2}} \int_0^\infty dS \int_0^\infty S\, d\alpha \int_0^\infty S\, d\beta \frac{1}{S^{n/2}} \delta\left(S(1-\alpha-\beta)\right) \exp\left\{-S[m^2 + \alpha\beta P^2]\right\}\,. \tag{3.5.28}$$

Since $\delta(Sx) = \delta(x)/S$, the S-integration involves

$$\int_0^\infty dS S^{1-n/2} \exp\left\{-S[m^2 + \alpha\beta P^2]\right\} = \Gamma\left(2 - \frac{n}{2}\right)[m^2 + \alpha\beta P^2]^{\frac{n}{2}-2}\,; \tag{3.5.29}$$

whence

$$F(-P^2) = -\frac{\mu^{n-4}}{(4\pi)^2}\Gamma\left(2 - \frac{n}{2}\right) \int_0^1 d\alpha \left[\frac{m^2 + \alpha(1-\alpha)P^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\,, \tag{3.5.30}$$

where we have used the δ-function to eliminate the β-integration. But what is μ? It is a "unit of mass" that was introduced to explicitly display the dimensions of $F(-P^2)$ with the ratio in the square brackets now being dimensionless. The role of the mass unit μ is an important one which will soon be made clear.

In our world $n = 4$. In this limit

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{\Gamma\left(3 - \frac{n}{2}\right)}{2 - \frac{n}{2}} \to \frac{1}{2 - \frac{n}{2}} + \Gamma'(1)\,, \tag{3.5.31}$$

where $\Gamma'(1) = -\gamma$ is Euler's constant. A *simple pole* in n appears in the dimensional continuation which reflects the *logarithmic divergence* of the original 4-dimensional k-integration. The divergent term is exhibited by writing

$$F(-P^2) = \mu^{n-4}\left\{\frac{1}{(4\pi)^2}\,\frac{2}{n-4} + F_{\text{ren}}(-P^2)\right\}\,, \tag{3.5.32}$$

where a factor of μ^{n-4} has been displayed so as to obtain the dimensionless function

$$F_{\text{ren}}(-P^2) = -\frac{1}{(4\pi)^2}\int_0^1 d\alpha \left\{\Gamma\left(2 - \frac{n}{2}\right)\left[\frac{m^2 + \alpha(1-\alpha)P^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2} + \frac{2}{n-4}\right\} \tag{3.5.33}$$

which is well defined in the $n \to 4$ limit. Note that only a "minimal subtraction" has been made — only the pole but nothing else has been subtracted. One could have, for example, removed instead the divergent quantity

$$\frac{\mu^{n-4}}{(4\pi)^{n/2}}\Gamma\left(2 - \frac{n}{2}\right)\,.$$

Such a subtraction would produce a different definition of the renormalized coupling constant as one can see by tracing through the development that is about to be made. We make only a minimal subtraction because this simplifies the general structure of the theory. Note, however, that the dimensional-dependent residue of the pole μ^{n-4} must be kept so as to maintain the correct dimensions for arbitrary n.

To exhibit at last the renormalization procedure, we note that generalizing the previous dimensional analysis to n-dimensional space-time shows that now the dimension of the scalar field is given by

$$[\phi]_d = M^{\frac{n}{2}-1}\,. \tag{3.5.34}$$

Since, as we are about to see, the coupling constant λ must be a divergent constant, we rename it and call it the "bare coupling" λ_0,

$$\lambda \to \lambda_0\,. \tag{3.5.35}$$

It has the dimensions

$$[\lambda_0]_d = M^{4-n}\,. \tag{3.5.36}$$

Replacing λ by λ_0, the scattering amplitude appears as

$$T = \lambda_0 + \lambda_0^2 \frac{\mu^{n-4}}{(4\pi)^2}\,\frac{3}{n-4} + \frac{1}{2}\lambda_0^2\mu^{n-4}\left[F_{\text{ren}}(s) + F_{\text{ren}}(t) + F_{\text{ren}}(u)\right]\,. \tag{3.5.37}$$

This is a physical quantity which must be finite. This is achieved by expressing the divergent bare coupling λ_0 (which carries dimensions) in terms of a finite renormalized coupling λ (which is dimensionless) according to

$$\lambda_0 = \lambda\mu^{4-n}\left[1 - \frac{3\lambda}{(4\pi)^2}\,\frac{1}{n-4}\right]\,. \tag{3.5.38}$$

This works to our level of accuracy, λ^2, because to this order

$$T = \mu^{4-n}\left\{\lambda + \frac{1}{2}\lambda^2\left[F_{\rm ren}(s) + F_{\rm ren}(t) + F_{\rm ren}(u)\right]\right\}. \tag{3.5.39}$$

The unit of mass μ was introduced to explicitly display the dimensions and to make the renormalized coupling parameter λ dimensionless. As we shall see in more detail later, this defines a "floating" renormalization scheme since μ can be chosen arbitrarily. But physically measurable results cannot depend upon μ, and in particular, the scattering amplitude T must be independent of the size of μ. Although we have rearranged terms so as to express the scattering amplitude T as a power series in the renormalized coupling λ with each power of λ multiplying a finite coefficient, T is, of course, still a function of λ_0 with no μ dependence. Hence λ_0 must be independent of the arbitrary unit of mass,[§]

$$\mu\frac{d}{d\mu}\lambda_0 = 0. \tag{3.5.40}$$

Since λ_0 is independent of μ, λ must depend upon μ. Since λ is finite and dimensionless, the dimensionless logarithmic derivative $\mu d\lambda/d\mu$ can only be a function[¶] of λ and n:

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda; n). \tag{3.5.41}$$

This is the "renormalization group equation". Inserting the explicit form of the second order coupling renormalization (3.5.38) in Eq. (3.5.40), it is easy to see that, through terms of second order,

$$\beta(\lambda; n) = (n-4)\lambda + \frac{3\lambda^2}{(4\pi)^2}. \tag{3.5.42}$$

We should note that, in general, as we shall explain in detail later, the structure of the coupling constant renormalization in dimensional continuation appears as

$$\lambda_0 = \lambda\mu^{4-n}[1 + \text{poles}], \tag{3.5.43}$$

where the "poles" are ascending poles in $(n-4)$: $(n-4)^{-1}$, $(n-4)^{-2}$, ..., whose residues are functions of λ. The 1 in the 1 + poles (rather than some arbitrary function of λ and n) is the hallmark that we are using

[§] This can be stated in more general terms: The generating function of all measurable quantities — the functional integral – depends only upon the bare coupling λ_0 which enters in the Lagrangian. Since μ is only an auxiliary parameter used to define the renormalized coupling λ, λ_0 must be independent of μ.

[¶] In general, to arbitrary order in λ, the coupling constant renormalization and thus the β-function is independent of any mass parameters so that $\beta(\lambda; n)$ cannot depend upon m/μ.

"minimal subtraction". It follows from this structure that in general

$$\beta(\lambda; n) = (n-4)\lambda + \beta(\lambda), \tag{3.5.44}$$

in which $\beta(\lambda)$ is independent of the dimensionality n.

It is worthwhile pausing to illustrate the utility of this renormalization scheme, particularly with regard to high-energy behavior. To our order, the renormalization group equation may be written as

$$\left[\mu\frac{d}{d\mu} + (n-4)\right]\frac{1}{\lambda} = -\frac{3}{(4\pi)^2}. \tag{3.5.45}$$

Hence the renormalized couplings at two different scales μ_1 and μ_0 are related by

$$\frac{1}{\lambda(\mu_1)} = \frac{1}{\lambda(\mu_0)}\left(\frac{\mu_0}{\mu_1}\right)^{n-4} + \frac{3}{(4\pi)^2}\,\frac{1}{n-4}\left[\left(\frac{\mu_0}{\mu_1}\right)^{n-4} - 1\right]. \tag{3.5.46}$$

These two coupling parameters give the *same* physical amplitudes — the change in μ in $\lambda(\mu)$ is precisely compensated by the change in the μ-dependence of the coefficients in the power series expansion in $\lambda(\mu)$. In the physical limit $n \to 4$,

$$\frac{1}{\lambda(\mu_1)} = \frac{1}{\lambda(\mu_0)} - \frac{3}{(4\pi)^2}\ln\left(\frac{\mu_1}{\mu_0}\right). \tag{3.5.47}$$

This notion of a "running coupling constant" can be exploited to immediately derive the (center-of-mass) fixed-angle, high-energy limit of the physical ($n = 4$) scattering amplitude

$$T = \lambda + \frac{1}{2}\lambda^2\left[F_{\rm ren}(s) + F_{\rm ren}(t) + F_{\rm ren}(u)\right]. \tag{3.5.48}$$

In this limit $s \to \infty$ with t/s and u/s remaining finite. Hence if we choose $\mu_1 = \sqrt{s}$, $F_{\rm ren}(s)$, $F_{\rm ren}(t)$, and $F_{\rm ren}(u)$ will remain finite in this limit which is equivalent to the limit $m^2 \to 0$ as one sees from the previous integral representation for $F_{\rm ren}(-P^2)$. Hence, with $n = 4$, the infinite energy limit is given by

$$T = \lambda(s) + \frac{1}{2}\lambda(s)^2[\text{finite}], \tag{3.5.49}$$

with Eq. (3.5.47) giving

$$\lambda(s) = \frac{\lambda(\mu_0)}{1 - \lambda(\mu_0)\,\frac{3}{(4\pi)^2}\,\frac{1}{2}\ln\left(\frac{s}{\mu_0^2}\right)}. \tag{3.5.50}$$

That is, to within our second order of accuracy,

$$T = \lambda(\mu_0) + \frac{1}{2}\lambda(\mu_0)^2\left[\frac{3}{(4\pi)^2}\ln\left(\frac{s}{\mu_0^2}\right) + \text{finite}\right]. \tag{3.5.51}$$

Since the running coupling $\lambda(s)$ becomes large as $s \to \infty$, this asymptotic expansion breaks down for sufficiently large s no matter how small is the coupling $\lambda(\mu_0)$ at the scale μ_0. The high-energy limit necessarily moves the $\lambda\phi^4$ theory away from the lowest order perturbative regime. Nonetheless, it is to be hoped that the point has been made: the high-energy limit is controlled by the running coupling constant $\lambda(s)$. Moreover, the renormalization of the coupling which yields the β-function and thus the running coupling is relatively easy to calculate — it is easier to isolate and compute the "infinite parts" [the poles in $(n-4)$] than the finite parts. So this way of computing the high-energy limit is a fast way. It gives of course, as it must, the same result that one obtains by a more pedestrian examination of the large s behavior of the function $F(s)$ which is evaluated in Eq. (3.5.66) below.‖

The renormalized coupling, as we have defined it, is not directly the result of the measurement of the scattering amplitude at some kinematical point. Consider, for example, the scattering amplitude at threshold, $s = 4m^2$, $t = u = 0$:

$$T_{\rm th} = \lambda + \frac{1}{2}\lambda^2\left[F_{\rm ren}(4m^2) + 2F_{\rm ren}(0)\right] . \tag{3.5.52}$$

This amplitude could be measured and thus used to define a "threshold coupling" $\lambda_{\rm th}$ by

$$\lambda_{\rm th} = T_{\rm th} . \tag{3.5.53}$$

This defines λ as a function of $\lambda_{\rm th}$ but, it should be noted, a function that depends upon m^2/μ^2.

Finally, let us compute $F_{\rm ren}(-P^2)$. In the $n = 4$ limit, the previous integral representation (3.5.33) becomes

$$F_{\rm ren}(-P^2) = \frac{1}{(4\pi)^2}\left\{\text{~}+\int_0^1 d\alpha \ln\left[\frac{m^2+\alpha(1-\alpha)P^2}{4\pi\mu^2}\right]\right\} . \tag{3.5.54}$$

This is a function of $s = -P^2$ which is analytic in the cut s plane whose analytic structure is best exhibited by a dispersion relation.** To obtain

‖ It should be mentioned that renormalization group methods can be employed to go beyond the first few orders of perturbation theory. These methods may be used to "sum the leading logs". For example, the denominator structure in Eq. (5.50) is the sum of all the terms in the exact $\lambda(s)$ which are of the form $[\lambda(\mu_0)]^p[\ln(s/\mu_0)]^q$ with equal powers, $p = q$. The next order in the perturbative expansion of $\beta(\lambda)$ adds a term of order λ to the right-hand side of Eq. (5.46). This addition produces terms with $p = q+1$ in the expansion of $\lambda(s)$ which are smaller if the logarithm is larger than the coupling $\lambda(\mu_0)$.

** The function $F(-P^2)$ is an example of many of the functions encountered in field theory which are analytic except for a cut along the real axis. These are "real analytic" functions in the sense that the discontinuity across the cut is the imaginary part of the function, and the function is real on those portions of the real axis away from the cut. A dispersion relation expresses such an analytic function with a cut in terms of an integral over the discontinuity across the the cut. Dispersion relations were first used to describe the frequency-dependent dielectric "constant", which is the Fourier transform of a retarded response function and hence has this analytic character. The imaginary part of the dielectric "constant" describes the absorption of an electromagnetic wave traveling through the medium; the real part describes the dispersion - the frequency-wave number relation and thus the group velocity of the wave. The relation is called a "dispersion relation" because it expresses the real part of the function, the dispersion, in terms of an integral over the imaginary part of the function, the absorption.

the dispersion relation, we introduce $1 = d\alpha/d\alpha$ into the integrand and integrate by parts to get

$$\int_0^1 d\alpha \ln\left[\frac{m^2+\alpha(1-\alpha)P^2}{4\pi\mu^2}\right] = \ln\left(\frac{m^2}{4\pi\mu^2}\right) - P^2\int_0^1 d\alpha \frac{\alpha(1-2\alpha)}{m^2+\alpha(1-\alpha)P^2} . \tag{3.5.55}$$

Placing

$$1 = \int_0^\infty ds'\delta\left(s' - \frac{m^2}{\alpha(1-\alpha)}\right) \tag{3.5.56}$$

inside the α-integral gives the desired dispersion relation

$$-\int_0^1 d\alpha \frac{\alpha(1-2\alpha)}{m^2+\alpha(1-\alpha)P^2} = \int_0^\infty ds' \frac{\rho(s')}{s'+P^2} , \tag{3.5.57}$$

with

$$\begin{aligned}\rho(s') &= -\int_0^1 d\alpha \frac{1-2\alpha}{1-\alpha}\delta\left(s' - \frac{m^2}{\alpha(1-\alpha)}\right) \\ &= -\frac{1}{s'}\int_0^1 d\alpha\, \alpha(1-2\alpha)\delta\left(\alpha(1-\alpha) - \frac{m^2}{s'}\right) .\end{aligned} \tag{3.5.58}$$

Factoring the quadratic form in the δ-function yields

$$\begin{aligned}\delta\left(\alpha(1-\alpha) - \frac{m^2}{s'}\right) = \frac{1}{\sqrt{1-4m^2/s'}}&\left[\delta\left(\alpha - \frac{1}{2} - \frac{1}{2}\sqrt{1-4m^2/s'}\right)\right. \\ &\left.+\delta\left(\alpha - \frac{1}{2} + \frac{1}{2}\sqrt{1-4m^2/s'}\right)\right] ,\end{aligned} \tag{3.5.59}$$

and a little algebra now shows that

$$\rho(s') = \begin{cases} 0 & , \quad s' < 4m^2 , \\ \frac{1}{s'}\sqrt{1-\frac{4m^2}{s'}} & , \quad s' \geq 4m^2 . \end{cases} \tag{3.5.60}$$

Tracing through the $i\epsilon$, one finds that s' plays a role akin to m^2 and the prescription $m^2 \to m^2 - i\epsilon$ implies that the representation valid for all

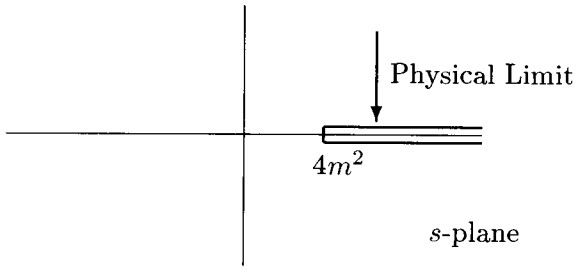


Fig. 3.11. Analytic structure of $F_{\rm ren}(s)$.

$-P^2 = s$ is given by

$$F_{\rm ren}(s) = \frac{1}{(4\pi)^2} \ln\left(\frac{m^2 e^\gamma}{4\pi\mu^2}\right) + \frac{s}{(4\pi)^2} \int_{4m^2}^{\infty} ds' \, \frac{\rho(s')}{s - s' + i\epsilon} \,. \tag{3.5.61}$$

This dispersion relation defines an analytic function in the entire cut s-plane as shown in Fig. 3.11.

As shown in Problem 17 at the end of this chapter, unitarity - the conservation of probability — relates the imaginary part of the forward scattering amplitude to the total cross section. Using the formula

$$\text{Im} \, \frac{1}{x - i\epsilon} = \pi\delta(x) \,, \tag{3.5.62}$$

above threshold, $s > 4m^2$, the second order scattering amplitude has an imaginary part involving

$$\begin{aligned} \text{Im} \, F_{\rm ren}(s) &= -\frac{s}{16\pi}\rho(s) \\ &= -\frac{1}{16\pi}\sqrt{1 - \frac{4m^2}{s}} \,. \end{aligned} \tag{3.5.63}$$

In view of the analytic structure of $F_{\rm ren}(s)$, it is uniquely determined by its discontinuity across the cut, $2i \, \text{Im} \, F_{\rm ren}(s)$, along with its asymptotic behavior and its value at one point. Now

$$\begin{aligned} \lim_{|s|\to\infty} s \int_{4m}^{\infty} ds' \frac{\rho(s')}{s - s'} &= \lim_{|s|\to\infty} \int_{\rm const.}^{\infty} ds' \left[\frac{1}{s'} - \frac{1}{s' - s}\right] \\ &= \ln s + \text{const.} \,, \end{aligned} \tag{3.5.64}$$

and

$$F_{\rm ren}(0) = \frac{1}{(4\pi)^2} \ln\left(\frac{m^2 e^\gamma}{4\pi\mu^2}\right) \,. \tag{3.5.65}$$

Therefore

$$F_{\rm ren}(s) = \frac{1}{(4\pi)^2} \left\{ \ln\left(\frac{m^2 e^\gamma}{4\pi\mu^2}\right) + \sqrt{1 - \frac{4m^2}{s}} \ln\left[\frac{\sqrt{s - 4m^2} + \sqrt{s}}{\sqrt{s - 4m^2} - \sqrt{s}}\right] - 2 \right\} \,, \tag{3.5.66}$$

since this function has the all the required properties.

As a simple application of this function, we note that it may be used to compute, to second order, the scattering amplitude at threshold. According to Eq. (3.5.53), this defines a physically measurable coupling constant $\lambda_{\rm th}$, and so, to this order,

$$\lambda_{\rm th} = \lambda(\mu) \left\{ 1 + \frac{1}{2} \frac{\lambda(\mu)}{(4\pi)^2} \left[3 \ln\left(\frac{m^2 e^\gamma}{4\pi\mu^2}\right) - 2 \right] \right\} \,. \tag{3.5.67}$$

It is easy to verify that $\lambda_{\rm th}$ is independent of μ (to second order). It should be noted that the relation between physically measurable constants and running couplings typically involve the logarithm $\ln(m^2/\mu^2)$ which appears here.

Problems

And now reader, bestir thyself – for though we will always lend thee proper assistance in difficult places, as we do not, like some others, expect thee to use the arts of divination to discover our meaning, yet we shall not indulge thy laziness where nothing but thy own attention is required; for thou art highly mistaken if thou dost imagine that we intended when we began this great work to leave thy sagacity nothing to do ... .

Henry Fielding

Review of Special Relativity

There was a young lady named Bright
Who could travel much faster than light
She set out one day
In a relative way
And returned on the previous night.

Unknown

An event in space-time is labeled by the four coordinates x^μ, with $x^0 = t$ the time and x^k, $k = 1, 2, 3$ the spatial coordinates. (The velocity of light is set to one, $c = 1$.) We use the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

and its inverse

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

so that (we always use the summation convention)

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu{}_\nu .$$

These tensors are used to raise and lower indices, e.g.

$$x_\mu = g_{\mu\nu} x^\nu = (-x^0, x^k) .$$

Superscripts are used for coordinates (and momenta) with the proper physical sign of the time component. (But note that $\partial_0 = \partial/\partial x^0$ is the time derivative with the right sign.) An inhomogeneous Lorentz transformation or Poincaré transformation is given by

$$x^\mu \to \bar{x}^\mu = \ell^\mu{}_\nu x^\nu + a^\mu ,$$

with the transformation matrix $\ell^\mu{}_\nu$ restricted to maintain

$$(\bar{x}^\mu - \bar{y}^\mu) g_{\mu\nu} (\bar{x}^\nu - \bar{y}^\nu) = (x^\mu - y^\mu) g_{\mu\nu} (x^\nu - y^\nu) .$$

Since the translation specified by a^μ is trivial, it suffices to consider only the (homogeneous) Lorentz transformation

$$x^\mu \to \bar{x}^\mu = \ell^\mu{}_\nu x^\nu .$$

We use the notation

$$x^2 = x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu .$$

1. a) Prove that if $\bar{x}^2 = x^2$ for all x^μ, then

$$g_{\mu\nu} \ell^\mu{}_\alpha \ell^\nu{}_\beta = g_{\alpha\beta} .$$

Defining matrices by

$$(\ell)_{\alpha\beta} = \ell^\alpha_\beta , \qquad (g)_{\alpha\beta} = g_{\alpha\beta} ,$$

this restriction may be expressed as

$$\ell^T g \ell = g ,$$

where the T superscript denotes matrix transposition.

b) Use this result to prove that

$$\det \ell = \pm 1 .$$

If $\det \ell = +1$, the transformation is said to be proper; if $\det \ell = -1$, the transformation is said to be improper. Note that

$$g_{00} = -1 = g_{\mu\nu} \ell^\mu{}_0 \ell^\nu{}_0 = (\ell^k{}_0)^2 - (\ell^0{}_0)^2 ;$$

whence

$$(\ell^0{}_0)^2 = 1 + (\ell^k{}_0)^2 \geq 1 .$$

Thus, the proper and improper sets are further split into two separate pieces, orthochronous (no time reversal): $\ell^0{}_0 \geq +1$, and anti-orthochronous (time reversal): $\ell^0{}_0 \leq -1$. Altogether, there are four disjoint sets of Lorentz transformations:

I			*II*		
$\det \ell$	$=$	$+1$	$\det \ell$	$=$	-1
$\ell^0{}_0$	$\geq$	$+1$	$\ell^0{}_0$	$\geq$	$+1$
III			*IV*		
$\det \ell$	$=$	-1	$\det \ell$	$=$	$+1$
$\ell^0{}_0$	$\leq$	-1	$\ell^0{}_0$	$\leq$	-1

Transformations in one component cannot be continuously deformed into transformations in other components. However, if one considers analytic continuations of Lorentz transformations in which the time is continued into the complex plane, then the set in category IV can be reached continuously from category I which includes the identity. This is the basis of the fundamental CPT theorem.

c) The tensor $\epsilon^{\alpha\beta\gamma\delta}$ is defined by the condition that it is completely antisymmetric in its indices with $\epsilon^{0123} = +1$. Prove that $\epsilon^{\alpha\beta\gamma\delta}$ is invariant under proper Lorentz transformations. That is, prove that

$$\epsilon^{\alpha\beta\gamma\delta} = \ell^\alpha{}_\mu \ell^\beta{}_\nu \ell^\gamma{}_\rho \ell^\delta{}_\sigma \epsilon^{\mu\nu\rho\sigma} .$$

What happens for improper transformations?

2. Show how the electric and magnetic fields **E** and **B** form components of an antisymmetrical field strength tensor $F^{\mu\nu} = -F^{\nu\mu}$, and that the charge ρ and current **j** form components of a current vector j^μ so as to put Maxwell's equations in the form

$$\partial_\nu F^{\mu\nu} = j^\mu , \qquad \partial_\nu {}^d F^{\mu\nu} = 0 .$$

Here

$$\partial_\nu = \frac{\partial}{\partial x^\nu} ,$$

while

$${}^d F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} ,$$

is the dual to the field strength tensor.*

* Note that if $\partial_\nu {}^d F^{\mu\nu} = k^\mu$, with $k^\mu \neq 0$, then magnetic charge would exist. The absence of magnetic charge implies a lack of symmetry of the Maxwell equations under a duality transformation.

3. a) Under a Lorentz transformation $x^\mu \to \bar{x}^\mu = \ell^\mu{}_\nu x^\nu$, a scalar field transforms as

$$\phi(x) \to \bar{\phi}(\bar{x}) = \phi(x) ;$$

a vector field transforms as

$$V^\mu(x) \to \bar{V}^\mu(\bar{x}) = \ell^\mu{}_\nu V^\nu(x) ;$$

a second rank tensor field transforms as

$$T^{\mu\nu}(x) \to \bar{T}^{\mu\nu}(\bar{x}) = \ell^\mu{}_\alpha \ell^\nu{}_\beta T^{\alpha\beta}(x) ;$$

and so forth.

Write a paragraph explaining in physical or kinematical terms what the previous paragraph means in terms of observers and so on.

b) Prove that the Maxwell equations as presented in Problem 2 are invariant in form under Lorentz transformations. Explain why $\phi = F^{\mu\nu} F_{\mu\nu}$ and $\psi = F_{\mu\nu} {}^d F^{\mu\nu}$ are scalars. Write ϕ and ψ in terms of **E** and **B**.

4. a) Prove that a spatial integral is rotationally invariant in the sense that if $f(\mathbf{r})$ is any function of **r** then

$$\int (d^3\mathbf{r}) f(\mathbf{r}) = \int (d^3\mathbf{r}) f(\bar{\mathbf{r}}) ,$$

where

$$\bar{r}^k = R^{kl} r^l ,$$

with R^{kl} a rotation matrix. Hint: Change variables to $\bar{\mathbf{r}}$ using the appropriate Jacobian factor and then remember that integration variables can be labeled by anything you please so long as they are labeled consistently with the same symbol throughout the integral.

b) The four-dimensional delta function can be defined by

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) .$$

Prove that it is Lorentz invariant, that is, that

$$\delta^{(4)}(\bar{p}) = \delta^{(4)}(p) ,$$

where $\bar{p}^\mu = \ell^\mu{}_\nu p^\nu$. This you can do by proving that

$$\int (d^4p) \delta^{(4)}(\bar{p}) f(p) = f(0)$$

for any function $f(p) = f(p^0, p^1, p^2, p^3)$.

c) The θ-function is defined by

$$\theta(x) = \begin{cases} 1, & x > 0 , \\ 0, & x < 0 . \end{cases}$$

A Lorentz transformation cannot exchange past and future events within the light cone. Hence if p^μ is the (time-like) four-momentum of a particle which transforms into $\bar{p}^\mu = \ell^\mu{}_\nu p^\nu$, then $\theta(p^0)$ is a Lorentz invariant,

$$\theta(\bar{p}^0) = \theta(p^0)\,.$$

Note that

$$\int (d^4p)\theta(p^0)\delta(p^2+m^2)f(p) = \int \frac{(d^3\mathbf{p})}{2E} f(p)\,,$$

where $E = \sqrt{\mathbf{p}^2+m^2}$, integrates $f(p)$ over the "mass shell" of p^μ, that is, integrates over only p^μ obeying $p^\mu p_\mu = -m^2$, with $p^0 > 0$. Prove that this integral is Lorentz invariant in the sense that

$$\int \frac{(d^3\mathbf{p})}{2E} f(p) = \int \frac{(d^3\mathbf{p})}{2E} f(\bar{p})\,,$$

where $\bar{p}^\mu = \ell^\mu{}_\nu p^\nu$.

5. Consider the integral

$$\Phi(P) = \int \frac{(d^3\mathbf{p}')}{2E'}\frac{(d^3\mathbf{p})}{2E}\,\delta^{(3)}(\mathbf{p}'+\mathbf{p}-\mathbf{P})\delta(E'+E-P^0)\,,$$

where $E' = \sqrt{\mathbf{p}'^2+m'^2}$, and $E = \sqrt{\mathbf{p}^2+m^2}$. This integral appears when one computes the rate at which one particle decays into two particles.

a) Prove that $\Phi(P)$ is Lorentz invariant, that $\Phi(\bar{P}) = \Phi(P)$, where $\bar{P}^\mu = \ell^\mu{}_\nu P^\nu$.

b) Compute the value of $\Phi(P)$ in terms of $P^\mu P_\mu$, m'^2, and m^2. Hint: Consider the frame where $\mathbf{P} = 0$.

6. a) The set of all Lorentz transformations form a *group*. Prove this. Clearly $\ell^\mu{}_\nu = \delta^\mu{}_\nu$ is the identity transformation. So all you need do is prove: (i) If ℓ_1 and ℓ_2 are two Lorentz transformations, then $\ell_3 = \ell_1\ell_2$ is a third Lorentz transformation; (ii) If ℓ is a Lorentz transformation, then its inverse exists. [Hint: The matrix form introduced above in Problem 1 should be useful.]

Infinitesimal Lorentz transformations are of the form

$$\ell^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,,$$

where the $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are infinitesimal numbers. Note that this antisymmetry implies that $g_{\mu\nu}\ell^\mu{}_\alpha\ell^\nu{}_\beta = g_{\alpha\beta}$ to first order in $\omega^{\mu\nu}$, and so this does give a Lorentz transformation. In matrix form

$$\ell_{\text{inf}} = 1 + \omega\,.$$

According to the group property, a sequence of such infinitesimal transformations can be compounded; for example one can form

$$\ell = (1+\omega)^N\,,$$

where N is a large number. In this way any finite transformation which is continuously connected to the identity can be produced. Thus, all the transformations which are in category I can be produced in this fashion. The transformations in the other categories can now be constructed by multiplying one of these transformations in category I by a spatial reflection or parity transformation,

$$\ell = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix},$$

or by a time-inversion transformation

$$\ell = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

In general, if U_1 and U_2 are two matrices or operations which represent some group, then

$$U_1^{-1}U_2U_1 = U_3$$

defines a third such representation. If U_2 is an infinitesimal element,

$$U_2 = 1 + i\epsilon_2 G_2\,,$$

where G_2 a "group generator". In this case

$$U_3 = 1 + i\epsilon_2 U_1^{-1}G_2U_1$$

is also an infinitesimal operation with the generator

$$G_3 = U_1^{-1}G_2U_1\,.$$

Suppose now that

$$U_1 = 1 + i\epsilon_1 G_1$$

is an infinitesimal operation. Then

$$G_3 = G_2 + i\epsilon_1[G_2\,,\,G_1]\,,$$

neglecting terms of higher order which we may do since ϵ_1 is small. This requires that the commutator $i[G_2\,,\,G_1]$ of two group generators lie in the space of generators. Thus, for a group with a finite number of linearly independent generators, $G_1, G_2, \ldots, G_N$, the general infinitesimal

operation is of the form

$$U = 1 + i\sum_{a=1}^{N} \epsilon_a G_a \,,$$

with the closure of the group operations expressed by

$$[G_a \,,\, G_b] = i\sum_c f_{abc} G_c \,.$$

The number f_{abc} are called the structure constants of the group. They, in fact, determine the entire part of the group which is continuously connected to the identity except for its global, topological aspects such as appear in the distinction between the rotation group $SO(3)$ and the unitary group $SU(2)$.

b) Determine the structure (constants) of the Poincaré group. To do this, you can define generators $\mathcal{P}_\mu$ and $\mathcal{J}_{\mu\nu}$ as differential operators by considering the transformation of a scalar field:

$$\begin{aligned}\bar{\phi}(x^\mu) &= \phi(x^\mu - \omega^\mu{}_\nu x^\nu - \epsilon^\mu) \\ &= \Big[1 - i\epsilon^\alpha \mathcal{P}_\alpha + \frac{i}{2}\omega^{\alpha\beta}\mathcal{J}_{\alpha\beta}\Big]\phi(x^\mu) \,.\end{aligned}$$

Compute all the commutators $[P_\mu \,,\, P_\nu]$, $[P_\alpha \,,\, J_{\mu\nu}]$, $[J_{\mu\nu} \,,\, J_{\rho\sigma}]$ using this differential operator realization of the generators $\mathcal{P}_\mu \,,\, \mathcal{J}_{\mu\nu}$.

It is important to note that the $\mathcal{P}_\mu$ and $\mathcal{J}_{\mu\nu}$ used in this problem form a simple differential realization of the abstract group generators — they are not operators that act on the vector space of physical states! However, the structure constants of a group are independent of any representation, and so they may be obtained by this simple differential realization.

7. a) Prove that

$$M^2 = -P^\mu P_\mu$$

is a Lorentz invariant – that it commutes with P_μ and $J_{\mu\nu}$.

b) The Pauli-Lubanski operator is defined by

$$W^\mu = {}^dJ^{\mu\nu}P_\nu = P_\nu {}^dJ^{\mu\nu} \,,$$

where

$${}^dJ^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\rho\sigma} \,.$$

Using the commutation relations of the Poincaré group generators, show that W^μ commutes with P^ν while the commutator of W^μ with $J^{\rho\sigma}$ is of the same form as the commutator of P^μ with $J^{\rho\sigma}$. [Hints: Consider the invariance of $\epsilon^{\alpha\beta\gamma\delta}$ under infinitesimal Lorentz transformations and the

fact that the commutators simply produce infinitesimal Poincaré transformations. Thus you should be able to work out the problem with very little actual calculation.] Show that

$$[W^\mu, W^\nu] = i\epsilon^{\mu\nu\rho\sigma}W_\rho P_\sigma \,.$$

Compute $W^\mu W_\mu$.

Since P^μ commutes with W^μ one can consider the action of W^μ in the state space with a fixed numerical value of P^μ. Note that in this subspace W^0 is essentially the helicity operator $\hat{\mathbf{p}}\cdot\mathbf{J}$, while if $\mathbf{p} = 0$, W^k is essentially the spin operator. (Check this!) In general, since $W^\mu W_\mu$ commutes with P_μ and $J_{\mu\nu}$, it is a Casimir operator (like $\mathbf{J}^2$ in angular momentum theory), and its eigenvalues can be used to characterize the representations of the Poincaré group.

8. Consider further the structure of the Lorentz group. To do this, define

$$J_k = \frac{1}{2}\epsilon_{klm}J_{lm} \,, \quad N_k = J_{k0} \,.$$

Compute the commutators $[J_k \,,\, J_l]$, $[J_k \,,\, N_l]$, and $[N_k \,,\, N_l]$. Now define

$$J_k{}^{(\pm)} = \frac{1}{2}(J_k \pm iN_k) \,,$$

with the factor of i showing that we are continuing to imaginary time. Compute $[J_k{}^{(+)} \,,\, J_l{}^{(+)}]$, $[J_k{}^{(+)} \,,\, J_l{}^{(-)}]$, and $[J_k{}^{(-)} \,,\, J_l{}^{(-)}]$. [By continuing to imaginary time we have altered the Lorentz group SO(3,1) to SO(4). You have now shown that SO(4) = SO(3) $\otimes$ SO(3) = SU(2) $\otimes$ SU(2).]

9. Define the Pauli spin matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,,$$

and thereby

$$X = x_\mu \sigma^\mu \,.$$

Compute $\det X$. Consider

$$X \to \bar{X} = \bar{x}_\mu \sigma^\mu = L^\dagger X L \,,$$

with $\det L = 1$. Explain how such 2×2 complex matrices L represent the Lorentz group. [Thus SO(3,1) = SL(2,C).]

10. a) Suppose that $j^\mu(x)$ is conserved

$$\partial_\mu j^\mu = 0 \,,$$

transforms as a four-vector, and vanishes at spatial infinity. A total charge is defined by

$$Q = \int (d^3\mathbf{r})\, j^0(\mathbf{r}, x^0)\,.$$

Prove that this charge is conserved,

$$\frac{d}{dx^0} Q = 0\,.$$

Hence

$$Q = \int (d^3\mathbf{r})\, j^0(\mathbf{r}, 0)\,.$$

In another inertial frame

$$\bar{Q} = \int (d^3\bar{\mathbf{r}})\, \bar{j}^{\,0}(\bar{\mathbf{r}}, 0)\,.$$

Since $\mathbf{r}$ is a dummy integration variable

$$\bar{Q} = \int (d^3\mathbf{r})\, \bar{j}^0(\mathbf{r}, 0)\,.$$

Suppose that the two frames differ by an infinitesimal Lorentz transformation. Prove that

$$\bar{Q} = Q\,.$$

Hence, by the group property, the charge is invariant under any finite Lorentz transformation which is continuously connected to the identity — any Lorentz transformation of category I.

b) Let $T^{\mu\nu}(x)$ be a symmetrical second rank tensor, $T^{\mu\nu} = T^{\nu\mu}$, which is conserved,

$$\partial_\mu T^{\mu\nu} = 0\,.$$

Define

$$P^\mu = \int (d^3\mathbf{r})\, T^{\mu 0}\,,$$

and

$$J^{\mu\nu} = \int (d^3\mathbf{r})\, [x^\mu T^{\nu 0} - x^\nu T^{\mu 0}]\,.$$

Prove that P^μ and $J^{\mu\nu}$ are conserved (constant in time). Prove that P^μ and $J^{\mu\nu}$ transform as a vector and tensor under Lorentz transformation of category I.

Note that this problem involves structures of the general form

$$F = \int (d\mathbf{r}) f^0(x)\,,$$

where $f^0(x)$ is the time component of a conserved quantity,

$$\partial_\mu f^\mu(x) = 0\,.$$

The time independence and Lorentz transformation properties of F can be made clearer if we note that the spatial volume at some fixed time may be thought of as a 3-dimensional hypersurface embedded in the 4-dimensional space-time world. It is a "time-slice" of the 4-space. Just as an element of the 2-dimensional surface embedded in a 3-dimensional space is described by a 3-vector dS_k, a 3-dimensional (hyper-)surface embedded in a 4-dimensional space is described by a 4-vector $d\sigma_\mu$. The slice at a constant time has only the time component $d\sigma_0 = (d\mathbf{r})$, and so we can write

$$F = \int d\sigma_\mu f^\mu(x)\,.$$

Evaluating F on a different time slice gives a change ΔF which may be expressed as a closed surface integral if we assume that $f^\mu(x)$ vanishes at spatial infinity,

$$\Delta F = \oint d\sigma_\mu f^\mu(x)\,.$$

Gauss's theorem now gives

$$\Delta F = \int (d^4x)\, \partial_\mu f^\mu(x) = 0\,.$$

The Lorentz transformation properties of F can be deduced in a similar way by noting that the evaluation of F in a new frame corresponds to an "inclined" time slice in the old frame plus an explicit Lorentz transformation on any other tensor indices in $f^\mu(x)$ that have so far been left implicit.

Having learned special relativity, let us turn to other problems.

11. a) Construct perfectly spatially localized states of a spinless particle $|\mathbf{r}, t\rangle$ in terms of some combination of scalar field operators acting upon the vacuum state. The statement "perfect spatial localization" means that

$$\langle \mathbf{r}', t|\mathbf{r}, t\rangle = \delta(\mathbf{r} - \mathbf{r}')\,.$$

b) Do such states have much physical significance? In particular compute $\langle \mathbf{r}', t'|\mathbf{r}, t\rangle$ and discuss its behavior when the space-time points $(\mathbf{r}', t')$ and $(\mathbf{r}, t)$ are in space-like relation, $(\mathbf{r}' - \mathbf{r})^2 - (t' - t)^2 > 0$.

12. The Green's function for a free scalar field is given by

$$\Delta_+(x-x') = \int \frac{(d^4p)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2+m^2-i\epsilon}.$$

Consider only space-like separations, $(x-x')^2 > 0$.

a) The Green's function is a relativistically invariant function of $(x-x')^2$. Hence you can imagine going to a frame in which $t-t'=0$. Use this possibility to rotate the p^0 integration

$$p^0 \to e^{i\theta} p_4$$

with p_4 kept real. Through what angles θ can you perform this rotation, keeping the value of the integration fixed? Rotate until p^0 becomes purely imaginary and

$$p^2 = \mathbf{p}^2 + p_4^2 > 0.$$

b) Now express

$$\frac{1}{p^2+m^2} = \int_0^\infty ds\, e^{-s(p^2+m^2)},$$

interchange integrals and carry out the p-integrations.

c) Use the result of part (b) to compute the zero mass ($m=0$) limit of the Green's function.

d) Look up in a book on Bessel functions to find the result of part b) expressed as a Bessel function.

13. The Stress-Energy Tensor

Consider a non-interacting, scalar field theory. Construct the stress-energy tensor $T^{\mu\nu}$ as follows:

a) Suppose that

$$T_{\mu\nu} = a\partial_\mu\phi\partial_\nu\phi + bg_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi + c\partial_\mu\partial_\nu\phi^2 + dg_{\mu\nu}\partial^2\phi^2 + eg_{\mu\nu}\phi^2.$$

Is this the most general form which is bilinear in the field operators? Prove your statement. Find constraints on the coefficients a, b, c, d, e by requiring that the stress tensor is conserved,

$$\partial_\mu T^{\mu\nu} = 0,$$

using the Klein-Gordon equation for the field ϕ.

b) Suppose further that $T_{\mu\nu}$ is "new and improved" so that its trace vanishes when the mass m of the field ϕ vanishes. Obtain further constraints on a, b, c, d, e.

c) The total four-momentum

$$P^\mu = \int (d^3\mathbf{r}) T^{0\mu}$$

gives

$$\partial_\mu\phi = i[\phi, P_\mu].$$

Write $\partial_0\phi$ in terms of Π and use this to completely determine $T^{\mu\nu}$.

14. General Relativity and the Stress-Energy Tensor

The proper and physically rigorous definition of the stress-energy tensor $T^{\mu\nu}(x)$ comes from the theory of general relativity, for $T^{\mu\nu}(x)$ is the source of the gravitational field. Let us see how this goes in the context of the simple scalar field theory. Under a general coordinate transformation

$$x^\mu \to \bar{x}^\mu = \bar{x}^\mu(x),$$

where $\bar{x}^\mu(x)$ are four arbitrary functions of x^λ, the scalar field transforms as

$$\phi(x) \to \bar{\phi}(\bar{x}) = \phi(x).$$

Under this general coordinate transformation, the metric tensor $g_{\mu\nu}(x)$ [which reduces to the familiar constant $g_{\mu\nu}$ in flat space time] transforms as a covariant, second rank tensor,

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\lambda}{\partial \bar{x}^\mu} \frac{\partial x^\kappa}{\partial \bar{x}^\nu} g_{\lambda\kappa}(x).$$

The tensor $g^{\mu\nu}(x)$ is defined to be the inverse of $g_{\mu\nu}(x)$,

$$g^{\mu\nu}(x) g_{\nu\sigma}(x) = \delta^\mu{}_\sigma.$$

A general relativistically valid form is obtained if it is invariant under general coordinate transformations.

a) Prove that the action

$$W = \int (d^4x) \sqrt{-g(x)}\, \mathcal{L},$$

where

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu}(x) \partial_\mu\phi(x) \partial_\nu\phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \Lambda,$$

and

$$g(x) = \det g_{\mu\nu}(x),$$

is invariant under general coordinate transformations.

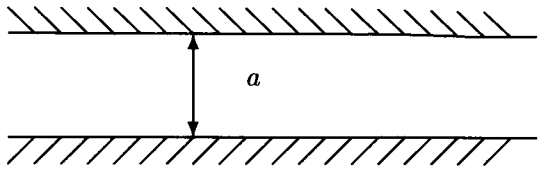


Fig. 3.12. Infinite, parallel plane mirrors a distance a apart.

b) The stress tensor in curved space-time is defined by the response of the action to a metric variation as follows:

$$\delta W = \frac{1}{2}\int (d^4x)\sqrt{-g(x)}\, T^{\mu\nu}(x)\delta g_{\mu\nu}(x)\,.$$

Use this definition to compute $T^{\mu\nu}(x)$. In particular, note that $T^{\mu\nu}$ contains a term of the form $\Lambda g^{\mu\nu}$. Since the stress tensor is the source in the Einstein field equation, this term is identified as the "cosmological constant" term. Show that the flat space limit gives the usual result without the "new, improved" addition.† [Remember that if X is a matrix, $\delta(XX^{-1}) = \delta(1) = 0$ so that $\delta X^{-1} = -X^{-1}\delta X X^{-1}$.]

15. Vacuum Stress Between Conducting Plates

Consider two "perfectly conducting", infinite plane "mirrors" which are parallel and a distance a apart as shown in Fig. 3.12. Suppose that a *massless* scalar field ϕ lives in the region between the two mirrors and suppose that ϕ vanishes on the surfaces of the mirrors. Set up a coordinate system such that its origin lies on the plane which is half-way between the two mirrors with the z axis perpendicular to the two mirrors. The purpose of this problem is to understand the stress-energy tensor $T_{\mu\nu}(t,x,y,z)$ in the region between the mirrors.

a) By virtue of the symmetry of this time-independent problem, the stress tensor must be composed of the tensor $g_{\mu\nu}$ and tensors composed from the unit vectors $\hat{n}_\mu$ and $\hat{z}_\nu$ defined by

$$\hat{n}^\mu = (1,0,0,0), \quad \hat{z}^\mu = (0,0,0,1)\,,$$

and it must be composed of these tensors multiplying functions which depend only upon the coordinate z. Write down the general form of $\langle T_{\mu\nu}\rangle$ based upon these considerations. Here $\langle\cdots\rangle$ denotes the expectation value in the vacuum space between the two mirrors.

† A parenthetical remark for the jaded sophisticate: The "new, improved" tensor $T^{\mu\nu}(x)$ is obtained by adding $\frac{1}{6}\phi^2 R$ to the Lagrange function, where R is the scalar curvature. This makes the theory invariant, except for the mass term, under a conformal transformation of the metric, $g_{\mu\nu}(x) \to e^{2\sigma(x)}g_{\mu\nu}(x)$, which thus gives $g_{\mu\nu}(x)T^{\mu\nu}(x) = 0$ if $m^2 = 0$.

b) Use the "new, improved" tensor $T_{\mu\nu}$ which is traceless and obtain the constraints on the general form found in part a) that follow from this vanishing trace.

c) The system (the geometry plus boundary conditions) is invariant under Lorentz boosts along the surfaces of the mirrors. Hence for these transformations we must have

$$\langle \bar{T}^{\mu\nu}\rangle = \langle T^{\mu\nu}\rangle\,.$$

Use this invariance and the previous results to show that the tensor structure of $\langle T^{\mu\nu}\rangle$ is uniquely determined. That is, show that $\langle T^{\mu\nu}\rangle$ is composed of a certain tensor involving $g_{\mu\nu}$, $\hat{n}_\mu$, $\hat{z}_\mu$ multiplying a single function $f(z)$.

d) Show that the conservation law $\partial_\mu T^{\mu\nu} = 0$ requires that $f(z)$ be a constant. Use dimensional analysis to determine this constant in terms of the single parameter of the problem that bears a dimension. The stress tensor is now completely determined except for an overall pure number which let us call γ.

e) The numerical constant γ later will be determined later. Assuming γ to be given, compute the energy density ϵ between the mirrors and the pressure p on one of the mirrors. Show that these results agree with the principle of virtual work, that when the mirror separation a is increased to $a + \delta a$ the change of the energy is given by the work done.

f) Compute the Green's function

$$G(x,\, x') = \langle iT\,(\phi(x)\phi(x'))\rangle\,.$$

Do this by first showing that

$$-\partial^2 G(x,\, x') = \delta(x - x')\,.$$

Then show that this Green's function equation together with the boundary condition $\phi = 0$ on a mirror surface are satisfied by an infinite sum of ordinary Green's functions Δ_+ where the coordinates of Δ_+ are displaced so as to represent an infinite set of image sources. That is, $G(x,\, x')$ can be constructed in terms of an infinite set of images just as the electrostatic problem of two parallel conducting plates can be solved by such an image construction. Give a simple physical reason for this.

g) Use this construction for $G(x,\, x')$ together with the explicit form for Δ_+ for massless particles found in Problem 12 to explicitly construct $\langle T^{\mu\nu}\rangle$. In doing this you must discard the member of the sum that gives the usual, infinite vacuum contribution. It is ill-defined as it stands, but a proper regularization must give zero for this term since the ordinary vacuum energy is zero. Sum the series to compute γ. (If you would

multiply your result by 2 you would get the effect produced by massless photons which have two polarization states. This effect has been measured experimentally.)

16. Consider the coupled field theory of two scalar fields ϕ and χ described by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g\phi\chi^2 .$$

If $M > 2m$ the ϕ-particle can decay into two χ-particles. Calculate the decay rate to first order in the coupling constant g. Do this by following closely the methods used in the text to calculate the lowest order scattering cross section in the $\lambda\phi^4$ theory. Explain what you are doing very carefully with many words — write an essay for this problem.

17. The Unitarity of the S-Matrix

a) The transition amplitude that describes the scattering of two spin zero, identical particles has the general structure

$$\langle p'q' + |pq-\rangle = \langle p'|p\rangle\langle q'|q\rangle + \langle p'|q\rangle\langle q'|p\rangle - i(2\pi)^4\delta^{(4)}(p'+q'-p-q)\,T .$$

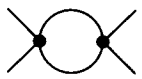
If the total energy of the system is below the threshold for inelastic processes, the two particle intermediate states effectively form a complete resolution of the identity, and

$$\frac{1}{2}\int \frac{(d^3\bar{\mathbf{p}})}{(2\pi)^3}\,\frac{1}{2\bar{p}^0}\,\frac{(d^3\bar{\mathbf{q}})}{(2\pi)^3}\,\frac{1}{2\bar{q}^0}\,\langle p'q'+|\bar{p}\bar{q}-\rangle\langle\bar{p}\bar{q}-|pq+\rangle$$
$$= \langle p'|p\rangle\langle q'|q\rangle + \langle p'|q\rangle\langle q'|p\rangle .$$

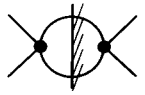
Insert the general structure above into this formula, compute the resulting phase space integral, and thereby determine the elastic unitarity relation for the scattering amplitude T. [Roughly, $S = 1 - iT$ and $SS^\dagger = 1$.]

b) Using the results in Section 5, check this unitarity constraint on T for the $\lambda\phi^4$ theory including all the terms up to and including order λ^2.

Note that the unitarity constraint relates the imaginary part of T in second order to the square of T in first order. In terms of Feynman graphs,

Im  = $\int$ $|^2$.

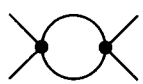
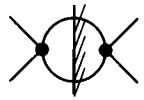
This can be written as

$\int$ $|^2$ =  .

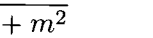
where

$= (2\pi)\theta(p^0)\delta(p^2+m^2)$.

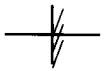
Thus, the imaginary parts of Feynman graphs can be interpreted in terms of "cut" graphs; in our example

Im  =  ,

in which

 $= \dfrac{-i}{p^2+m^2}$

is replaced by

 $= (2\pi)\theta(p^0)\delta(p^2+m^2)$.

18. Consider the scalar field theory described by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 .$$

a) Forget about renormalization effects and take the non-relativistic limit of the functional integral as in Section 1 of the text by using

$$\phi(x) = \frac{1}{\sqrt{2m}}[\psi(\mathbf{r},\,t)e^{-imt} + \psi^\dagger(\mathbf{r},\,t)e^{+imt}] .$$

Explain the structure of your result and in particular identify the two-particle non-relativistic potential $V(\mathbf{r}-\mathbf{r}')$ which follows from this reduction.

b) Look up the formula for the scattering amplitude in first Born approximation $f^{(1)}$ in some book on quantum mechanics. Compute $f^{(1)}$ for the $V(\mathbf{r}-\mathbf{r}')$ found above and obtain the corresponding differential cross section $d\sigma/d\Omega$ for the scattering of the two identical particles in their center-of-mass frame. Compare with the result for $d\sigma/d\Omega$ given in Eq. (3.4.44).

19. Consider the field theory described by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}M^2\chi^2 + \frac{1}{2}g\phi\chi^2 .$$

Note that this theory is the scalar analog of quantum electrodynamics with ϕ playing the role of a scalar (massive) "photon" and χ playing the

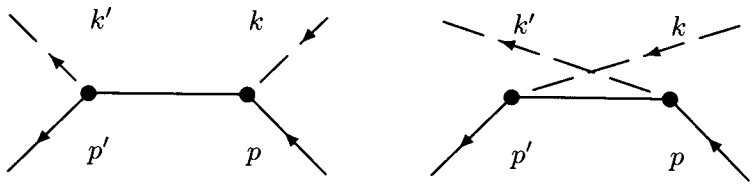


Fig. 3.13. Compton-like graphs.

role of a scalar "electron". Remember that the scattering amplitude T is defined by

$$\langle \cdots + | \cdots - \rangle = \langle \quad | \quad \rangle \langle \quad | \quad \rangle - i(2\pi)^4 \delta(\cdots) T \,.$$

a) Expand the functional integral to second order in g (using the methods explained in the text) to compute the lowest order scattering amplitude T for $\chi - \phi$ scattering. This is the scalar analog for Compton scattering. Explain how this amplitude corresponds to the two Feynman graphs in Fig. 3.13. Defining

$$\begin{aligned} s &= -(p+k)^2 = -(p'+k')^2 \,, \\ t &= -(p'-p)^2 = -(k'-k)^2 \,, \\ u &= -(k'-p)^2 = -(p'-k)^2 \,, \end{aligned}$$

prove that

$$s + t + u = 2m^2 + 2M^2 \,.$$

Express your result for T in terms of s, t, and u.

b) Compute, in a similar way, the lowest order scattering amplitude T for $\chi - \chi$ scattering. Write down the Feynman graphs for this process and explain how they arise. Express your result for T in terms of appropriately defined s, t, u. Using the text, write down the formula for the center-of-mass frame differential cross section $d\sigma/d\Omega$ in terms of $|T|^2$.

c) Write down the functional integral for the theory. Perform the $[d\phi]$ integration in closed form. Take the non-relativistic limit of the resulting $[d\chi]$ integration as was done in Problem 18. You will encounter

$$\int dt (d^3\mathbf{r}) dt' (d^3\mathbf{r}') \psi^* \psi(\mathbf{r}, t) \Delta_+(\mathbf{r} - \mathbf{r}', t - t') \psi^* \psi(\mathbf{r}', t') \,,$$

where Δ_+ is the propagator of the free ϕ field. This propagator involves $e^{-iE(k)(t-t')}$ where $E(k) \geq m$ and thus it has a phase that varies rapidly in comparison with the variation of $\psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t)$ and $\psi^*(\mathbf{r}', t)\psi(\mathbf{r}', t')$

which correspond to non-relativistic motion. Hence you can change integration variables to

$$\tau = t' - t \,, \qquad T = \frac{1}{2}(t' + t)$$

and neglect the τ variation of the $\psi^*\psi$ and thereby carry out the integration. Do this and thereby derive a theory with a static potential $V(\mathbf{r} - \mathbf{r}')$. Compute $V(\mathbf{r} - \mathbf{r}')$. What is the name of $V(\mathbf{r} - \mathbf{r}')$? What is the limit of $V(\mathbf{r} - \mathbf{r}')$ when $m \to 0$?

d) Compute the scattering amplitude in first Born approximation for the non-relativistic limit found in part c) above. Use the $V(\mathbf{r} - \mathbf{r}')$ found there and the usual formula of quantum mechanics which you used in Problem 18. Compare this result with the non-relativistic limit of the $d\sigma/d\Omega$ found in part b) above.

4
Elementary processes

The preceding chapter showed that the coupling constant must be renormalized. In the present chapter we shall describe, to lowest order in the coupling, the other basic or elementary processes that give divergent results. These divergencies are compensated by renormalizing other parameters of the theory, the mass and field strength. To render the theory completely well defined, another renormalization is needed — the addition of an (infinite) constant Λ_0 to the Lagrange function. This constant corresponds to an infinite renormalization of the "zero-point energy" — in rough terms, to zeroth order in the coupling, the field theory corresponds to an infinite collection of harmonic oscillators with the a-th oscillator having a zero-point or ground state energy given by $\frac{1}{2}\omega_a$, and the sum $\Sigma_a \frac{1}{2}\omega_a$ diverges. Higher order terms in the coupling give additional zero-point divergencies. This divergence was previously hidden in a re-definition of the measure $[d\phi]$. However, the divergence depends upon the scalar field's mass, and it is best to define the functional integral, and thereby the theory, in a mass-independent fashion. This requires that the zero-point divergence be displayed explicitly. Moreover, when gravitational couplings are introduced into theory, this additional constant in the Lagrange function appears as a contribution to the "cosmological constant"* Λ_0. The exceedingly small value of this constant on the scale set by elementary particles is an outstanding problem in contemporary physics, and this puzzle further motivates our discussion of Λ_0.

With these considerations in mind, we now write the Lagrange function as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\,\lambda_0\,\phi^4 + \Lambda_0 + \rho_0\phi\,. \tag{4.0.1}$$

The parameters m_0^2, λ_0, Λ_0 and $\rho_0(x)$ are loosely referred to as "infi-

* This was described in Problem 14 of Chapter 3.

nite", bare parameters. They are in fact made perfectly well defined by regulating the theory — which we do by continuing the theory to a space-time dimensionality $n \neq 4$. Subtractions are then performed — the renormalization process — to relate the bare parameters to renormalized parameters which remain finite when the regularization is removed. In our case, pole terms in $n-4$ are removed from the bare quantities such that the result is finite in the $n \to 4$ limit. It should be noted that the field ϕ which appears in the Lagrange function (0.1) is the "dummy" integration variable of the functional integral. This field is not renormalized. Indeed, its scale is set by the factor $\frac{1}{2}$ in front of the "kinetic energy" term $(\partial_\mu\phi)^2$. It is the source function $\rho_0(x)$ that generates the insertions of the quantum field in matrix elements. We shall denote this quantum field operator by $\phi_0(x)$. The renormalization of the quantum field amplitude, the "wave-function" renormalization, appears as a scale change of the source $\rho_0(x)$. This source acts as just another bare parameter of the theory, and it is renormalized just as are the others.

4.1 Euclidean space-time

Previously it was noted that the scattering amplitude is an analytic function of its momentum variables so that it could be evaluated for space-like momenta and then analytically continued back for the general case. This remark was exploited by making a continuation to Euclidean space as in $k_0 \to i\,k_4$. It is an exploitation because in this way positive denominators are obtained: One need not worry about the proper integration contours to route about poles and, more importantly, one can exponentiate these denominators [as in Eq. (3.5.21)] with well-defined integrals since the denominators are now always positive. We pause here to further explain the continuation to Euclidean space because this will simplify the calculations of the various elementary processes performed in the following sections.

The free propagator in n-dimensional Minkowski space time is given by

$$\Delta_+(x) = \int \frac{dk_0}{2\pi} \int \frac{(d^{n-1}\mathbf{k})}{(2\pi)^{n-1}}\, \frac{e^{i(\mathbf{k}\cdot\mathbf{r}+k_0x^0)}}{\mathbf{k}^2 - k_0^2 + m_0^2 - i\epsilon}\,. \tag{4.1.1}$$

We may write

$$k_0 = k_4\, e^{i\,\theta}\,, \tag{4.1.2}$$

and starting from $\theta = 0$, continue to small positive values of θ. This retains a negative imaginary part in the denominator and so the denominator never vanishes for any k-values throughout this continuation — the potential pole is never crossed. Hence this gives an analytic continuation that does not alter the value of the integral if $x^0 = 0$ or, more generally, by

the Lorentz invariance of the integral, for space-like x^μ, $x^2 > 0$. However, with $x^0 \neq 0$ and $\theta > 0$, the $ik_0 x^0$ term in the exponential in Eq. (4.1.1) acquires a non-vanishing real part that gives rise to a divergence when $|k_4| \to \infty$. To avoid this divergence we must simultaneously continue to complex times,

$$x^0 = x_4 e^{-i\theta} . \tag{4.1.3}$$

Once we have $\theta > 0$, the $i\epsilon$ term in the denominator in Eq. (4.1.1) is not needed to keep this denominator from vanishing, and we can set $\epsilon = 0$. It is easy to see that the denominator never vanishes if θ is restricted to the range

$$0 < \theta < \pi . \tag{4.1.4}$$

The original propagator in Minkowski space-time is recovered in the limit $\theta \to 0$. The value $\theta = \pi/2$ is preferred because this transforms the theory to Euclidean space-time:

$$k^0 = -k_0 \to -ik_4, \quad x^0 \to -i x_4 . \tag{4.1.5}$$

In this Euclidean space $k^2 = k_4^2 + \mathbf{k}^2 \geq 0$, $x^2 = x_4^2 + \mathbf{r}^2 \geq 0$, and $kx = k_4 x_4 + \mathbf{k} \cdot \mathbf{r}$. And the propagator (4.1.1) is continued into the Euclidean Green's function

$$\Delta_E(x) = \int \frac{(d_E^n k)}{(2\pi)^n} \frac{e^{ikx}}{k^2 + m_0^2} , \tag{4.1.6}$$

with

$$\Delta_+(x) \to i\Delta_E(x) , \tag{4.1.7}$$

where the arrow, which denotes the continuation, is actually an equality if x^μ is space-like.

The continuation to Euclidean space is clear if one considers the functional integral representation of the generating functional,

$$Z[\rho] = \int [d\phi] \exp\left\{ i \int (d^n x) \mathcal{L} \right\} . \tag{4.1.8}$$

The continuation $x^0 \to -i x_4$ produces

$$Z[\rho] = \int [d\phi] \exp\left\{ - \int (d_E^n x) \mathcal{L}_E \right\} , \tag{4.1.9}$$

where $-\mathcal{L}_E$ is given by Eq. (4.0.1) evaluated in Euclidean space. Except for the source term, this defines a positive-definite function of the field ϕ, and we have essentially $\mathcal{L}_E > 0$. The crucial point is that when $(\partial_\mu \phi)^2$ or ϕ^2 becomes large, $\mathcal{L}_E$ becomes large and the functional integral is (formally) convergent. One sees, at least in a formal way, that the continuation $x^0 \to -i x_4$ changes the oscillating exponential into a damped

exponential. Note that the interaction term dominates for large ϕ with this damping requiring that the coupling constant be positive, $\lambda_0 \geq 0$. At this point it is worth recalling that the mass term in the Minkowski version (4.1.8) of the functional integral appears in the form

$$\exp\left\{ -\frac{i}{2} m_0^2 \int (d^n x) \phi^2 \right\} ,$$

which motivated the replacement $m_0^2 \to m_0^2 - i\epsilon$ to get a damped exponential. This $-i\epsilon$ prescription is equivalent to the continuation $x^0 \to -i x^4$.

Finally, we should note that the continuation to Euclidean space gets rid of bothersome factors of i. In particular, the Euclidean two-point correlation function is defined as

$$\frac{\delta^2}{\delta\rho_0(x)\delta\rho_0(x')} Z[\rho]\Big|_{\rho=0} = \langle \phi_0(x)\phi_0(x') \rangle . \tag{4.1.10}$$

In the free-field limit

$$\langle \phi_0(x)\phi_0(x') \rangle^{(0)} = \Delta_E(x - x') . \tag{4.1.11}$$

Moreover, just as in Minkowski space, in the free-field limit one has contraction rules as illustrated in

$$\begin{aligned} \langle \phi_0(1)\,\phi_0(2)\phi_0(3)\phi_0(4) \rangle^{(0)} &= \Delta_E(1-2)\Delta_E(3-4) \\ &+ \Delta_E(1-3)\Delta_E(2-4) + \Delta_E(1-4)\Delta_E(2-3) , \end{aligned} \tag{4.1.12}$$

but now with no bothersome factors of i.

4.2 Cosmological constant

Although the cosmological constant Λ_0 is related to an "infinite collection of zero-point energies", it is in fact, a Lorentz invariant since it appears as an addition to the Lagrange function. It is related to "vacuum graphs". To see how this goes, we consider here the renormalization of Λ_0 that must be done even in the lowest-order, free-field theory. We shall first work out the renormalization in a manifestly Lorentz invariant form and then express it as a sum of zero-point energies. It suffices to set $\rho_0(x) = 0$ as well as $\lambda_0 = 0$ and examine the vacuum transformation function from the remote past to the distant future (in Euclidean space),

$$\langle 0+ | 0- \rangle = \int [d\phi] \exp\left\{ - \int (d_E^n x) \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 - \Lambda_0 \right] \right\} . \tag{4.2.1}$$

This is analogous to the harmonic oscillator amplitude calculated in Chapter 1 (Section 5) by varying the spring constant. Here we compute the variation with the mass parameter m_0^2. Since the functional integral with

Fig. 4.1. Vacuum loop graph.

additional field variables in the integrand produces the matrix element with the insertion of the corresponding quantum operators,

$$\frac{\partial}{\partial m_0^2}\langle 0+|0-\rangle = -\frac{1}{2}\int (d_E^n x)\langle 0+|\phi_0(x)^2|0-\rangle\,. \tag{4.2.2}$$

The matrix element appearing on the right-hand side is essentially the Green's function at coincident space-time points $x = x'$, $\Delta_E(0)$, except that $\Delta_E(x-x')$ must be written as a normalized expectation value when $\langle 0+|0-\rangle$ departs from unity. Hence

$$\frac{\partial}{\partial m_0^2}\langle 0+|0-\rangle = -\frac{1}{2}\int (d_E^n x)\Delta_E(0)\langle 0+|0-\rangle\,, \tag{4.2.3}$$

or

$$\frac{\partial}{\partial m_0^2}\ln\langle 0+|0-\rangle = -\frac{1}{2}\int (d_E^n x)\Delta_E(0)\,. \tag{4.2.4}$$

The Euclidean Green's function (4.1.6) at coincident points gives

$$\Delta_E(0) = \int \frac{(d_E^n k)}{(2\pi)^n}\,\frac{1}{k^2+m_0^2}\,. \tag{4.2.5}$$

The result (4.2.4) for the mass variation of the vacuum transformation in conjunction with the evaluation (4.2.5) of the Green's function can be expressed in graphical terms. The mass variation is described by a disconnected closed loop as shown in Fig. 4.1. As we are doing all along in this chapter, graphs are displayed to give an intuitive picture of the physical process taking place. The precise, quantitative explanation of Feynman graphs is deferred until the next chapter.

Returning to compute $\Delta_E(0)$, we go through a series of steps that should be becoming a familiar litany: Exponentiate the denominator in the integrand,

$$\frac{1}{k^2+m_0^2} = \int_0^\infty ds\, e^{-s(k^2+m_0^2)}\,, \tag{4.2.6}$$

interchange integrals, and calculate the resulting Gaussian integral:

$$\Delta_E(0) = \frac{1}{(4\pi)^{n/2}}\int_0^\infty ds\, s^{-n/2}\, e^{-s\,m_0^2}\,. \tag{4.2.7}$$

A scale change of the integration variable puts this in the standard form

of the integral representation of the gamma function and so

$$\Delta_E(0) = \frac{(m_0^2)^{\frac{n}{2}-1}}{(4\pi)^{n/2}}\,\Gamma\Big(1-\frac{n}{2}\Big)\,. \tag{4.2.8}$$

Remembering the presence of the constant Λ_0 in the definition (4.2.1) of vacuum amplitude, Eq. (4.2.4) is now trivially integrated to yield

$$\langle 0+|0-\rangle = \exp\Big\{-\int (d_E^n x)\,\mathcal{E}\Big\}\,, \tag{4.2.9}$$

where

$$\mathcal{E} = \frac{m_0^n}{(4\pi)^{n/2}}\,\frac{1}{n}\Gamma\Big(1-\frac{n}{2}\Big) - \Lambda_0\,. \tag{4.2.10}$$

No integration constant (other than Λ_0) can appear here since it would be a dimensional parameter, and the theory contains none other than the mass m_0. The vacuum amplitude is obviously rendered finite if Λ_0 is chosen such as to make $\mathcal{E} = 0$, giving $\langle 0+|0-\rangle = 1$. This, however, is too restrictive a requirement. The space-time integral in Eq. (4.2.9) defines an $n-1$ dimensional spatial volume V times the total elapsed (Euclidean) time T,

$$\int (d_E^n x) = VT\,, \tag{4.2.11}$$

and the vacuum amplitude may be written as

$$\langle 0+|0-\rangle = e^{-ET}\,, \tag{4.2.12}$$

which identifies E as the total vacuum energy. This total energy appears as a spatial volume times an energy density, $\mathcal{E}$,

$$E = V\mathcal{E}\,. \tag{4.2.13}$$

Note that although the constant defined in Eq. (4.2.10) plays the role of an energy density it is, as we emphasized before, a Lorentz invariant as, of course, any constant must be. The point is that, as discussed in Problem 14 of the previous chapter, a constant term $-\mathcal{E}$ in the Lagrange function gives a stress-energy contribution $T^{\mu\nu} = -g^{\mu\nu}\,\mathcal{E}$. The vacuum amplitude involves $T^{00} = \mathcal{E}$, but there is also a spatial stress $T^{kl} = -\delta^{kl}\,\mathcal{E}$ so that the constant $\mathcal{E}$ also appears as a pressure $p = -\mathcal{E}$.

Before performing the necessary renormalization of the cosmological constant, let us pause to show that the lowest-order vacuum energy density $\mathcal{E}$ which has just been calculated is indeed just the sum of zero-point energies. These are the zero-point energies of the harmonic oscillators which describe each of the field modes. To do this, we perform the p_4-integral in the Fourier representation (4.2.5) of $\Delta_E(0)$. This is the Euclidean space equivalent of performing the p^0 integral in Minkowski space.

The integral is easily done by contour integration with the result

$$\Delta_E(0) = \int \frac{(d^{n-1}\mathbf{k})}{(2\pi)^{n-1}} \frac{1}{2\sqrt{\mathbf{k}^2 + m_0^2}} . \tag{4.2.14}$$

Integrating this result with respect to m_0^2 gives

$$\mathcal{E} = \int \frac{(d^{n-1}\mathbf{k})}{(2\pi)^{n-1}} \frac{1}{2} \sqrt{\mathbf{k}^2 + m_0^2} \; - \Lambda_0 . \tag{4.2.15}$$

This is exactly the (continuum limit of the) mode sum* of zero-point energies (minus the cosmological constant). It is, however, an awkward form to work with, and so we return to the previous result (4.2.10) to perform the renormalization.

The theory could be restricted to a finite space-time volume, and so it is only necessary that the energy density $\mathcal{E}$ be finite, not zero. The gamma function in Eq. (4.2.10) has poles at $n = 2$ and $n = 4$ corresponding to the fact that the momentum integral in Eq. (4.2.5) diverges logarithmically in two dimensions and quadratically in four dimensions. Using the recursion relation

$$\Gamma(z+2) = z(z+1)\Gamma(z) \tag{4.2.16}$$

and $\Gamma(1) = 1$, the pole in four dimensions is given by

$$\Gamma(1 - \frac{n}{2}) \simeq \frac{2}{n-4} . \tag{4.2.17}$$

Therefore, the vacuum energy density is made finite (in four dimensions) if the cosmological constant is renormalized according to

$$\Lambda_0 = \mu^{n-4} \left\{ \frac{1}{2} \frac{m_0^4}{(4\pi)^2} \frac{1}{n-4} + \Lambda \right\} . \tag{4.2.18}$$

This formula requires some explanation. Only *integer* powers of m_0^2 appear in the counter terms because the counter terms correspond to the subtraction of the divergent pieces coming from the high momentum (k^2) behavior of integrals that involve integer powers of m_0^2/k^2. The mass parameter m_0^2 appears in the Lagrange function just as another coupling parameter. Counter terms such as in Eq. (4.2.18) are to be constructed in terms of integer power of the coupling parameters. Hence a unit of mass μ must be introduced as has been done. This mass unit also multiplies the renormalized cosmological constant Λ so as to keep it at the mass dimension four. The remaining structure is dictated by the requirement of minimal subtraction which we employ — only the pole term is removed

* Remember that $\sum_n \to V \int (d\mathbf{k})/(2\pi)^{n-1}$.

from the gamma function, and the residue of the pole is evaluated at $n = 4$ except that power of the mass unit, μ^{n-4}, must vary so as to maintain the correct dimensionality away from $n = 4$.

As do all the renormalized parameters, the renormalized cosmological constant obeys a renormalization group equation. Since the theory is defined by Λ_0 which knows nothing of the mass unit, $\mu\, d\Lambda_0/d\mu = 0$, and Eq. (4.2.18) requires that

$$\mu \frac{d\Lambda}{d\mu} = (4-n)\Lambda - \frac{1}{2} \frac{m^4}{(4\pi)^2} . \tag{4.2.19}$$

Here we have written the notation for the renormalized mass m rather than the bare mass m_0 that was used in the counter term of Eq. (4.2.18). In the free theory with which we work here, the two masses are of course the same. This notational care has been taken because renormalization group equations such as Eq. (4.2.19) must involve only finite, renormalized quantities. On the other hand, counter terms, such as that in Eq. (4.2.18), may be expressed in part in terms of bare parameters. It proves convenient to renormalize the cosmological constant Λ using the bare mass m_0 when higher order terms in the coupling constant λ are evaluated, and the notation used in Eq. (4.2.18) anticipates this.

For the sake of completeness, let us record the renormalized vacuum energy that is obtained when Eq. (4.2.18) is inserted in Eq. (4.2.10) and the limit $n \to 4$ taken:

$$\mathcal{E} = \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\ln \left(\frac{m^2}{4\pi\mu^2} \right) + \gamma - \frac{3}{2} \right] - \Lambda , \tag{4.2.20}$$

where $\gamma \simeq 0.5772$ is the Euler-Mascheroni constant.

4.3 Mass renormalization

Self-energy or mass corrections follow from examining the "dressed" or fully interacting propagator

$$\begin{aligned} G_0(x-x') &= \langle \phi_0(x)\, \phi_0(x') \rangle \\ &= \int \frac{(d_E^n p)}{(2\pi)^n} e^{ip(x-x')} G_0(p) . \end{aligned} \tag{4.3.1}$$

Here $\phi_0(x)$ is the quantum field inserted by the variation of the bare source function $\rho_0(x)$, it is the "bare field", and the subscript 0 is used as always to denote unrenormalized quantities. The self-energy function $\Sigma_0(p)$ is related to the propagator by

$$G_0(p) = \frac{1}{p^2 + m_0^2 + \Sigma_0(p)} . \tag{4.3.2}$$

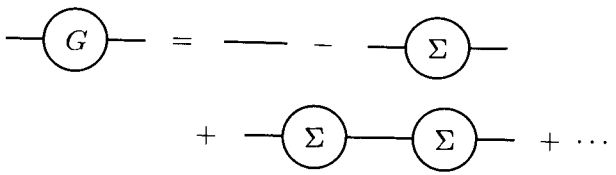


Fig. 4.2. Graphical representation of the fully interacting propagator expanded in powers of the self-energy function.

Expanding in powers of $\Sigma(p)$ is equivalent to the graphical series sketched in Fig. 4.2. This sketch shows that the Green's function is not "irreducible" - it contains graphs that are broken in two by cutting a single line. It will be shown in Chapter 6 that the self-energy function $\Sigma(p)$ is irreducible. Clearly, it is $\Sigma(p)$ that should be calculated as a power series in the coupling λ, not $G(p)$; an expansion of $G(p)$ in powers of λ produces a series of higher powers of the denominator $(p^2 + m_0^2)$ giving rise to spurious higher order poles in $(p^2 + m_0^2)$.

The functional integral representation for the fully interacting propagator reads

$$\langle \phi_0(x)\phi_0(x')\rangle = \int [d\phi]\phi(x)\phi(x')\exp\Big\{ - \int (d_E^n x) \Big[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\, m_0^2\,\phi^2 + \frac{1}{4!}\,\lambda_0\,\phi^4 - \Lambda_0\Big]\Big\}, \quad (4.3.3)$$

with the understanding that either Λ_0 is chosen to give $\langle 0 + |0-\rangle = 1$ or that the propagator is normalized by dividing by this vacuum amplitude. The first-order self energy function $\Sigma_0^{(1)}(p)$ may be calculated by expanding the functional integral to first order in λ_0. The resulting functional integral defines a free-field matrix element and one has the first-order Green's function correction $G_0^{(1)}$ in the form

$$G_0^{(1)}(x - x') = -\frac{1}{4!}\,\lambda_0 \int (d_E^n \bar{x})\langle \phi(x)\phi(x')\phi^4(\bar{x})\rangle^{(0)}. \quad (4.3.4)$$

The contraction rules for the free-field theory now produce

$$G_0^{(1)}(x - x') = -\frac{1}{2}\,\lambda_0\,\Delta_E(0)\int (d_E^n \bar{x})\Delta_E(x - \bar{x})\Delta_E(\bar{x} - x'), \quad (4.3.5)$$

where disconnected vacuum graphs have been discarded since they are canceled by the normalization which has just been discussed. In graphical terms, this first-order correction appears as in Fig. 4.3. Since the Fourier transform of a convolution is the simple product of the Fourier

Fig. 4.3. First-order propagator correction.

transformed terms,

$$G_0^{(1)}(p) = -\frac{1}{2}\,\lambda_0\,\Delta_E(0)\Big[\frac{1}{p^2 + m_0^2}\Big]^2. \quad (4.3.6)$$

Comparing this with the expansion of the denominator in Eq. (4.3.2), we conclude that, to first order,

$$\Sigma_0^{(1)}(p) = \frac{1}{2}\,\lambda_0\,\Delta_E(0). \quad (4.3.7)$$

The number $\Delta_E(0)$ is given in Eq. (4.2.8). Remembering that in lowest order $\lambda_0 = \lambda\mu^{4-n}$, and replacing m_0 by m in this first-order correction, we now have

$$\Sigma_0^{(1)}(p) = m^2\frac{\lambda}{(4\pi)^2}\Big(\frac{m^2}{4\pi\mu^2}\Big)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\Big(1 - \frac{n}{2}\Big). \quad (4.3.8)$$

The gamma function has the pole shown in Eq. (4.2.17) near four dimensions. Since the combination $m_0^2 + \Sigma_0^{(1)}(p)$ appears in the propagator structure (4.3.2), a finite result is achieved by writing

$$m_0^2 = m^2\Big[1 - \frac{\lambda}{(4\pi)^2}\,\frac{1}{n-4}\Big]. \quad (4.3.9)$$

This yields a finite, renormalized self energy given by

$$\Sigma^{(1)}(p) = m^2\frac{\lambda}{(4\pi)^2}\Big[\Big(\frac{m^2}{4\pi\mu^2}\Big)^{\frac{n}{2}-2}\frac{1}{2}\,\Gamma\Big(1 - \frac{n}{2}\Big) - \frac{1}{n-4}\Big], \quad (4.3.10)$$

and to our order

$$G(p) = \frac{1}{p^2 + m^2 + \Sigma^{(1)}(p)}. \quad (4.3.11)$$

In the limit $n \to 4$:

$$\Sigma^{(1)}(p) = \frac{1}{2}\,\frac{\lambda}{(4\pi)^2}\,m^2\Big[\ln\Big(\frac{m^2}{4\pi\mu^2}\Big) + \gamma - 1\Big]. \quad (4.3.12)$$

A minimal subtraction has been performed in writing the counter term structure (4.3.9). Just as the coupling λ is a running parameter not directly related to experimental measurement, the renormalized mass m^2 is a running mass, not the physical mass of the particle created by the ϕ field. Since the theory is defined by bare parameters that know nothing

of the arbitrary mass unit μ one has, as before, $\mu\, d\, m_0^2/d\mu = 0$. Recalling that $\mu\, d\,\lambda/d\mu = (n-4)\lambda$ to lowest order, it now follows from Eq. (4.3.9) that

$$\mu \frac{dm^2}{d\mu} = \delta(\lambda) m^2 , \tag{4.3.13}$$

where, to order λ,

$$\delta(\lambda) = \frac{\lambda}{(4\pi)^2} . \tag{4.3.14}$$

Thus m^2 is indeed a running mass.

In the simple first-order example that we have been discussing, the self energy $\Sigma(p)$ is independent of the momentum p (which is not true in higher orders). In this case, the physical particle mass m_{phy} is given by

$$m_{\text{phy}}^2 = m^2 + \Sigma^{(1)} , \tag{4.3.15}$$

for this gives the position of the propagator pole,

$$G(p) = \frac{1}{p^2 + m_{\text{phy}}^2} , \tag{4.3.16}$$

and thus determines the energy-momentum relation for the long-term particle propagation just as in the free particle case discussed near the beginning of the previous chapter. This is the only effect produced by the lowest-order self-energy function. One must go on the the next order (λ^2) to get a non-trivial momentum dependence which, as will be seen in Section 5, requires a further wave function renormalization. Note that since $\lambda\, \mu^{4-n} = \lambda_0$ is independent of μ, it follows immediately from Eq. (4.3.10) that

$$\mu \frac{d}{d\mu} \Sigma^{(1)} = -\delta(\lambda) m^2 . \tag{4.3.17}$$

Hence the physical mass is independent of the arbitrary mass unit μ as it must be,

$$\mu \frac{dm_{\text{phy}}^2}{d\mu} = 0 . \tag{4.3.18}$$

4.4 Cosmological constant corrected

The inclusion of the field self-interaction modifies the renormalization of the cosmological constant of the non-interacting theory. This is the simplest example of how a higher order renormalization works. Expanding

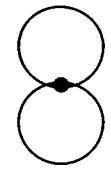


Fig. 4.4. First-order disconnected vacuum graph.

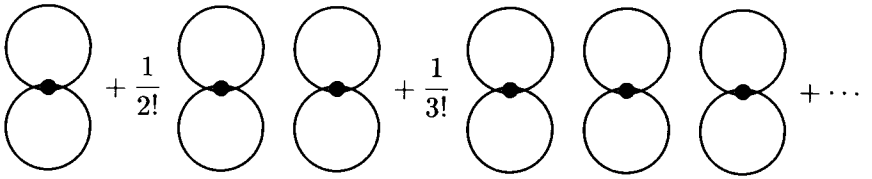


Fig. 4.5. A part of the vacuum amplitude consists of this graphical sum.

the functional integral representation to first order in λ_0 gives the correction to the vacuum amplitude

$$\begin{aligned} \langle 0+|0-\rangle^{(1)} &= -\frac{\lambda_0}{4!} \int (d_E^n x) \langle 0+|\, \phi_0^4(x) |0-\rangle^{(0)} \\ &= -3\, \frac{\lambda_0}{4!} \int (d_E^n x) [\Delta_E(0)]^2 \langle 0+|0-\rangle^{(0)} . \end{aligned} \tag{4.4.1}$$

This corresponds to multiplying the unperturbed amplitude by the disconnected graph shown in Fig. 4.4. We anticipate that *parts* of the higher order corrections form the sum depicted in the graphical series shown in Fig. 4.5. This graphical series corresponds to the expansion of the exponential

$$\langle 0+|0-\rangle \simeq \langle 0+|0-\rangle^{(0)} \exp\left\{ -3\, \frac{\lambda_0}{4!} \int (d_E^n x) [\Delta_E(0)]^2 \right\} . \tag{4.4.2}$$

The expansion of Eq. (4.4.2) does, of course, give the first-order correction of Eq. (4.4.1). We shall show later that the higher orders do combine to give this structure — plus other corrections. Remembering Eq. (4.2.8) for $\Delta_E(0)$, the definition (4.2.9) of the vacuum energy density, and the zero order result (4.2.10) for this energy density, we now have through first order

$$\mathcal{E} = \frac{m_0^n}{(4\pi)^{n/2}} \frac{1}{n} \Gamma\Big(1 - \frac{n}{2}\Big) + \frac{\lambda_0}{8} \left[\frac{m_0^{n-2}}{(4\pi)^{n/2}} \Gamma\Big(1 - \frac{n}{2}\Big) \right]^2 - \Lambda_0 . \tag{4.4.3}$$

To put this expression in the renormalized form, we need to first replace the bare mass by the renormalized mass given in Eq. (4.3.9) which yields,

to order λ,

$$m_0^n = m^n\left[1 - \frac{n}{2}\frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right]. \tag{4.4.4}$$

Making this replacement and performing some algebraic rearrangement gives

$$\begin{aligned}\mathcal{E} =& \frac{m^n}{(4\pi)^{n/2}}\frac{1}{n}\Gamma\Big(1-\frac{n}{2}\Big) - \frac{1}{2}\mu^{n-4}\frac{m^4}{(4\pi)^2}\frac{1}{n-4} \\ &+\frac{1}{2}\mu^{n-4}\frac{\lambda m^4}{(4\pi)^4}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\Big(1-\frac{n}{2}\Big) - \frac{1}{n-4}\right]^2 \\ &+\frac{1}{2}\mu^{n-4}\frac{m^4}{(4\pi)^2}\frac{1}{n-4}\left[1 - \frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right] - \Lambda_0\,. \end{aligned} \tag{4.4.5}$$

The first two lines of this equation are finite in the limit $n \to 4$. The vacuum energy is thus rendered finite by renormalizing the cosmological constant according to

$$\Lambda_0 = \mu^{n-4}\left\{\frac{1}{2}\frac{m^4}{(4\pi)^2}\frac{1}{n-4}\left[1 - \frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right] + \Lambda\right\}, \tag{4.4.6}$$

where Λ is finite in the $n \to 4$ limit. It is important to note that although this renormalization involves a double pole in $(n-4)$, the residue of this double pole has no n-dependence except for the overall factor of μ^{n-4}; it does not involve, for example, $(m/\mu)^{n-4}$ or $(4\pi)^{n-4}$. In particular, if the double pole had a residue involving $(m/\mu)^{n-4}$, then in the limit $n \to 4$ there would be a single-pole counter term of the form $\ln(m/\mu)/(n-4)$ which would violate our claim that counter terms only depend upon integer powers of the mass.

To see how the renormalization group equation for Λ goes, let us first use the mass renormalization formula (4.3.9) to rewrite the counter term in the standard structure involving the bare mass

$$\Lambda_0 = \mu^{n-4}\left\{\frac{1}{2}\frac{m_0^4}{(4\pi)^2}\frac{1}{n-4}\left[1 + \frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right] + \Lambda\right\}. \tag{4.4.7}$$

Using $\mu\, d\Lambda_0/d\mu = 0$, $\mu\, dm_0^2/d\mu = 0$, and $\mu\, d\lambda/d\mu = (n-4)\lambda$ gives

$$\begin{aligned}\mu\frac{d\Lambda}{d\mu} =& (4-n)\Lambda - \frac{1}{2}\frac{m_0^4}{(4\pi)^2}\left[1 + 2\frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right] \\ =& (4-n)\Lambda - \frac{1}{2}\frac{m^4}{(4\pi)^2}\,, \end{aligned} \tag{4.4.8}$$

which is finite as it must be. Note that there is no correction to the renormalization group equation for the cosmological constant Λ to first order in λ. Equation (4.4.8) is identical to the zeroth order result given in Eq. (4.2.19).

4.5 Wave function renormalization

The need for a wave function renormalization first arises in order λ^2. Thus we need to compute the propagator to this order,

$$\begin{aligned}G_0(x-x') =& \langle\phi_0(x)\,\phi_0(x')\rangle \\ \simeq& G_0^{(0)}(x-x') + G_0^{(1)}(x-x') + G_0^{(2)}(x-x')\,. \end{aligned} \tag{4.5.1}$$

Expanding the functional integral representation as should be now familiar, the second order correction is given by the free-field matrix element

$$G_0^{(2)}(x-x') = \frac{1}{2}\left(\frac{-\lambda_0}{4!}\right)^2\int (d_E^n y)(d_E^n z)\langle\phi(x)\phi(x')\phi^4(y)\phi^4(z)\rangle^{(0)}\,. \tag{4.5.2}$$

Before plunging ahead to compute the essential second order corrections to the propagator, it is worth a pause to examine the graphical structure of the theory in a little more detail. We recall that the free propagator is represented as a simple line while the first-order correction was shown in Fig. 4.3. In computing this first order correction we omitted a disconnected vacuum graph. This term is obtained when the fields $\phi(x)\phi(x')$ are contracted in Eq. (4.3.4) to produce the free-field propagator multiplied by

$$-\frac{\lambda_0}{4!}\int (d_E^n\bar{x})\langle\phi^4(\bar{x})\rangle^{(0)}\,.$$

This factor is precisely the correction to the vacuum amplitude given in Eq. (4.4.1) corresponding to the graph of Fig. 4.4. It is removed by defining the fully interacting propagator to be properly normalized — to be the vacuum expectation value of the two field operators divided by the vacuum amplitude. This is equivalent to adjusting the cosmological constant so as to attain a unit vacuum amplitude. There are several disconnected vacuum graphs in the evaluation of the second-order correction (4.5.2). Again these arise when the field $\phi(x)$ is contracted with $\phi(x')$. They have the form shown in Fig. 4.6. The first term, the "square of the figure eight graph", is itself a disconnected graph. Working out the combinational factors (Problem 1), one finds that it is just $\frac{1}{2}$ times the square of the first-order correction of Eq. (4.4.1). Adding this second-order term to the first-order correction gives the first two terms of the expansion of the exponential of the first-order correction as was asserted in Eq. (4.4.2). This is an illustration of a general theorem that will be proven later on: The vacuum amplitude is the exponential of a sum of purely connected graphs. The remaining vacuum graphs shown above contain divergent terms that are removed by an order λ^2 renormalization of the cosmological constant. There remains the one final disconnected graph appearing in Fig. 4.7. It corresponds to the first-order vacuum amplitude multiply-

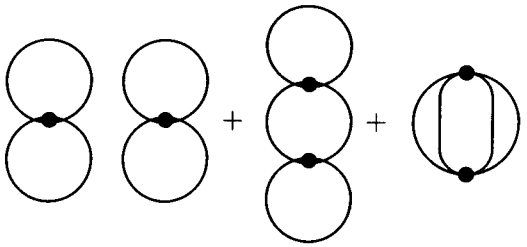


Fig. 4.6. Second-order disconnected vacuum graphs.

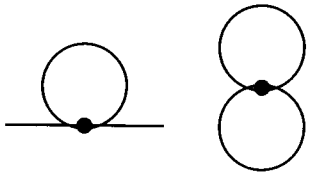


Fig. 4.7. First-order disconnected vacuum graph multiplying first-order propagator correction.

ing the first-order propagator correction, and it is removed by a proper normalization or by a definition of the cosmological constant. (Problem 1 shows that the combinational factors work out correctly for this to be the case.)

We may now turn to the connected graphs. First we note that performing the various contractions in Eq. (4.5.2) gives a term represented by the graph of Fig. 4.8. This graph is not "single-particle irreducible" since it is cut into two disconnected graphs by cutting a single line. Such terms were already discussed in the introduction of the self-energy function $\Sigma_0(p)$ in Eq. (4.3.2). The combinational factors work out (Problem 1) so that this term is just that arising from the expansion of Eq. (4.3.2) to order $\Sigma_0(p)^2$, with $\Sigma_0(p)$ evaluated in first order. There now remain only two connected, single-particle irreducible graphs that contribute to $\Sigma_0(p)$ in second order. The contribution corresponding to Fig. 4.9 will be denoted as $\Sigma_0^{(2,1)}(p)$. Working out the algebra, one finds (Problem 1)

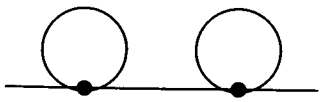


Fig. 4.8. Iteration of first-order self-energy correction which is not single-particle irreducible.

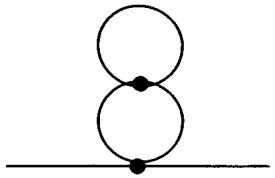


Fig. 4.9. "Double-bubble" self-energy graph.

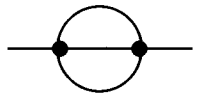


Fig. 4.10. Two loop self-energy graph.

that

$$\Sigma_0^{(2,1)}(p) = -\frac{1}{4}\lambda_0^2 \int (d_E^n y)\, \Delta_E(y)^2\, \Delta_E(0)\,. \tag{4.5.3}$$

The contribution corresponding to Fig. 4.10 will be denoted as $\Sigma_0^{(2,2)}(p)$. Working out the algebra, one finds (Problem 1) that

$$\Sigma_0^{(2,2)}(p) = -\frac{1}{6}\lambda_0^2 \int (d_E^n x) e^{-ipx} \Delta_E(x)^3\,. \tag{4.5.4}$$

It is easy to compute $\Sigma_0^{(2,1)}(p)$. Using the Fourier representation (4.1.6) one has

$$\begin{aligned}\int (d_E^n y)\Delta_E(y)^2 &= \int \frac{(d_E^n k)}{(2\pi)^n} \frac{1}{(k^2+m_0^2)^2} \\ &= -\frac{\partial}{\partial m_0^2} \int \frac{(d_E^n k)}{(2\pi)^n} \frac{1}{k^2+m_0^2} \\ &= -\frac{\partial}{\partial m_0^2}\, \Delta_E(0)\,. \end{aligned} \tag{4.5.5}$$

The number $\Delta_E(0)$ was calculated in Eq. (4.2.8), and remembering that $z\Gamma(z) = \Gamma(z+1)$, we have

$$\int (d_E^n y)\Delta_E(y)^2 = \frac{(m_0^2)^{\frac{n}{2}-2}}{(4\pi)^{n/2}} \Gamma\Big(2-\frac{n}{2}\Big)\,. \tag{4.5.6}$$

Again using Eq. (4.2.8) we now have

$$\Sigma_0^{(2,1)}(p) = -\frac{1}{4}m^2\Big[\frac{\lambda}{(4\pi)^2}\Big]^2 \Big(\frac{m^2}{4\pi\mu^2}\Big)^{n-4} \Gamma\Big(2-\frac{n}{2}\Big)\Gamma\Big(1-\frac{n}{2}\Big)\,, \tag{4.5.7}$$

where the lowest order values $\lambda_0 = \lambda\,\mu^{4-n}$ and $m_0^2 = m^2$ have been used

as may be done here. To obtain the renormalized result, let us first define

$$\bar{\Sigma}^{(2,1)}(p) = -m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2\left\{\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(2-\frac{n}{2}\right)+\frac{1}{n-4}\right\}$$
$$\left\{\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right)-\frac{1}{n-4}\right\}, \qquad (4.5.8)$$

which is finite in the $n \to 4$ limit. In terms of this finite function

$$\Sigma_0^{(2,1)}(p) = \bar{\Sigma}^{(2,1)}(p) - m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{n-4}\left\{\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\right.$$
$$\left.\frac{1}{2}\left[\Gamma\left(2-\frac{n}{2}\right)-\Gamma\left(1-\frac{n}{2}\right)\right]+\frac{1}{n-4}\right\}. \qquad (4.5.9)$$

Before going on to deal with the remaining second-order self-energy function $\Sigma_0^{(2,2)}(p)$, let us return to the lowest order self energy given in Eq. (4.3.8). Since we are now working to second order, we must use the bare quantities in the evaluation of this formula and write

$$\Sigma_0^{(1)}(p) = \frac{\lambda_0}{(4\pi)^{n/2}}(m_0^2)^{\frac{n}{2}-1}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right). \qquad (4.5.10)$$

To compute this through the terms of second order requires the use of the coupling constant renormalization given in Eq. (3.5.39) which is reproduced here:

$$\lambda_0 = \lambda\mu^{4-n}\left[1-\frac{3\lambda}{(4\pi)^2}\frac{1}{n-4}\right]. \qquad (4.5.11)$$

Moreover, the lowest-order mass renormalization of Eq. (4.3.9) is needed which gives

$$(m_0^2)^{\frac{n}{2}-1} = (m^2)^{\frac{n}{2}-1}\left[1-\frac{\lambda}{(4\pi)^2}\frac{\frac{n}{2}-1}{n-4}\right]. \qquad (4.5.12)$$

The lowest-order renormalized self energy is given by

$$\Sigma^{(1)}(p) = m^2\frac{\lambda}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right)-\frac{1}{n-4}\right]. \qquad (4.5.13)$$

Using this function we may write

$$\Sigma_0^{(1)}(p) = \Sigma^{(1)}(p) + m^2\frac{\lambda}{(4\pi)^2}\frac{1}{n-4} - m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{n-4}$$
$$\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\left[\frac{3}{2}\Gamma\left(1-\frac{n}{2}\right)-\frac{1}{2}\Gamma\left(2-\frac{n}{2}\right)\right]. \qquad (4.5.14)$$

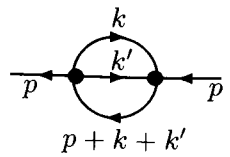


Fig. 4.11. Two loop self-energy graph in momentum space.

In adding this to the previous calculation (4.5.9) of $\Sigma_0^{(2,1)}(p)$ some cancellation of terms occur, and one finds that

$$\Sigma_0^{(1)}(p)+\Sigma_0^{(2,1)}(p) = \Sigma^{(1)}(p) + m^2\frac{\lambda}{(4\pi)^2}\frac{1}{n-4} + \bar{\Sigma}^{(2,1)}(p)$$
$$-m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{n-4}\left\{\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\Gamma\left(1-\frac{n}{2}\right)+\frac{1}{n-4}\right\}. \qquad (4.5.15)$$

We should note that the pole of $\Gamma\left(1-\frac{n}{2}\right)$ in the curly brackets multiplying the factor $(m^2/4\pi\mu^2)^{\frac{n}{2}-2}$ produces a remainder involving $\ln(m^2/4\pi\mu^2)$ in the $n \to 4$ limit. This logarithmic remainder in turn appears as a residue of the single pole which remains. On the other hand, we have previously asserted that only integer powers of m^2 appear in the pole terms. This, in fact, is the case. We shall find that there is a term in $\Sigma_0^{(2,2)}(p)$ which cancels this untoward behavior. Looking back on the calculations that have been made so far, one can see that already such cancellations have been made in adding $\Sigma_0^{(1)}(p)$ and $\Sigma_0^{(2,1)}(p)$.

We turn now to compute $\Sigma_0^{(2,2)}(p)$. The usual method is to write the Green's functions $\Delta_E(x)$ in Eq. (4.5.4) in terms of their Fourier transforms and perform the spatial integration to secure

$$\Sigma_0^{(2,2)}(p) = -\frac{1}{6}\lambda^2\mu^{8-2n}\int\frac{(d_E^n k)}{(2\pi)^n}\frac{(d_E^n k')}{(2\pi)^n}$$
$$\frac{1}{k^2+m^2}\frac{1}{k'^2+m^2}\frac{1}{(p+k+k')^2+m^2}. \qquad (4.5.16)$$

This corresponds to the Feynman graph in momentum space shown in Fig. 4.11. In this case, however, a different method is more convenient. This method uses a parametric representation of the coordinate space Green's function which is obtained by the usual steps of exponentiating the denominator, completing the square, and performing the resulting Gaussian integrations:

$$\Delta_E(x) = \int \frac{(d_E^n k)}{(2\pi)^n} \frac{e^{ikx}}{k^2+m^2}$$
$$= \int_0^\infty ds e^{-sm^2} \int \frac{(d_E^n k)}{(2\pi)^n} \exp\left\{-s\left(k - \frac{ix}{2s}\right)^2 - \frac{x^2}{4s}\right\}$$
$$= \frac{1}{(4\pi)^{n/2}} \int_0^\infty ds s^{-n/2} e^{-sm^2 - x^2/4s} . \quad (4.5.17)$$

Using the product of three such representations for $\Delta_E(x)^3$, the Fourier transform in Eq. (4.5.4) now appears, after interchanging integrals, as the Fourier transform of a simple Gaussian function

$$\int (d_E^n x) e^{-ipx} \exp\left\{-\frac{x^2}{4}\left(\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\right)\right\}$$
$$= (4\pi)^{n/2} \left(\frac{s_1s_2+s_1s_3+s_2s_3}{s_1s_2s_3}\right)^{-n/2}$$
$$\exp\left\{-\frac{s_1s_2s_3}{s_1s_2+s_1s_3+s_2s_3}p^2\right\} , \quad (4.5.18)$$

and we get

$$\Sigma_0^{(2,2)}(p) = -\frac{1}{6}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{(4\pi\mu^2)^{n-4}}$$
$$\int_0^\infty ds_1 ds_2 ds_3 (s_1s_2+s_1s_3+s_2s_3)^{-n/2}$$
$$\exp\left\{-(s_1+s_2+s_3)m^2 - \frac{s_1s_2s_3}{s_1s_2+s_1s_3+s_2s_3}p^2\right\} . \quad (4.5.19)$$

It is a non-trivial task to compute this parameter integral for arbitrary dimensionality n. (Translation: I don't know how to do it.) All that we shall do here is to compute the terms which diverge when $n \to 4$ so as to understand how the renormalization goes. The divergencies in Eq. (4.5.19) appear when the s-variables approach zero. Therefore, if the exponential is expanded in powers of p^2, only the first few powers of p^2 involve divergent integrals since the coefficient $s_1s_2s_3(s_1s_2+s_1s_3+s_2s_3)^{-1}$ is of order s_a, $a = 1,2,3$, when s_a vanishes. A closer examination shows that only the integrals multiplying $(p^2)^0$ and $(p^2)^1$ diverge at $n = 4$. That this is so is easily seen by expanding the usual momentum space representation (4.5.16) in powers of p^μ; the terms of higher order than p^2 have sufficiently high powers of k and k' in the denominators to give convergent ($n = 4$) integrals when $|k| \to \infty$, $|k'| \to \infty$. As we shall see, the fact that only the constant and order p^2 terms diverge means that the theory is rendered finite with only a mass and wave function

renormalization in addition to the coupling renormalization.

The constant piece of $\Sigma^{(2,2)}$ in the p^2 expansion is given by

$$\Sigma_0^{(2,2)}(0) = -\frac{1}{6}m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{n-4} I_{-\frac{n}{2}} , \quad (4.5.20)$$

where it proves convenient to view the integral defining $I_{-\frac{n}{2}}$ as a special case of

$$I_l = (m^2)^{3+2l} \int_0^\infty ds_1 ds_2 ds_3 (s_1s_2+s_1s_3+s_2s_3)^l e^{-(s_1+s_2+s_3)m^2} . \quad (4.5.21)$$

The divergent terms in Eq. (4.5.20) may be computed with the use of a recursion relation obeyed by the general coefficient I_l. To derive this recursion relation, we first note that scaling the s-variables by $m^2 s \to s$, shows that I_l is a pure number which is thus independent of the mass parameter,

$$\frac{\partial}{\partial m^2} I_l = 0 . \quad (4.5.22)$$

Carrying out this derivative on Eq. (4.5.21) and then taking $m^2 \to 1$ gives

$$(3+2l)I_l - \int_0^\infty ds_1 ds_2 ds_3 (s_1s_2+s_1s_3+s_2s_3)^l (s_1+s_2+s_3) e^{-(s_1+s_2+s_3)} = 0 . \quad (4.5.23)$$

Since

$$(s_1+s_2+s_3)(s_1s_2+s_1s_3+s_2s_3)^l$$
$$= \frac{1}{2l+2}\left(\frac{\partial}{\partial s_1}+\frac{\partial}{\partial s_2}+\frac{\partial}{\partial s_3}\right)(s_1s_2+s_1s_3+s_2s_3)^{l+1} , \quad (4.5.24)$$

an integration by parts may be performed to obtain the recursion relation

$$(3+2l)I_l - \frac{3}{2l+2}I_{l+1} = -\frac{3}{2l+2}\int_0^\infty ds_1 ds_2 (s_1s_2)^{l+1} e^{-(s_1+s_2)}$$
$$= -\frac{3}{2l+2}\Gamma(l+2)^2 . \quad (4.5.25)$$

This recursion relation is put in a neater form by writing

$$I_l = \frac{1}{3^l}\Gamma(2l+2)J_l . \quad (4.5.26)$$

Using $z\Gamma(z) = \Gamma(z+1)$, it is easy to see that

$$J_l = J_{l+1} - \frac{3^{l+1}\Gamma(l+2)^2}{\Gamma(2l+4)} . \quad (4.5.27)$$

We need to compute J_l for $l = -n/2 \simeq -2$. The recursion relation (4.5.27) together with Eq. (4.5.26) gives

$$I_{-n/2} = 3^{n/2}\Gamma(2-n)J_{1-n/2} - 3\frac{\Gamma(2-n)}{\Gamma(4-n)}\Gamma\left(2-\frac{n}{2}\right)^2 . \tag{4.5.28}$$

The recursion relation (4.5.27) also gives $J_{-1} = J_0 - 1$, while

$$J_0 = I_0 = \left[\int_0^\infty ds\, e^{-s}\right]^3 = 1 , \tag{4.5.29}$$

and so $J_{-1} = 0$. Hence in the limit $n \to 4$, the first term in Eq. (4.5.28) involves a simple pole in $n-4$ multiplying a residue which vanishes — this term has a finite limit. Accordingly, we may now use Eq. (4.5.28) to evaluate Eq. (4.5.20) and obtain

$$\Sigma_0^{(2,2)}(0) = \frac{1}{8}m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2\left(\frac{m^2}{4\pi\mu^2}\right)^{n-4}\left(\frac{n-2}{n-3}\right)\Gamma\left(1-\frac{n}{2}\right)^2 + \text{ finite}. \tag{4.5.30}$$

To extract the singular terms in Eq. (4.5.30) we write

$$\begin{aligned}
&\frac{1}{4}\left(\frac{n-2}{n-3}\right)\left(\frac{m^2}{4\pi\mu^2}\right)^{n-4}\Gamma\left(1-\frac{n}{2}\right)^2 \\
&= \left(\frac{n-2}{n-3}\right)\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4}\right]^2 \\
&\quad + \left(\frac{n-2}{n-3}\right)\frac{1}{n-4}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4}\right] \\
&= \frac{2}{n-4}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4} - \frac{1}{2}\right] + \text{ finite}.
\end{aligned} \tag{4.5.31}$$

Therefore

$$\Sigma_0^{(2,2)}(0) = m^2\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{n-4}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4} - \frac{1}{2}\right] + \text{ finite}. \tag{4.5.32}$$

This is the result to be combined with the terms displayed in Eq. (4.5.15) to produce the complete self energy at $p^2 = 0$,

$$\Sigma_0^{(1)}(0) + \Sigma_0^{(2)}(0) = \Sigma_0^{(1)}(0) + \Sigma_0^{(2,1)}(0) + \Sigma_0^{(2,2)}(0) . \tag{4.5.33}$$

We find that the terms involving $\Gamma\left(1-\frac{n}{2}\right)$ cancel. As we have remarked before, this cancellation must occur so as to avoid a single pole with a residue involving $\ln m^2$. The sum gives

$$\Sigma_0^{(1)}(0)+\Sigma_0^{(2)}(0) = m^2\left\{\frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + \text{ finite}\right] - \left[\frac{\lambda}{(4\pi)^2}\right]^2\left[\frac{2}{(n-4)^2} + \frac{1}{2}\frac{1}{n-4} + \text{ finite}\right]\right\} . \tag{4.5.34}$$

It remains to calculate the divergent coefficient of the term of order p^2 in the momentum expansion of the self-energy function $\Sigma_0^{(2,2)}(p)$ given in Eq. (4.5.19). Rather than doing this directly, we note that this divergent term is independent of the mass parameter m^2. Hence it may be obtained from the massless theory $m^2 = 0$. In this case the complete self-energy function is easily computed. Moreover, the massless theory is of interest in its own right and, in particular, it gives the high-energy limit where the mass is of no importance. To calculate in the massless case, we return to the Green's function representation (4.5.17), set $m^2 = 0$ and write $s = 1/t$ to obtain the standard representation of the Γ-function and

$$\Delta_E(x) = \frac{1}{(4\pi)^{n/2}}\left(\frac{4}{x^2}\right)^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}-1\right) . \tag{4.5.35}$$

The Fourier transform of the cube of this Green's function is needed according to Eq. (4.5.4). We represent

$$\left(\frac{4}{x^2}\right)^{\frac{3n}{2}-3} = \frac{1}{\Gamma\left(\frac{3n}{2}-3\right)}\int_0^\infty dt\, t^{\frac{3n}{2}-4}\, e^{-tx^2/4} , \tag{4.5.36}$$

and take the Fourier transform of the Gaussian to obtain

$$\begin{aligned}
\Sigma_0^{(2,2)}(p) &= -\frac{1}{6}\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{(4\pi\mu^2)^{n-4}}\frac{\Gamma\left(\frac{n}{2}-1\right)^3}{\Gamma\left(\frac{3n}{2}-3\right)}\int_0^\infty dt\, t^{n-4}\, e^{-p^2/t} \\
&= -\frac{1}{6}\left[\frac{\lambda}{(4\pi)^2}\right]^2 p^2\left(\frac{p^2}{4\pi\mu^2}\right)^{n-4}\frac{\Gamma\left(\frac{n}{2}-1\right)^3\Gamma(3-n)}{\Gamma\left(\frac{3n}{2}-3\right)} .
\end{aligned} \tag{4.5.37}$$

In the limit $n \to 4$,

$$\Sigma_0^{(2,2)}(p^2) = -\frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 p^2\left\{\frac{1}{n-4} + \ln\left(\frac{p^2}{4\pi\mu^2}\right) + \text{const.}\right\} . \tag{4.5.38}$$

Note that the divergence of the theory is reflected in two ways: (1) The coefficient of p^2 diverges; (2) At large p^2 the free-field propagator behavior is altered by the appearance of an extra factor of $\ln p^2$. The

two traits are intimately connected. The field interaction ϕ^4 is at a single point. This is required by Lorentz invariance — the interaction cannot be smeared over a finite spatial region and still maintain this invariance. The point-like interaction does not become weak at high energies. At high energies, the particle wavelengths become very small, and an interaction of finite spatial extent becomes weak because it averages to a small quantity when integrated over an oscillating wave that is rapidly varying. But this does not happen with a point interaction whose effects do not diminish at high energies. This strong interaction at high energies means that virtual processes do not damp, and this results in the divergence of the coupling renormalization. Similarly, the large momentum behavior of the inverse Green's function, which would approach the free-particle behavior $p^2 + m_0^2$ if the interaction became weak at high energies, is modified by the $\lambda^2 p^2 \ln p^2$ term in Eq. (4.5.38). We should also remark that we have been able to take the massless limit because we introduced the arbitrary mass unit μ. If some scheme were used that effectively identifies μ with the particle mass m, then this mass would appear in the logarithm in Eq. (4.5.38), and the massless limit could not be taken. There would be a spurious divergence in the massless limit.

We are at long last in the position to describe the mass and wave function renormalization to order λ^2. In view of Eqs. (4.5.34) and (4.5.38), the inverse of the interacting Green's function has the structure, to order λ^2,

$$\begin{aligned} G_0(p)^{-1} =& p^2\left\{1 - \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{n-4}\right\} \\ &+ m_0^2 + m^2\left\{\frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + \text{finite}\right]\right\} \\ &- \left[\frac{\lambda}{(4\pi)^2}\right]^2\left\{m^2\left[\frac{2}{(n-4)^2} + \frac{1}{2}\,\frac{1}{n-4}\right]\right. \\ &\left. + \text{finite function of } p^2\right\}. \end{aligned} \tag{4.5.39}$$

The divergence multiplying p^2 is removed by making an overall "wave function" renormalization z_1^2. One sets

$$G_0(p) = z_1^2\, G(p)\,, \tag{4.5.40}$$

where $G(p)$ is the finite, renormalized Green's function. Comparing Eq. (4.5.40) with Eq. (4.5.39) and making a minimal subtraction we see that

$$z_1^2 = 1 + \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{n-4}\,. \tag{4.5.41}$$

Multiplying Eq. (4.5.39) by this factor gives, to order λ^2,

$$\begin{aligned} G(p)^{-1} =& p^2 + m_0^2 + m^2\left\{\frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + \text{finite}\right]\right\} \\ &- \left[\frac{\lambda}{(4\pi)^2}\right]^2\left\{m^2\left[\frac{2}{(n-4)^2} + \frac{5}{12}\,\frac{1}{n-4}\right] + \text{finite}\right\}, \end{aligned} \tag{4.5.42}$$

and the minimally subtracted mass renormalization is identified as

$$m_0^2 = m^2\left\{1 - \frac{\lambda}{(4\pi)^2}\,\frac{1}{n-4} + \left[\frac{\lambda}{(4\pi)^2}\right]^2\left[\frac{2}{(n-4)^2} + \frac{5}{12}\,\frac{1}{n-4}\right]\right\}. \tag{4.5.43}$$

The Green's functions with the 0 subscript are the vacuum values of the "bare" fields ϕ_0. For example,

$$G_0(x - x') = \langle \phi_0(x)\phi_0(x')\rangle\,. \tag{4.5.44}$$

The Green's functions without this subscript denote the vacuum values of the renormalized field ϕ which are finite (in the $n \to 4$ limit). For example,

$$G(x - x') = \langle \phi(x)\phi(x')\rangle\,. \tag{4.5.45}$$

Comparing these with Eq. (4.5.40) gives

$$\phi_0(x) = z_1\phi(x)\,. \tag{4.5.46}$$

It should be emphasized that, although this renormalization produces finite Green's functions, it is an intermediate, "running" renormalization. The renormalized field ϕ does not have the correct amplitude to create single-particle states with unit norm.* The interacting field ϕ acting on the vacuum produces many short-lived excitations in addition to a stable single-particle state with whatever normalization it may have. But with the minimally subtacted wave function renormalization, the single-particle component does not have a unit amplitude. Another way to say this is that the renormalized propagator pole at the physical mass value does not have a unit residue. The minimally subtracted wave function renormalization is like the coupling and mass renormalizations which do not produce parameters that are directly related to experimental measurements. The wave function renormalization z_1 is best thought of as corresponding to a renormalization of the source $\rho_0(x)$. Variational derivatives with respect to the "bare" source $\rho_0(x)$ produce insertions of the "bare" field $\phi_0(x)$. Variational derivatives with respect to the renormalized source $\rho(x)$ produce insertions of the renormalized field $\phi(x)$. Hence

* See Chapter 6, Section 1 for a full discussion of the construction of the correctly normalized single-particle states.

the renormalization of the two source functions are the inverse of the field renormalizations, and they are related by

$$\rho_0(x) = \rho(x) z_1^{-1} . \tag{4.5.47}$$

In the leading order which has been calculated,

$$\rho_0(x) = \rho(x)\left\{1 - \frac{1}{24}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{n-4}\right\} . \tag{4.5.48}$$

Just as for any other coupling parameter $\rho(x)$ obeys a renormalization group equation

$$\mu \frac{d}{d\mu} \rho(x) = \gamma(\lambda)\, \rho(x) . \tag{4.5.49}$$

As with the other couplings, the bare source is independent of the scale mass, $\mu\, d\rho_0/d\mu = 0$. Hence, using the lowest-order result $\mu\, d\lambda/d\mu = (n-4)\lambda$, Eq. (4.5.48) yields the leading order form

$$\gamma(\lambda) = \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 . \tag{4.5.50}$$

The behavior of the mass parameter m^2 under a change in the scale mass μ is described by

$$\mu \frac{dm^2}{d\mu} = \delta(\lambda) m^2 . \tag{4.5.51}$$

Setting $\mu\, dm_0^2/d\mu = 0$ in Eq. (4.5.43) gives, to order λ^2,

$$\delta(\lambda)\left[1 - \frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right] = \mu \frac{d}{d\mu}\left[\frac{\lambda}{(4\pi)^2}\right]\frac{1}{n-4} - \mu \frac{d}{d\mu}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \left[\frac{2}{(n-4)^2} + \frac{5}{12}\frac{1}{n-4}\right] . \tag{4.5.52}$$

Clearly a detailed cancellation is needed to remove all the poles and produce a function $\delta(\lambda)$ which is independent of n. That this is indeed the case follows from Eqs. (3.5.42) and (3.5.45) which give

$$\mu \frac{d}{d\mu}\left[\frac{\lambda}{(4\pi)^2}\right] = (n-4)\left[\frac{\lambda}{(4\pi)^2}\right] + 3\left[\frac{\lambda}{(4\pi)^2}\right]^2 , \tag{4.5.53}$$

and thus

$$\delta(\lambda) = \left[\frac{\lambda}{(4\pi)^2}\right] - \frac{5}{6}\left[\frac{\lambda}{(4\pi)^2}\right]^2 . \tag{4.5.54}$$

Problems

"They say that genius is an infinite capacity for taking pains," he remarked with a smile.

Sherlock Holmes

1. Work out all the second order corrections to the Green's function of the $\lambda\phi^4$ theory which Section 5 of the text asserted were worked out in this problem. That is, calculate all the factors for the graphs and verify the initial equations used in the text.

2. Consider the coupled field theory of two scalar fields ϕ and χ described by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}M_0^2\chi^2 + \frac{1}{2}g_0\phi\chi^2 .$$

Let $G(p)$ be the propagator for the ϕ-field, with

$$G(p) = \frac{1}{p^2 + m_0^2 + \Sigma(p)} .$$

The purpose of this problem is to compute the self energy $\Sigma(p)$ to the lowest non-trivial order so as to illustrate and augment the material covered in the text. Note that thus $\Sigma(p)$ corresponds to the graph of Fig. 4.12. Work out this problem in the following steps.

a) Explain the computation of this graph in configuration space in detail. Explain where the numerical factors come from, what the lines stand for, what the vertex dots stand for, etc. Do all of this by starting at the beginning with the functional integral and its expansion in powers of g.

b) Explain in detail how this is transformed into momentum space and what the lines, dots, and so forth stand for.

c) Regulate the theory by continuing to n dimension. Introduce a scale mass μ and relate g_0 to a g times an appropriate power of μ, with g chosen to have the dimensions of g_0 at $n = 4$. Write down the momentum space integral for $\Sigma(p)$ in n-dimensional Euclidean space time.

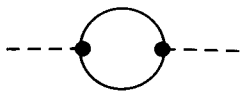


Fig. 4.12. Self-energy graph with $\phi\chi^2$ interaction.

d) Identify the divergent pole term and remove it by mass renormalization with minimal subtraction. Compute

$$\mu \frac{dm^2}{d\mu} = ?$$

How is the physical mass m_{phy} defined? How is it related to m and the finite remainder of $\Sigma(p)$?

e) You now have the form

$$G(p) = \frac{1}{p^2 + m^2 + \Sigma_{\text{ren}}(p)},$$

with $\Sigma_{\text{ren}}(p)$ finite. Show that this finite self energy may be expressed as a dispersion integral,

$$\Sigma_{\text{ren}}(p) = \Sigma_{\text{ren}}(0) + p^2 \int_{s_0}^{\infty} ds \frac{\sigma(s)}{s + p^2}.$$

Compute $\sigma(s)$. What is the value of the integration end point s_0? Why is s_0 what it is?

f) Suppose that $m^2_{\text{phy}} > 4M^2$ so that the ϕ-particle can decay into two χ-particles. Compute the imaginary part

$$\text{Im}\,\Sigma(p^2 = -m^2_{\text{phy}}) = ?$$

Compare this result with the decay rate Γ computed in Problem 16 of Chapter 3.

3. High energy electron-positron collisions produce a virtual photon (γ^*) which turns into a quark-antiquark pair which finally turn into observed hadrons. Since the energy is very high, the mass of the electron can be neglected. The purpose of this problem is to give a scalar analog of the process

$$e^+e^- \to \gamma^* \to q\bar{q}.$$

The analogy is very imperfect as the angular and energy dependence come out wrong, but the basic ideas are there in the analog. So consider a massless scalar field ψ (the "electron") which is coupled to a massless scalar field ϕ (the "photon"), with a coupling $\frac{1}{2}g\phi\psi^2$. The "photon" field ϕ is then coupled to the "quark" field χ via $\frac{1}{2}g\phi\chi^2$. The scattering amplitude is described by the graph of Fig. 4.13.

a) Work in the center of mass $\mathbf{p} + \mathbf{p}' = 0$. (This is how e^+e^- colliders work.) Compute the differential cross section $d\sigma/d\Omega$ for the production of a "quark" at the polar angles θ, ϕ with the z-axis taken as the beam direction.

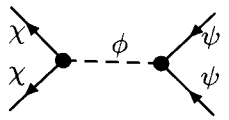


Fig. 4.13. Scalar analog of quark-antiquark production in electron-positron annihilation.

b) Compute the total cross section σ.

c) Compare σ with $\text{Im}\,\Sigma(p^2)$ of Problem 2. Use the results and discussion of Problem 17 of the previous chapter to explain why this comparison makes sense, and write down a graph to explain this comparison.

5
Renormalization

The character of relativistic quantum field theory will continue to be illustrated by the simple example of scalar field theory with the $\lambda\phi^4$ interaction. Previously we have computed a few basic processes by working out the perturbative development provided by the functional integral representation. Based upon this experience, we can now systematize what has been done in the form of Feynman rules that enable one to compute an amplitude in momentum space directly from its graphical representation. With this in hand, we then turn to describe the character of the divergencies which appear in the theory and how these divergencies are removed by renormalization. We shall present only a very rough outline of how the renormalization process works. A detailed description and proof of the renormalizability of the theory is a non-trivial task whose methods are of little use in other contexts, and so this we will not do. We shall, however, describe in some detail the nature of the parameter renormalization in the minimally subtracted, dimensionally regulated scheme, for this leads directly to the renormalization group which is a very useful tool. The behavior that the renormalization group implies for a general field theory — a theory which could have a fixed point of the renormalization group or a theory such as quantum chromodynamics which is asymptotically free — will be described. Composite operators are introduced, and the relationship of the stress-energy tensor to the renormalization group is described. The chapter concludes with a sketch of the operator product expansion.

5.1 Euclidean space Feynman rules

The graphical rules that have been used need to be made more explicit and codified. As was remarked before, it simplifies the analysis to work in Euclidean space and then continue back into Minkowski space-time

after a calculation has been performed. And as has been done all along, the essential elements of the theory are illustrated by working with the simple scalar field theory with the $\lambda\phi^4$ interaction. A general (interacting) N-point Green's function in Euclidean space is defined by

$$G_0(x_1, x_2, \cdots, x_N) = \langle \phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_N)\rangle \,. \tag{5.1.1}$$

Here the angular bracket symbol denotes the analytic continuation of the time-ordered product in Minkowski space-time into Euclidean space. This continuation is made explicit by using the functional integral representation in Euclidean space with the additional functional field variables $\phi(x_1)\ \phi(x_2)\cdots\phi(x_N)$ in the integrand. The Green's functions are translationally invariant. They are not changed by translating each coordinate x_c^μ, $c = 1, 2, \cdots, N$ to $x_c^\mu + a^\mu$,

$$G_0(x_1 + a, x_2 + a, \cdots, x_N + a) = G_0(x_1, x_2, \cdots, x_N) \,. \tag{5.1.2}$$

This translational invariance requires that the Fourier transform into momentum space contains a δ-function factor that conserves the total momentum. We shall adopt an outgoing momentum convention for every field so that the Fourier representation takes the form

$$G_0(x_1, x_2, \cdots, x_N) = \int \prod_{a=1}^{N} \frac{(d_E^n p_a)}{(2\pi)^n} (2\pi)^n \delta^{(n)}\left(\sum_{b=1}^{N} p_b\right) G_0(p_1, p_2, \cdots, p_N) \exp\left\{ i \sum_{c=1}^{N} p_c^\mu x_{c\mu} \right\} . \tag{5.1.3}$$

So as to implement the dimensional regularization scheme, our work is done in an n-dimensional Euclidean space. The inverse transform is given by

$$(2\pi)^n \delta^{(n)}(\Sigma p)\, G_0(p_1, p_2, \cdots, p_N) = \int \prod (d_E^n x) \exp\left\{ -i \sum px \right\} G_0(x_1, x_2, \cdots, x_N) \,. \tag{5.1.4}$$

This is represented by the graph shown in Fig. 5.1.

The notion of an "outgoing momentum convention" needs some explanation. The point is that the Fourier transform may be defined either with "positive" exponentials $\exp(ipx)$ or "negative" exponentials $\exp(-ipx)$. We chose positive exponentials in Eq. (5.1.3). To see in a heuristic way that this corresponds to "outgoing" momentum, we temporarily revert to Minkowski space-time. For the free-field theory, Eq. (3.2.58) states that the annihilation operator has the structure

$$\phi^{(+)}(p) \sim \int (d\mathbf{r}) e^{-ipx} \phi(x) \,. \tag{5.1.5}$$

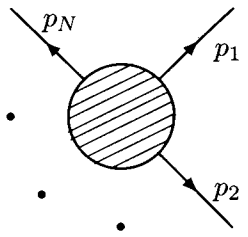


Fig. 5.1. Graphical representation of the Green's function. Each line corresponds to a field carrying the momentum indicated. The outward pointing arrows indicate that these are outgoing momenta.

In the free-field theory, the N-particle bra ("final") state has the construction

$$\langle p_1\,,\,p_2\,,\cdots,p_N| = \langle 0|\phi^{(+)}(p_1)\phi^{(+)}(p_2)\cdots\phi^{(+)}(p_N)\,. \tag{5.1.6}$$

Hence — at least as far as the sign conventions go - Eq. (5.1.4) corresponds roughly to the correspondence

$$\langle p_1\,,\,p_2\,,\cdots,p_N|0\rangle \sim G_0(p_1\,,\,p_2\,,\cdots,p_N)\,. \tag{5.1.7}$$

This is the reason for saying that an outgoing momentum convention has been adopted. Of course some of the p_a^μ must have negative time components to get a non-vanishing amplitude, and these correspond to particles in the initial state.

With these conventions in hand, we can now outline the Feynman rules in the Euclidean momentum space. They can be motivated by considering the first-order correction to the four-point Green's function. As usual, the interaction term in the functional integral representation may be expanded to write this correction in terms of a free-field matrix element,

$$G_0^{(1)}(x_1,\,x_2,\,x_3,\,x_4) = -\frac{\lambda_0}{4!}\int (d_E^n y)\,\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y)\rangle^{(0)}\,. \tag{5.1.8}$$

The connected contribution to this function is evaluated by contracting each of the four $\phi(x_a)$ fields with one of the four $\phi(y)$ fields using the Euclidean propagator

$$\begin{aligned}\Delta_E(x-y) &= \langle\phi(x)\phi(y)\rangle^{(0)} \\ &= \int \frac{(d_E^n p)}{(2\pi)^n}\,\frac{1}{p^2+m_0^2}\,e^{ip(x-y)}\,.\end{aligned} \tag{5.1.9}$$

The Fourier transform (5.1.4) just "cancels" each of these Fourier integrals that represent the four propagators obtained in the contractions of Eq. (5.1.8), and we represent each of the resulting factors by a line with

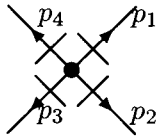
the correspondence

$$\underline{\qquad p \qquad} \;=\; \frac{1}{p^2+m_0^2}\,. \tag{5.1.10}$$

The remaining y-integration produces a factor of

$$\int (d_E^n y)\;e^{-i(p_1+p_2+p_3+p_4)y} = (2\pi)^n\delta^{(n)}(p_1+p_2+p_3+p_4)\,. \tag{5.1.11}$$

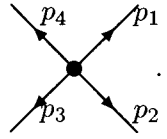
Since such a factor always appears with the insertion of an interaction term, a graphical notation is invented to represent it. Including the associated factor involving the coupling strength we write



$$= -\lambda_0(2\pi)^n\delta^{(n)}(p_1+p_2+p_3+p_4)\,. \tag{5.1.12}$$

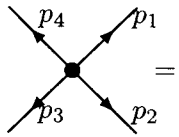
The little slashes or bars on the lines are placed to indicate that no propagator factors occur here — the lines are used only to represent the momentum flow. Note that the conventional factor 1/4! which accompanies the the coupling λ_0 in the Lagrange function is omitted here. The Fourier transform of the Green's function correction (5.1.8) has now been evaluated except for the overall combinatorial factor: 4 ways to contract $\phi(x_1)$ with one of the four $\phi(y)$, 3 ways to contract $\phi(x_2)$ with one of the remaining three $\phi(y)$, 2 ways to contract $\phi(x_3)$ with one of the remaining two $\phi(y)$, and finally a unique contraction of $\phi(x_4)$ with the last $\phi(y)$. The overall combinatorial factor is 4!, which just cancels the conventional factor of 1/4! originally put in the interaction term. This cancellation is the reason for the convention of placing this factor in the Lagrange function. In graphical terms

$$(2\pi)^n\delta^{(n)}(p_1+p_2+p_3+p_4)\,G_0^{(1)}(p_1,\,p_2,\,p_3,\,p_4) =$$



. (5.1.13)

Here each of the lines of the vertex factor (5.1.12) are multiplied by propagator factors which removes the little slashes and so



$$= -\lambda_0(2\pi)^n\delta^{(n)}(p_1+p_2+p_3+p_4)$$

$$\frac{1}{p_1^2+m_0^2}\,\frac{1}{p_2^2+m_0^2}\,\frac{1}{p_3^2+m_0^2}\,\frac{1}{p_4^2+m_0^2}\,. \tag{5.1.14}$$

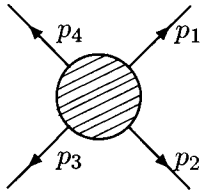
It should be emphasized that the graphs have now been given a precise, quantitative meaning.

Graphs with closed loops are illustrated by going on to the next order of the four-point Green's function and considering

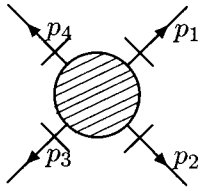
$$G_0^{(2)}(x_1, x_2, x_3, x_4) = \frac{1}{2}\left(\frac{-\lambda_0}{4!}\right)^2 \int (d_E^n y)(d_E^n z) \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y)\phi^4(z)\rangle^{(0)} . \quad (5.1.15)$$

The overall factor of $\frac{1}{2}$ comes from the expansion of the exponential of the interaction term to second order. To compute this contribution to the Green's function, one performs all the contractions using the propagator (5.1.9). Neglecting the disconnected graphs and self-energy insertions, a typical term involves the 4 ways $\phi(x_1)$ can be contracted with one of the four $\phi(y)$ and the 3 ways that $\phi(x_2)$ can be contracted with the three remaining $\phi(y)$ with similar contractions of $\phi(x_3)$ and $\phi(x_4)$ with $\phi^4(z)$ giving another factor of $4 \cdot 3$. Interchanging $\phi^4(y) \leftrightarrow \phi^4(z)$ gives another factor of 2. The remaining contractions of $\phi^2(y)$ with $\phi^2(z)$ produce a factor of 2 and so the overall combinatorial factor is given by $(4 \cdot 3)^2 \cdot 2 \cdot 2 = (4!)^2$. There are, of course, three different contributions corresponding to the interchanges $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_3$ corresponding to three separate Feynman graphs. The contractions of the $\phi(x_a)$ and the overall Fourier transform produce external propagator factors which are removed by using the graphical notation*

$$(2\pi)^n \delta^{(n)}(p_1 + p_2 + p_3 + p_4) G_0(p_1, p_2, p_3, p_4) =$$

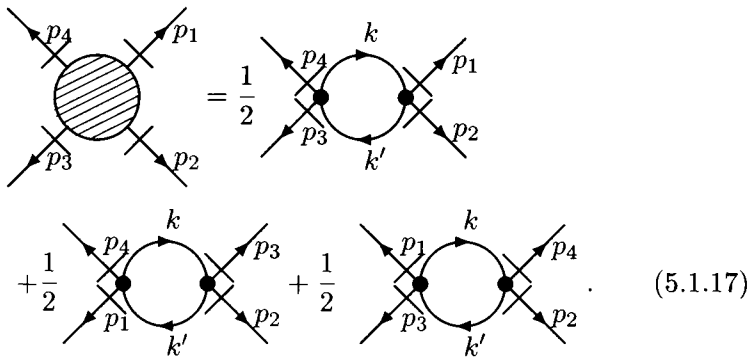


$$= \times \frac{1}{p_1^2 + m_0^2} \frac{1}{p_2^2 + m_0^2} \frac{1}{p_3^2 + m_0^2} \frac{1}{p_4^2 + m_0^2} . \quad (5.1.16)$$



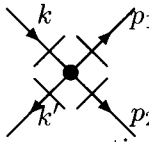
* This notation for removing the factors associated with the non-interacting propagators is convenient for the present work. It should be noted, however, that the fully truncated amplitudes are the more basic quantities. These involve the removal of the fully interacting, complete propagator factors associated with each of the external lines, with the resulting amplitudes truly single particle irreducible with respect to the external lines. This will be discussed in the next chapter.

Representing the remaining two factors of the propagator $\langle \phi(y)\phi(z)\rangle^{(0)}$ by the Fourier transform (5.1.9), the y and z integration produce δ-functions as in the vertex graph (5.1.12). It should be clear that the second order scattering contribution is given by the graphical decomposition



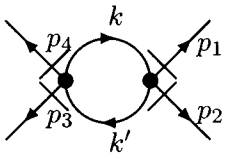
(5.1.17)

Here the internal lines labeled with k and k' involve an integration over $(d_E^n k)/(2\pi)^n$ and $(d_E^n k')/(2\pi)^n$ coming from the Fourier representation (5.1.9) of the two $\langle \phi(y)\phi(z)\rangle^{(0)}$ propagators. The line labeled by k', for example, is associated with the factor $\exp\{ik'(y-z)\}$. This corresponds to the outgoing momentum k' coming from the right-hand vertex but an incoming momentum going into the left-hand vertex. Thus, reversing the sense of the arrow on a line corresponds to reversing the sign of the corresponding momentum in the momentum-conserving δ-function attached to the vertex. Thus, for example,



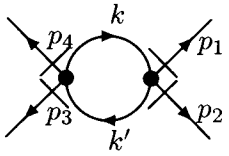
$$= -\lambda_0 (2\pi)^n \delta^{(n)}(p_1 + p_2 + k' - k) . \quad (5.1.18)$$

With these conventions, a typical graph above has the precise, quantitative form

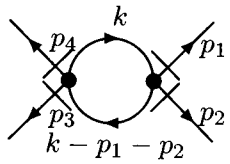


$$= \int \frac{(d_E^n k)}{(2\pi)^n} \int \frac{(d_E^n k')}{(2\pi)^n} (-\lambda_0)(2\pi)^n \delta^{(n)}(p_3 + p_4 + k - k') \frac{1}{k^2 + m_0^2} \frac{1}{k'^2 + m_0^2} (-\lambda_0)(2\pi)^n \delta^{(n)}(p_1 + p_2 + k' - k) . \quad (5.1.19)$$

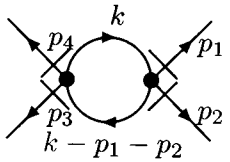
Using one of the δ-functions to constrain the argument of the other δ-function, the latter may be replaced by $\delta^{(n)}(p_1+p_2+p_3+p_4)$. This must, of course, be the case since the Green's function contains an overall factor of the δ-function of total momentum conservation. We shall adopt the convention that when the internal lines are labeled in such a way as to conserve momentum at each vertex (which may entail taking into accout the overall conservation of momentum carried by all the external lines coming into the graph), then the overall factor of the δ-function of total momentum conservation is removed. The remaining δ-functions from the vertices are used to "cancel" some of the integrations associated with internal lines. Thus we write



$$= \quad \times(2\pi)^n\delta^{(n)}(p_1+p_2+p_3+p_4)\,, \qquad (5.1.20)$$



in which



$$= (-\lambda_0)^2 \int \frac{(d_E^n k)}{(2\pi)^n}\,\frac{1}{k^2+m_0^2}\,\frac{1}{(k-p_1-p_2)^2+m_0^2}\,. \qquad (5.1.21)$$

The extension of these examples should be clear, and we shall not belabor it. The only remaining problem is to compute combinatorial factors. While general rules can be given to calculate these factors, it is not worth the trouble to remember them, and so we shall not emphasize them. For any given problem, the combinatorial factors can always be obtained by simply enumerating all the possible contractions, which is easy enough, although at times a little tedious. For example, it should now be clear that the evaluation of a rather complicated contribution to a Green's function shown in Fig. 5.2, including the proper overall factor, is given by[†]

[†] It is not hard to check that the the overall combinatorial factor is the factor of $\frac{1}{2}$ appearing in Eq. (5.1.22). In general, although we shall not prove it, the combinatorial factors work out to give an overall factor of $1/S$, where S is called the "symmetry factor" of the graph. It is the order of the symmetry (permutation) group of the graph when all the external lines and vertices are held fixed. For the graph in Fig. 5.2, the symmetry group consists of the identity and the interchange of the lines labeled by k' and k'', and thus $S = 2$. The graphs in Eq. (1.17) also have $S = 2$ corresponding to the symmetry $k \leftrightarrow k'$ there. The symmetry factor $S = 3!$ appears in Eq. (4.5.16) because the internal lines in Fig. 4.11 can be placed in 3! different orderings. A graph of order λ_0^n is obtained by expanding the exponential of the interaction Lagrangian to n-th order, giving a factor of $(1/4!)^n\,1/n!$. To give a rough explanation of the appearance of the symmetry factor S, we note that there are $n!$ equivalent orderings of the resulting 4-point vertices, and the overall $1/n!$ factor is canceled. Moreover, each of the four vertex legs in each of the n vertices can be contracted in 4! different ways, canceling the overall factor of $(1/4!)^n$. All these cancellations occur unless there are symmetries that make this enumeration over count; hence the factor $1/S$.

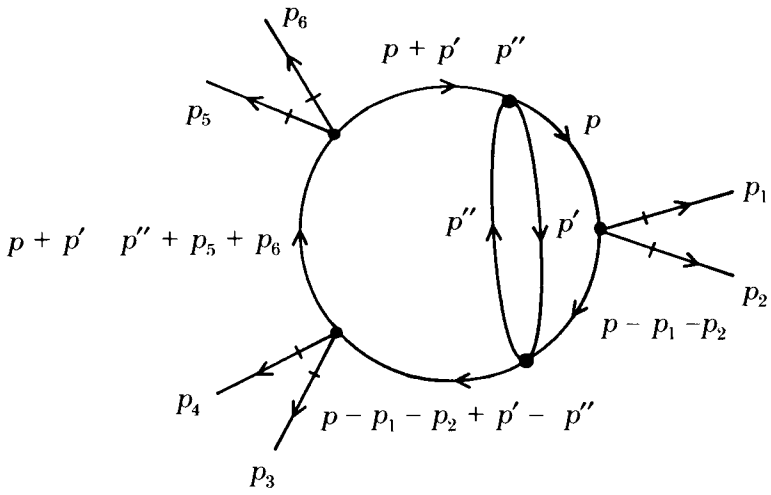


Fig. 5.2. A more complicated example.

$$\frac{1}{2}(-\lambda_0)^5 \int \frac{(d_E^n p)}{(2\pi)^n}\,\frac{(d_E^n p')}{(2\pi)^n}\,\frac{(d_E^n p'')}{(2\pi)^n}\,\frac{1}{p^2+m_0^2}\,\frac{1}{(p-p_1-p_2)^2+m_0^2}$$
$$\frac{1}{(p-p_1-p_2+p'-p'')^2+m_0^2}\,\frac{1}{(p+p'-p''+p_5+p_6)^2+m_0^2}$$
$$\frac{1}{(p+p'-p'')^2+m_0^2}\,\frac{1}{p'^2+m_0^2}\,\frac{1}{p''^2+m_0^2}\,. \qquad (5.1.22)$$

Each internal line has a propagator factor with the labeled momentum. One starts with an arbitrary internal line and assigns to it a directed momentum k. The momentum flow through a vertex attached to this line is then examined. (In the graph above, we chose to go clockwise starting with the upper, right-hand segment of the graph, but this was an arbitrary choice, and we could have chosen any other internal line and also gone in the counter-clockwise direction.) If this vertex is connected

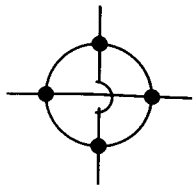


Fig. 5.3. A non-planar, three-loop graph.

to only one other internal line, then the conservation of the momentum flow through this vertex determines the momentum assigned to this next internal line. Otherwise, one or more new independent momentum variables (k', k'') are assigned to the lines whose momenta are not determined by the conservation of momenta at the vertex. One progresses in this way throughout the graph until the momenta of all the internal lines have been assigned. The overall conservation of the external momenta flowing into the graph, the momenta carried by the external lines, may be used to relabel the momentum carried by an internal line. (For example, the line labeled by $k + k' - k'' + p_5 + p_6$ could equally well be labeled by $k + k' - k'' - p_1 - p_2 - p_3 - p_4$.) Any other labeling of the internal momenta yields exactly the same result. Such a relabeling corresponds to some sequence of translations of the momentum variables. (For example, we may translate k by the addition of $p_1 + p_2$ to relabel the line carrying momentum $k - p_1 - p_2$ with the label k.) These translations can include internal as well as external momentum displacements.

In general, the number of independent internal momentum variables corresponds to the number of "independent closed loops" that a graph contains. The graph in Fig. 5.2 is a three-loop graph in a fairly obvious way. This is because it is a "planar graph" — it can be drawn with no crossing lines other than those connected by a vertex. The situation is a little more subtle for a non-planar graph such as that shown in Fig. 5.3.

5.2 Divergence structure

1 Power counting

We turn now to investigate the general character of the ultraviolet divergencies which appear in quantum field theory. We start by examining a few simple examples in which the divergencies are exhibited by introducing a cutoff K that limits the large loop momenta. We shall concentrate on the leading divergencies which appear in the limit of large momenta. Thus we may neglect the mass terms and the external momenta in the propagator factors that correspond to the internal lines in a graph. The

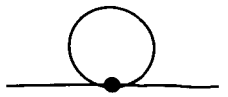


Fig. 5.4. Lowest order self-energy graph.

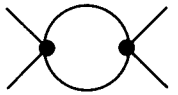


Fig. 5.5. Scattering amplitude loop graph.

self-energy loop shown in Fig. 5.4 has the divergent character

$$\int^K (d^n k) \frac{1}{k^2} \sim K^{n-2} . \tag{5.2.1}$$

It is logarithmically divergent in two dimensions, $n = 2$, and quadratically divergent in four dimensions, $n = 4$. This is reflected in the dimensional continuation method where the simple loop produces the gamma function $\Gamma(1 - \frac{n}{2})$ which has poles at $n = 2$ and $n = 4$ — the fact that there is already a pole at $n = 2$ signals that the pole at $n = 4$ corresponds to a quadratic divergence. The scattering amplitude of Fig. 5.5 has the divergent character

$$\int^K (d^n k) \left(\frac{1}{k^2}\right)^2 \sim K^{n-4} . \tag{5.2.2}$$

It is logarithmically divergent in four dimensions. Correspondingly, the dimensional continuation method produces $\Gamma(2 - \frac{n}{2})$ whose first pole appears at $n = 4$. In four dimensions, one-loop graphs with a larger number of vertex insertions converge. For example the graph of Fig. 5.6 behaves as

$$\int^K (d^n k) \left(\frac{1}{k^2}\right)^3 \sim K^{n-6} , \tag{5.2.3}$$

and it converges at $n = 4$. However, if space-time had six dimensions, this graph would have a logarithmic divergence. In addition to the dimension-

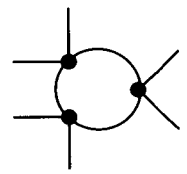


Fig. 5.6. A convergent loop graph.

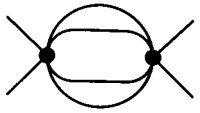


Fig. 5.7. A highly divergent graph.

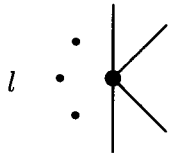


Fig. 5.8. The number of vertex legs is denoted by l.

ality of space-time, the degree of divergence also depends on the number of fields that are coupled together by the interaction. For example, a ϕ^6 coupling produces the graph in Fig. 5.7, which has four propagator factors and three independent loop momenta. The leading divergence appears when all the loop momenta become large. This region may be treated by considering the three n-vector loop momenta to be components of a $3n$-vector, and so the graph has the divergence behavior

$$\int^K (d^{3n}k)\left(\frac{1}{k^2}\right)^4 \sim K^{3n-8}. \tag{5.2.4}$$

It has a quartic divergence in four dimensions. The divergence structure worsens if the dimensionality of space-time is increased or if the number of fields coupled together is increased.

The simple examples which have been discussed illustrate the general method of "power counting" which shall now be described. We consider the general case in n space-time dimensions with an interaction* ϕ^l that couples l fields together. It is represented by the vertex graph with l external legs sketched in Fig. 5.8. These vertices appear within graphs with N external lines as indicated in Fig. 5.9 which give pieces of the N-point Green's function $G(p_1, p_2, \cdots, p_N)$. A particular graph has I internal lines connecting V_l l-leg vertices as illustrated in Fig. 5.10. Each of the I internal lines corresponds to an integration $\int(d^nk)$ and, since we are concerned only with the large momentum behavior, a propagator factor $1/k^2$. Each of the V_l vertices has a momentum conserving $\delta^{(n)}(k_1 + k_2 + \cdots + k_l)$. The product of the V_l δ-functions produce one $\delta^{(n)}(p_1 + p_2 +$

* The interaction could also involve derivatives of the field. Such interactions produce additional momentum factors in the loop integrals and thus worsen the divergence structure. We do not consider these derivative interactions so as to simplify the discussion — it is not difficult to generalize our considerations so as to include them.

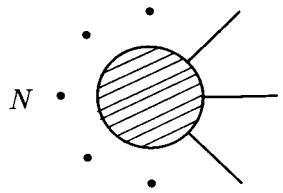


Fig. 5.9. The number of external legs in a graph is denoted by N.

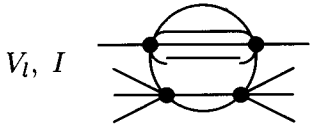


Fig. 5.10. The number of internal lines in a graph is denoted by I. The number of l-leg vertices is called V_l.

$\cdots + p_N)$ that gives the overall momentum conservation in the Green's function $G(p_1, p_2, \cdots p_N)$ which does not constrain the integration over the momentum of the I internal lines. Hence L, the number of independent $\int(d^nk)$ integrations, is given by

$$L = I - (V_l - 1). \tag{5.2.5}$$

This number is the number of "loops" in the graph. We now have the power counting result in hand. The degree of divergence D - with the graph diverging as K^D — is given by

$$D = n\,L - 2I. \tag{5.2.6}$$

The first term on the right-hand side counts the number of momentum integrations, the second the number of propagator factors $1/k^2$. We have yet to make use of a topological relation: Each vertex "radiates" l lines with some of these l lines giving external lines, the remaining lines being internal lines. Each internal line connects two vertices, it is shared by two vertices. Hence

$$l\,V_l = N + 2I. \tag{5.2.7}$$

Combining Eqs. (5.2.5), (5.2.6) and (5.2.7) gives

$$D = n - \frac{1}{2}(n-2)N + \frac{1}{2}V_l[(n-2)(l-2) - 4]. \tag{5.2.8}$$

This expresses the degree of divergence of a given graph with N external lines in an n-dimensional space-time in terms of the number of vertices V_l

which couple l fields together. That is, for a given graph with N external lines and with n and l fixed, D is essentially dependent upon the number of vertices V_l. Another way to state the significance of the index D is to note that if all the internal momenta k are scaled by a common factor Λ, then the entire graph scales as Λ^D. The index D is properly called the *superficial degree of divergence*. If $D \geq 0$, the graph certainly diverges. On the other hand, if $D < 0$ it may converge but, as we shall see shortly, this is not a sufficient condition for convergence.

Before proceeding to discuss the nature of the divergencies in more detail, it is instructive to give an alternative derivation of the index D based on dimensional analysis. We recall that since the action integral is dimensionless, $m^2\phi^2$ must have the mass dimension M^n and so the dimension of the field appears as

$$[\phi]_d = M^{(n-2)/2}. \tag{5.2.9}$$

Again requiring the action to be dimensionless constrains the dimensions of the ϕ^l interaction coupling constant λ_l. With $\lambda_l \phi^l$ having the dimension M^n,

$$[\lambda_l]_d = M^P, \tag{5.2.10}$$

where

$$P = -\frac{1}{2}[(l-2)(n-2) - 4]. \tag{5.2.11}$$

The N-point Green's function has the dimensionality

$$[G(p_1, p_2, \cdots, p_N)\delta^{(n)}(\Sigma p)]_d = M^{-nN} M^{N(n-2)/2}, \tag{5.2.12}$$

where the first factor comes from the Fourier transform and the second factor from the dimensionality of the fields. Removing the dimensional factor M^{-n} associated with $\delta^{(n)}(\Sigma p)$ yields

$$[G(p_1, p_2, \cdots, p_N)]_d = M^Q, \tag{5.2.13}$$

where

$$Q = n - \frac{1}{2}(n+2)N. \tag{5.2.14}$$

This dimensionality is accounted for by the dimensions of the coupling factors minus the dimensions of the propagator factors for the N external lines plus the degree of divergence D of the graph — the graph must behave as K^D so as to soak up the extra dimensions. Thus

$$Q = V_l P - 2N + D. \tag{5.2.15}$$

Using Eqs. (5.2.14) and (5.2.11), one sees that this is just the result (5.2.8). Note that if $D < 0$, then dimensional factors involving the external momenta and the masses make up the difference in the counting of dimensions. But if $D > 0$, the extra dimensions can only arise from divergencies in the loop integrations.

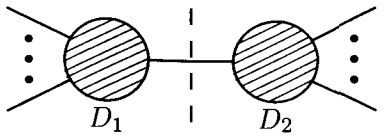


Fig. 5.11. Single-particle reducible graph.

To see that D does measure only a "superficial" degree of divergence, we consider the class of "single-particle reducible" graphs. A graph is single-particle reducible if it can be cut into two separate pieces by cutting a single line as shown in Fig. 5.11. Such graphs contain two subgraphs with superficial degrees of divergence D_1 and D_2 joined by a propagator line which is not part of a loop integral. Since the intermediate propagator contributes -2 to the dimension, the overall superficial degree of divergence for the complete graph is given by

$$D = D_1 + D_2 - 2. \tag{5.2.16}$$

If a graph is single particle reducible, it may contain divergent subgraphs even though the overall degree of divergence is negative. This is why the degree is called "superficial". The problem is easily dealt with by considering only single-particle irreducible graphs, only graphs that cannot be broken in two by cutting a single line.

This is not, however, the end of the story. Single-particle irreducible graphs may still diverge even though $D < 0$. This happens with graphs which contain a subgraph that diverges. A subgraph is defined by fixing some subset of the internal loop momenta and considering those lines to be external lines. The remaining loop graphs comprise the subgraph. An example of such a subgraph divergence is provided by the graph in Fig. 5.2 in four dimensions, $n = 4$. It follows from Eq. (5.2.8) — or from a simple examination of the graph — that $D = -2$. The divergent subgraph is exhibited by fixing the overall loop momentum k and the momentum $k - p_1 - p_2 + k' - k''$ on the line at the bottom of the graph. Then k'' becomes a linear function of k' and the subloop labeled by k' and k'' has a degree of divergence = 0; this subgraph corresponds to the one-loop correction to the scattering amplitude which has a logarithmic divergence in four dimensions.

The necessary and sufficient condition for a general graph to be finite is easily stated.

Theorem: A graph is finite if its superficial degree of divergence is negative and the superficial degrees of divergence of

all its subgraphs are also negative.

We shall not prove this very plausible theorem.

2 Renormalization counter terms

The divergencies which we have encountered are removed, order by order, by the following procedure. First the theory must be regulated in some manner so as to make it well defined. This we do by the continuation to a space-time of n dimensions, but other methods could be employed. The bare parameters are then expanded in terms of the renormalized parameters with the expansion containing counter terms that diverge when the regulation is removed. In our case, the counter terms appear as poles of various orders in $n-4$ which diverge when $n \to 4$. The divergencies brought about by the appearance of the counter terms (which involve higher powers of the coupling) in lower order graphs are chosen so that they cancel the divergencies in higher order graphs, with the sum, order-by-order in perturbation theory, remaining finite when the regulator is removed. The remarkable thing is that once the counter terms have been chosen to make a few basic processes finite in the limit where the regulator is removed, the same counter terms make all processes finite in this limit.

To see how this goes, we first consider the very simple case of two space-time dimensions. With $n = 2$, the superficial degree of divergence is given by

$$D = 2(1 - V_l)\,. \tag{5.2.17}$$

In this case D is independent of the number of external lines and also does not depend upon the number of legs attached to the vertex. Graphs with more than one vertex are superficially convergent. Graphs with one vertex are superficially logarithmically divergent. To be specific, let us consider $l = 4$ giving a $\lambda\phi^4$ interaction. Note that with $n = 2$, the field ϕ is dimensionless, and so the coupling λ has the dimensions of mass squared. The simplest graph is the self-energy loop of Fig. 5.4 which contains the logarithmically divergent loop integral $\int (d^2k)/k^2$. This divergence is removed by a mass renormalization

$$m_0^2 = m^2 - \delta m^2\,. \tag{5.2.18}$$

Regulating the theory by dimensional continuation and using minimal subtraction gives

$$\delta m^2 = \text{const.}\,\frac{\lambda}{n-2}\,. \tag{5.2.19}$$

Expanding the propagator to first order in δm^2 produces the correction

$$\frac{1}{k^2+m^2}\,\delta m^2\,\frac{1}{k^2+m^2}\,, \tag{5.2.20}$$

corresponding to the graph of Fig. 5.12. This correction cancels the divergence of the self-energy loop of Fig. 5.4. The divergencies in the iteration of the self-energy graph shown in Fig. 5.13 are cancelled by the δm^2 correction shown in Fig. 5.15 and by the expansion of the propagator to second order in δm^2 shown in Fig. 5.16. The sum of these graphs and their higher-order iterations corresponds to the expansion of the propagator in powers of the single-particle irreducible self-energy function, with this function approximated by the simple loop which is renormalized by the subtraction of δm^2.

The self-energy function has the second-order correction exhibited in Fig. 5.16. The divergence brought about by the top loop is canceled by the mass counter term correction to the simple loop graph of Fig. 5.4 shown in Fig. 5.17. The sum of the two graphs is finite. Such cancellations always occur. For example, the divergence brought about by the bubble inside the graph in Fig. 5.18 is canceled by the mass counter term correction to the lower-order graph shown in Fig. 5.19. Again, the sum of the two graphs is finite. Graphs with several divergent bubbles are made finite in the same way by the subtractions which occur when the bare mass terms in the propagators in lower-order graphs are expanded in terms of the renormalized mass with the counter term given in Eq. (5.2.19). Thus, the mass renormalization of Eq. (5.2.18) together with the counter term (5.2.19) renders the two-dimensional theory with the $\lambda\phi^4$ interaction finite.†

Although the renormalization of the two-dimensional theory is particularly simple, the general nature of this renormalization does hold for arbitrary dimensions. We shall simply state the result.

> *Theorem*: All the necessary parameter renormalizations are contained in the superficially divergent ($D \geq 0$) graphs. The divergent terms are polynomials of finite order in the external momenta, the momenta carried into or out of the graph by the external particles. Using dimensional regularization with minimal subtraction, the coefficients in these polynomials contain

† The disconnected vacuum graphs also need renormalization, but these usually play no role. It should be noted that the situation in a two-dimensional space-time is peculiar in that the degree of divergence $D = 2(1 - V_l)$ is independent of the number l of fields which are coupled together by the interaction. The reason for this comes from the fact that the field ϕ is dimensionless in two space-time dimensions. Hence all the coupling constants in an interaction $\lambda_l\,\phi^l$ have the same dimension, $[\lambda_l]_d = M^2$. It follows that any interaction in two dimensions is renormalizable. This is made explicit in the working of Problem 2.

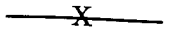


Fig. 5.12. Mass renormalization counter term.

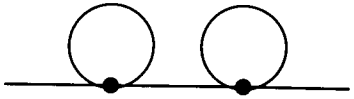


Fig. 5.13. Iteration of the lowest order self-energy graph.

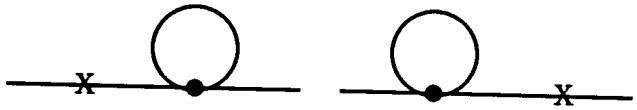


Fig. 5.14. Mass counter term corrections.

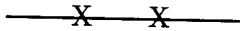


Fig. 5.15. Mass renormalization counter term iterated.

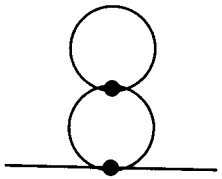


Fig. 5.16. Second-order self-energy graph.

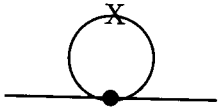


Fig. 5.17. Counter term correction of self-energy loop.

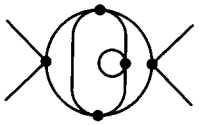


Fig. 5.18. Bubble subgraph divergence.

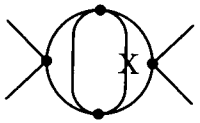


Fig. 5.19. Mass counter term correction to previous graph.

positive integer powers of the parameters of the theory (times poles in $n-4$). The coefficients contain powers of parameters that bear dimension (such as the squared mass) such that the mass dimension of each term in the polynomial is the superficial degree of divergence D. Subgraph divergencies in graphs which are superficially convergent ($D<0$) are canceled by expanding the bare parameters in lower-order graphs in terms of the finite, renormalized parameters with divergent counter terms.

The renormalization process works as follows: Suppose that the theory has been renormalized to some order. Going on to the next order, any superficially convergent graph will be finite if the appropriate subtractions are included. These are the subtractions in lower-order graphs brought about by the expansion of bare parameters in terms of renormalized parameters and counter terms calculated to the previous order. The divergencies in the remaining superficially divergent graphs are removed by additional counter terms giving parameter renormalizations at the new order.

Let us now return to the $\lambda\phi^4$ theory in four dimensions. The general formula (5.2.8) for the degree of superficial divergence gives in this case

$$D = 4 - N\,. \tag{5.2.21}$$

This informs us that only graphs with $N=2$ or 4 external lines have basic divergencies. These are removed by mass, wave function, and coupling constant renormalizations.

Now suppose, for example, that there is a ϕ^6 interaction. Then, with $n=4$ and $l=6$, Eq. (5.2.8) gives

$$D = 4 - N + 2\,V_6\,. \tag{5.2.22}$$

A graph with N external lines will always diverge if it contains a sufficient number of vertices V_6. For example, taking $N=8$, we find a divergence in a loop with two vertices, $V_6=2$. This infinity is removed by the introduction of a new ϕ^8 interaction with a divergent coupling. But going on to higher orders, new $\phi^{10}, \phi^{12}, \cdots$ interactions must be introduced to render the theory finite. Thus, although we started with a theory with only one coupling constant, an infinite number of coupling constants must be introduced in order to renormalize the theory. Each of these has an infinite part to make the theory finite, but after the infinities have been canceled there remains an infinite number of arbitrary finite parameters. Such a theory is called *non-renormalizable*. It follows from Eq. (5.2.8) that theories in four dimensions with an interaction ϕ^l with $l \geq 5$ are non-renormalizable. Such theories involve coupling constants whose di-

mensions are those of a negative power of mass. Referring to Eqs. (5.2.10) and (5.2.11), we see that, for $n = 4$,

$$[\lambda_l]_d = M^{(4-l)} . \tag{5.2.23}$$

This dimensionality accounts for the non-renormalizability: The perturbation theory for any N-point Green's function must diverge in some sufficiently high order since a large power of λ_l entails a large inverse mass dimension that can only be compensated by a positive power of the momentum cut off K. In general, if a theory has a coupling constant which has the dimension of some negative power of mass, then the theory is non-renormalizable.‡

To complete the catalog, we note that in dimensions higher than four, $n > 4$, Eq. (5.2.8) shows that the degree of divergence D involves the number of vertices V_l with a positive coefficient. Hence theories in higher dimensional space-time worlds are not renormalizable. The only notable exception (excluding $n = 5, l = 3$) is $n = 6$, $l = 3$ which gives $D = 6-2N$. The ϕ^3 theory in six-dimensional space-time is renormalizable; it is made finite by coupling and mass renormalizations. However, although the renormalized perturbation theory makes sense and exists term by term, the full theory cannot exist since the interaction ϕ^3 is not positive-definite. The energy of this theory is not bounded from below, and hence the theory must be unstable. Nonetheless, the ϕ^3 theory in six dimensions is of interest since its perturbative development has the asymptomatic freedom described in the following section of this chapter. In this connection, see Problem§ 3.

One important aspect of renormalization has yet to be discussed. It is illustrated by a theory containing two scalar fields ϕ_1, ϕ_2 in four dimensions. Suppose that the theory initially contained a self interaction of the form $\lambda_1 \phi_1^4$ and a coupling between the fields $\lambda_{12} \phi_1^2 \phi_2^2$, but no self interaction for the field ϕ_2. In order λ_{12}^2 there is a one-loop contribution to the $\phi_2 - \phi_2$ elastic scattering amplitude with the ϕ_1-field propagating about the loop as shown in Fig. 5.20. This graph diverges. The divergence is removed by a counter term multiplying ϕ_2^4. Thus, although there was initially no $\lambda_2 \phi_2^4$ self interaction, the coupling between the two fields induces one. It is not meaningful to take the renormalized coupling λ_2 to

‡ Although non-renormalizable theories can play no fundamental role in physics, they can be useful in describing processes over a limited energy range as has been discussed previously. Thus, although most of our work will be with renormalizable theories, one should keep in mind that the others are sometimes useful.

§ Our discussion has been for the interaction ϕ^l. The power counting method can be extended to interactions containing derivatives. For example, it is easy to see that the interaction $\phi^2(\partial_\mu \phi)^2$ is not renormalizable in four dimensions as are the other interactions involving derivatives.

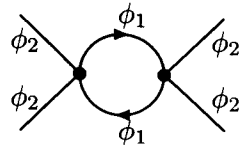


Fig. 5.20. Lowest-order loop graph involving the interaction between the two ϕ_1 ϕ_2 fields which yields a self interaction for the ϕ_2 field alone. This graph diverges and must be renormalized. Thus it is not consistent to initially omit a coupling describing the self interaction of the ϕ_2 field.

vanish with the bare coupling $\lambda_{2,0}$ given only by the counter terms needed to render the theory finite. The presence of these counter terms implies that the β-function for λ_2 does not vanish. Hence, even if λ_2 were chosen to vanish for some unit of mass μ, it would not vanish for another mass unit μ'. In general, a theory must contain all the dimensionless coupling constants which are associated with all possible renormalizable interactions. It is consistent, however, to take parameters bearing dimensions to vanish if they are renormalized in a homogeneous manner. For example, if the theory contains a single mass parameter, it is consistent to take this mass to vanish since it is renormalized only by a multiplicative factor. On the other hand, in the ϕ_1, ϕ_2 theory which was just considered, it is not consistent to take one running mass to be finite and the other to vanish, say $m_1(\mu) \neq 0$, $m_2(\mu) = 0$, since the masses mix under renormalization. Already in first order in λ_{12} the ϕ_2-field propagator has a one-loop self energy graph of the sort shown in Fig. 5.4 with the ϕ_1 propagator comprising the loop line. This graph gives a mass to the ϕ_2 field which must be renormalized. This renormalization gives $m_2(\mu)$ a non-vanishing δ-function, and so $m_2(\mu)$ must change when the mass unit μ is changed; it is not possible to assert that $m_2(\mu) = 0$. It is, of course, possible to assume that the physical mass of the ϕ_2 particle does vanish, but this requires a "fine tuning" of the running parameters of the theory.

5.3 Parameter renormalization

The renormalization of the $\lambda\phi^4$ theory to lowest order was described in Chapter 4. In the course of that work, the connection of the renormalization process to the renormalization group was introduced. We now present a general description of the character of the renormalization of the parameters which enter into the theory and describe the renormalization group more fully. The work which is about to be done has several motivations. The structure of renormalization with dimensional continuation will be made clear and precise. This work will aid the later study of

the constraints which the renormalization group imposes on Green's functions. These constraints will tell us how the theory behaves in asymptotic limits. Finally, the work in this section is needed for the later discussion of composite operators and the operator product expansion, topics which further reveal the nature of quantum field theory and which are also of considerable practical utility.

We start with the renormalization of the coupling λ. Previously we calculated the first two terms in the perturbative expansion of the scattering amplitude as shown in Eq. (3.5.15) of Chapter 3. The first term is a "tree" graph and the second a one-loop graph. Higher order corrections correspond to graphs with several closed loops. Using the dimensional continuation method we found that the lowest order coupling renormalization has the structure

$$\lambda_0 = \lambda\mu^{4-n}\left[1 + \frac{a_{12}\lambda}{n-4}\right]. \tag{5.3.1}$$

The correction of the tree graph by the divergent pole involving the coefficient a_{12} cancels the divergencies in the one-loop graphs. Going on to the next order gives

$$\lambda_0 = \lambda\mu^{4-n}\left[1 + \frac{a_{12}\lambda + a_{13}\lambda^2}{n-4} + \frac{a_{23}\lambda^2}{(n-4)^2}\right]. \tag{5.3.2}$$

Here the pure numbers a_{13} and a_{23} are determined by requiring that their tree contribution cancels the divergent terms in the two-loop graphs plus the divergencies in the one-loop graphs brought about by the renormalization of their parameters. Going on to higher order continues the "triangular structure" displayed in Eq. (5.3.2) in which the higher order poles start out with a higher order of the coupling λ. In general we write

$$\lambda_0 = \lambda\mu^{4-n} F_\lambda(\lambda; n), \tag{5.3.3}$$

in which the function $F_\lambda(\lambda; n)$ has the form

$$F_\lambda(\lambda; n) = 1 + \text{poles}. \tag{5.3.4}$$

This is the structure of *minimal subtraction*: Except for the unit of mass μ which displays the dimensions away from $n = 4$, the bare and renormalization couplings differ only by a series of pure pole terms whose residues are functions only of the renormalized coupling λ.

As has been discussed several times before, the bare coupling λ_0 is an independent parameter in the theory which knows nothing of the arbitrary unit of mass μ,

$$\mu\frac{d\lambda_0}{d\mu} = 0. \tag{5.3.5}$$

On the other hand, the renormalization relation (5.3.3) forces λ to be a running coupling with

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda; n). \tag{5.3.6}$$

Since λ is finite, β must be a finite function of the renormalized coupling λ and the space-time dimensionality n. In particular, it cannot contain poles in $(n-4)$ but rather only positive powers of $(n-4)$. To examine the consequences of these conditions, we first write

$$F_\lambda(\lambda; n) = \exp\{f(\lambda; n)\}, \tag{5.3.7}$$

where $f(\lambda; n)$ is a series of poles in $(n-4)$ of increasing order with no constant term. Using Eq. (5.3.7) in Eq. (5.3.3) and then taking $\mu d/d\mu$ of the result with the conditions (5.3.5) and (5.3.6) produces

$$(4-n)\lambda + \beta(\lambda; n)\left[1 + \lambda\frac{\partial}{\partial\lambda} f(\lambda; n)\right] = 0. \tag{5.3.8}$$

Since $f(\lambda; n)$ has only poles in $(n-4)$, $\beta(\lambda; n)$ must have the structure

$$\beta(\lambda; n) = (n-4)\lambda + \beta(\lambda), \tag{5.3.9}$$

where $\beta(\lambda)$ is independent of n. Then writing

$$f(\lambda; n) = \frac{f_1(\lambda)}{n-4} + \frac{f_2(\lambda)}{(n-4)^2} + \cdots \tag{5.3.10}$$

and examining the terms in Eq. (5.3.8) that have no pole one sees that

$$\beta(\lambda) = -\lambda^2\frac{\partial f_1(\lambda)}{\partial\lambda}. \tag{5.3.11}$$

Expanding the exponential in Eq. (5.3.7) in powers of $f(\lambda)$ shows that $f_1(\lambda)$ is also the residue of the simple pole in $(n-4)$ in $F_\lambda(\lambda; n)$. Hence we find that the β–function is completely determined by the simple pole of the renormalization function $F_\lambda(\lambda; n)$.

Equation (5.3.8) now reduces to an infinite number of constraints: The total residue of each pole $(n-4)^{-k}$ must vanish for all k. The vanishing of the first-order pole requires that

$$\frac{1}{n-4}\left\{\beta(\lambda)\lambda\frac{\partial f_1(\lambda)}{\partial\lambda} + \lambda^2\frac{\partial f_2(\lambda)}{\partial\lambda}\right\} = 0, \tag{5.3.12}$$

or, in view of Eq. (5.3.11),

$$\frac{\partial f_2(\lambda)}{\partial\lambda} = \frac{1}{\lambda^3}\beta(\lambda)^2. \tag{5.3.13}$$

Since $f_2(0) = 0$, this determines $f_2(\lambda)$. Thus $f_2(\lambda)$ is fixed by $\beta(\lambda)$ which in turn is given by the residue of the first-order pole $f_1(\lambda)$; the second-order pole is completely determined by the first-order pole. And so it

goes for every higher-order pole. It is easy to get the complete solution. Placing the form (5.3.9) in Eq. (5.3.8) gives

$$\Big[(n-4)\lambda+\beta(\lambda)\Big]\frac{\partial f(\lambda;n)}{\partial\lambda}=-\frac{\beta(\lambda)}{\lambda}\,, \tag{5.3.14}$$

which has the solution

$$f(\lambda;n)=-\int_0^\lambda\frac{d\lambda'}{\lambda'}\,\frac{\beta(\lambda')}{(n-4)\lambda'+\beta(\lambda')}\,. \tag{5.3.15}$$

Expanding the denominator in the integral in powers of $\beta(\lambda')$ produces a series of ascending poles in $(n-4)$. Expanding the exponential in Eq. (5.3.7) then expresses $F_\lambda(\lambda;n)$ as unity plus an ascending series of poles. The higher-order poles involve higher powers of $\beta(\lambda)$ and therefore higher powers of the coupling λ. To any given order in λ, there are only a finite number of poles.

Putting together the formulae gives

$$\lambda_0=\lambda\,\mu^{4-n}\exp\left\{-\int_0^\lambda\frac{d\lambda'}{\lambda'}\,\frac{\beta(\lambda')}{(n-4)\lambda'+\beta(\lambda')}\right\}. \tag{5.3.16}$$

This defines the running coupling $\lambda(\mu)$. Writing down similar expressions for $\lambda(\mu_2)$ and $\lambda(\mu_1)$ and dividing the two produces

$$1=\frac{\lambda(\mu_2)}{\lambda(\mu_1)}\left(\frac{\mu_2}{\mu_1}\right)^{4-n}\exp\left\{-\int_{\lambda(\mu_1)}^{\lambda(\mu_2)}\frac{d\lambda'}{\lambda'}\,\frac{\beta(\lambda')}{(n-4)\lambda'+\beta(\lambda')}\right\}. \tag{5.3.17}$$

This expression is finite in the $n\to4$ limit and is, in fact, the solution to the renormalization group equation (5.3.6). To see how this goes, we add and subtract $(n-4)\lambda'$ in the numerator to isolate

$$-\int_{\lambda(\mu_1)}^{\lambda(\mu_2)}\frac{d\lambda'}{\lambda'}=\ln\left[\frac{\lambda(\mu_1)}{\lambda(\mu_2)}\right], \tag{5.3.18}$$

whose exponential cancels the first factor in Eq. (5.3.17). Taking the logarithm of the resulting equality and striking out a common factor of $(n-4)$ yields

$$\ln\left(\frac{\mu_2}{\mu_1}\right)=\int_{\lambda(\mu_1)}^{\lambda(\mu_2)}\frac{d\lambda'}{(n-4)\lambda'+\beta(\lambda')}\,. \tag{5.3.19}$$

This (implicitly) determines $\lambda(\mu_2)$ in terms of $\lambda(\mu_1)$ and μ_2/μ_1.

The mass renormalization appears as

$$m_0^2=m^2\,F_m(\lambda;n)\,, \tag{5.3.20}$$

in which $F_m(\lambda;n)$ is equal to unity plus an ascending series of poles. Alternatively, one may write

$$F_m(\lambda;n)=\exp\{g(\lambda;n)\}\,, \tag{5.3.21}$$

where $g(\lambda;n)$ only contains pole terms. Again we have

$$\mu\frac{dm_0^2}{d\mu}=0\,, \tag{5.3.22}$$

while

$$\mu\frac{dm^2}{d\mu}=m^2\delta(\lambda;n)\,, \tag{5.3.23}$$

with $\delta(\lambda;n)$ finite in the $n\to4$ limit. Using Eq. (5.3.21) in Eq. (5.3.20) and requiring that $\mu\,d\,m_0^2/d\mu=0$ gives

$$\mu\frac{dm^2}{d\mu}+m^2\frac{\partial g}{\partial\lambda}\,\mu\frac{d\lambda}{d\mu}=0\,, \tag{5.3.24}$$

or

$$\delta(\lambda;n)=-[(n-4)\lambda+\beta(\lambda)]\,\frac{\partial g(\lambda;n)}{\partial\lambda}\,. \tag{5.3.25}$$

Writing

$$g(\lambda;n)=\frac{g_1(\lambda)}{n-4}+\frac{g_2(\lambda)}{(n-4)^2}+\cdots \tag{5.3.26}$$

in Eq. (5.3.25), we conclude that $\delta(\lambda;n)=\delta(\lambda)$ must be independent of the dimensionality n with

$$\delta(\lambda)=-\lambda\frac{\partial g_1(\lambda)}{\partial\lambda}\,. \tag{5.3.27}$$

Therefore, just as was the case for $\beta(\lambda)\,,\delta(\lambda)$ is determined by the residue of the simple pole in the mass renormalization. Equation (5.3.25) is now easily integrated to put the mass renormalization in the form

$$F_m\,(\lambda;n)=\exp\left\{-\int_0^\lambda d\lambda'\,\frac{\delta(\lambda')}{(n-4)\lambda'+\beta(\lambda')}\right\}. \tag{5.3.28}$$

The expansion of this result in powers of $\beta(\lambda')$ produces an ascending series of poles. Again we see that all the higher-order poles are determined by the simple poles. This explains the apparently "miraculous cancellation" noted at the end of the previous chapter. We can now see in detail how the structure of the second-order mass renormalization (4.5.44) is obtained. To second order we found [Eq. (4.5.54)] that

$$\delta(\lambda)=\left[\frac{\lambda}{(4\pi)^2}\right]-\frac{5}{6}\left[\frac{\lambda}{(4\pi)^2}\right]^2, \tag{5.3.29}$$

and [Eq. (3.5.43)] that

$$\beta(\lambda)=\frac{3\lambda^2}{(4\pi)^2}\,. \tag{5.3.30}$$

To this order

$$
\begin{aligned}
&-\int_0^\lambda d\lambda' \frac{\delta(\lambda')}{(n-4)\lambda'+\beta(\lambda')} \\
&\simeq -\frac{1}{n-4}\int_0^\lambda d\lambda' \frac{\delta(\lambda')}{\lambda'} + \frac{1}{(n-4)^2}\int_0^\lambda d\lambda' \frac{\delta(\lambda')\beta(\lambda')}{\lambda'^2} \\
&\simeq -\frac{1}{n-4}\left\{\left[\frac{\lambda}{(4\pi)^2}\right] - \frac{5}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2\right\} + \frac{1}{(n-4)^2}\frac{3}{2}\left[\frac{\lambda}{(4\pi)^2}\right]^2 .
\end{aligned}
\tag{5.3.31}
$$

Placing this in the exponent in Eq. (5.3.28) and expanding out the exponential to order λ^2 gives precisely the result (4.5.43) found previously in Chapter 4.

Equation (5.3.23) may be solved to express the mass at one scale, $m^2(\mu_2)$, in terms of the mass at another scale, $m^2(\mu_1)$. But as was the case with the running of the coupling, this is already done in the form of the mass renormalization (5.3.20) as evaluated by Eq. (5.3.28). Expressing m_0^2 alternatively in terms of $m^2(\mu_2)$ and then $m^2(\mu_1)$ and taking the ratio of the two results gives

$$
m^2(\mu_2) = m^2(\mu_1)\exp\left\{\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda' \frac{\delta(\lambda')}{(n-4)\lambda'+\beta(\lambda')}\right\}. \tag{5.3.32}
$$

The renormalization of the cosmological constant appears as

$$
\Lambda_0 = \mu^{n-4}\{\Lambda + m_0^4 F_\Lambda(\lambda;n)\}. \tag{5.3.33}
$$

This formula requires some discussion. The scale mass factor μ^{n-4} is placed as an overall factor so as to keep the dimensions of the renormalized cosmological constant Λ fixed at the dimension which the bare constant Λ_0 has at $n = 4$, namely M^4. Renormalization counter terms can only appear at this dimension and so the factor of m_0^4 must appear times the purely numerical function $F_\Lambda(\lambda;n)$. In the minimal subtraction scheme, this function contains only an ascending series of poles with no constant term. It is purely a matter of convention and convenience to choose the factor here to involve the bare mass, m_0^4, rather than the renormalized mass, m^4. Since the two differ by the factor $F_m(\lambda;n)^2$ which is equal to unity plus an ascending series of poles, using m^4 would just redefine the function $F_\Lambda(\lambda;n)$. Finally, we should remark that the renormalization of the cosmological constant (5.3.33) is an example of the general structure of the renormalization scheme in which all possible dimensional parameters appear so long as they have positive integral powers with which to make up the dimension (at $n = 4$) of the quantity being renormalized. In this case, the only available dimensional parameter is m^2.

Just as in our previous discussion, it is easy to see that the condition

$$
\mu\frac{d\Lambda_0}{d\mu} = 0, \tag{5.3.34}
$$

together with the constraint of minimal subtraction and the finiteness of $\mu\, d\Lambda/d\mu$ require that

$$
\mu\frac{d\Lambda}{d\mu} = -(n-4)\Lambda + m^4\beta_\Lambda(\lambda), \tag{5.3.35}
$$

with $\beta_\Lambda(\lambda)$ independent of n. Here, of course, the renormalized mass factor must appear. Using $\mu\, d\Lambda_0/d\mu = 0$ and $\mu dm_0^4/d\mu = 0$ in Eq. (5.3.33) and rewriting m_0^4 in terms of m^4 by Eq. (5.3.20), we get

$$
\begin{aligned}
&\left\{(n-4) + \left[(n-4)\lambda+\beta(\lambda)\right]\frac{\partial}{\partial\lambda}\right\}F_\Lambda(\lambda;n) \\
&\qquad = -F_m(\lambda;n)^{-2}\,\beta_\Lambda(\lambda).
\end{aligned}
\tag{5.3.36}
$$

This inhomogeneous first-order differential equation has an integrating factor — a solution to the homogeneous equation —

$$
I(\lambda;n) = \frac{1}{\lambda}\exp\left\{\int_0^\lambda \frac{d\lambda'}{\lambda'}\frac{\beta(\lambda')}{(n-4)\lambda'+\beta(\lambda')}\right\}, \tag{5.3.37}
$$

in terms of which the solution with the proper $\lambda \to 0$ boundary condition is given by

$$
\begin{aligned}
F_\Lambda(\lambda;n) = -I(\lambda;n)\int_0^\lambda d\lambda'\, I(\lambda';n)^{-1} \\
\times F_m(\lambda';n)^{-2}\frac{\beta_\Lambda(\lambda')}{(n-4)\lambda'+\beta(\lambda')}.
\end{aligned}
\tag{5.3.38}
$$

To verify that the correct boundary condition has been imposed, we note that $\beta(\lambda)$ starts out at order λ^2 so that $I(\lambda;n) \to 1/\lambda$ for small λ. Hence, using $F_m(\lambda;n) \to 1$, we obtain the $\lambda \to 0$ limit

$$
F_\Lambda(\lambda;n) = -\frac{1}{\lambda}\int_0^\lambda d\lambda' \frac{\beta_\Lambda(0)}{n-4} = -\frac{\beta_\Lambda(0)}{n-4}, \tag{5.3.39}
$$

which is correct. Note that since $I(\lambda;n)$ and $F_m(\lambda;n)$ differ from their $\lambda \to 0$ limit by an ascending series of poles,

$$
F_\Lambda(\lambda;n) = \frac{1}{\lambda}\int_0^\lambda d\lambda'\beta_\Lambda(\lambda')\frac{1}{n-4} + \cdots, \tag{5.3.40}
$$

where the dots stand for higher order poles. Again $\beta_\Lambda(\lambda)$ is related to the simple pole of $F_\Lambda(\lambda;n)$. This explains the structure of the first-order renormalization of the cosmological constant found in the previous chap-

ter, Eq. (4.4.6). The simple pole term in that formula identifies

$$\beta_\Lambda(\lambda) \simeq -\frac{1}{2}\frac{1}{(4\pi)^2}, \tag{5.3.41}$$

with no first-order correction. Working through Eq. (5.3.38) one can see that the leading order λ^2 term in the β-function makes no contribution while the lowest-order $\delta(\lambda)$ in $F_m(\lambda';n)$ generates the second-order pole found in Eq. (4.4.6).

We finally come to the wave function renormalization which, as emphasized before, is better considered as a renormalization of the source term,

$$\rho_0(x) = \rho(x)F_\rho(\lambda;n). \tag{5.3.42}$$

Going through the familiar litany

$$\mu\frac{d\rho_0(x)}{d\mu} = 0, \tag{5.3.43}$$

while

$$\mu\frac{d\rho(x)}{d\mu} = \gamma(\lambda;n)\rho(x), \tag{5.3.44}$$

with $\gamma(\lambda;n)$ finite in the $n \to 4$ limit. Hence Eq. (5.3.42) gives

$$\begin{aligned}\mu\frac{d}{d\mu}F_\rho(\lambda;n) &= \Big[(n-4)\lambda + \beta(\lambda)\Big]\frac{\partial}{\partial\lambda}F_\rho(\lambda;n)\\ &= -\gamma(\lambda;n)F_\rho(\lambda;n).\end{aligned} \tag{5.3.45}$$

Minimal subtraction, $F_\rho(\lambda;n) = 1+$ poles, requires that $\gamma(\lambda;n) = \gamma(\lambda)$ be independent of n and that the solution of Eq. (5.3.45) is given by

$$F_\rho(\lambda;n) = \exp\left\{-\int_0^\lambda d\lambda'\,\frac{\gamma(\lambda')}{(n-4)\lambda' + \beta(\lambda')}\right\}. \tag{5.3.46}$$

The pole expansion produces

$$F_\rho(\lambda;n) = 1 - \frac{1}{n-4}\int_0^\lambda \frac{d\lambda'}{\lambda'}\gamma(\lambda') + \cdots. \tag{5.3.47}$$

Thus, as usual, $\gamma(\lambda)$ is determined by the residue of the simple pole in the wave function renormalization, and all the other poles are then determined. Yet again we can solve the renormalization group equation (5.3.44) for the source by taking the ratio of the counter term expression (5.3.42) for two values of the unit of mass, μ_2 and μ_1, with corresponding source functions $\rho(x;\mu_2)$ and $\rho(x;\mu_1)$. This gives

$$\rho(x;\mu_2) = \rho(x;\mu_1)\exp\left\{\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda'\,\frac{\gamma(\lambda')}{(n-4)\lambda' + \beta(\lambda')}\right\}. \tag{5.3.48}$$

5.4 Renormalization group

The generating functional $Z[\rho_0]$ yields all the Green's functions of the bare field by functional differentiation with respect to the bare source $\rho_0(x)$. Differentiating instead with respect to the renormalized source produces Green's functions of the renormalized field, and these functions are finite. Writing all of the bare parameters of the theory in terms of renormalized parameters with the infinite renormalization factors $F(\lambda;n)$ yields a finite theory. All this has been said just to emphasize that the functional integral defines a generating functional

$$Z[\rho_0;\lambda_0,m_0^2] = Z[\rho;\lambda,m^2,\mu^2] \tag{5.4.1}$$

which is finite and well defined. Since the functional written in terms of the bare parameters knows nothing of the arbitrary mass unit μ, we must have

$$\mu\frac{d}{d\mu}Z[\rho;\lambda,m^2,\mu^2] = 0. \tag{5.4.2}$$

Using the chain rule for derivatives and the renormalization group equations (5.3.6), (5.3.23), and (5.3.44) for λ, m^2, and $\rho(x)$ gives

$$\begin{aligned}\Big\{\mu\frac{\partial}{\partial\mu} + \beta(\lambda;n)\frac{\partial}{\partial\lambda} + \delta(\lambda)m^2\frac{\partial}{\partial m^2}\\ +\gamma(\lambda)\int(d_E^n x)\rho(x)\frac{\delta}{\delta\rho(x)}\Big\}Z[\rho;\lambda,m^2,\mu^2] = 0.\end{aligned} \tag{5.4.3}$$

The renormalization group equation for the N-point Green's function follows from Eq. (5.4.3) by the functional differentiation $\delta/\delta\rho(x_1)\cdots\delta/\delta\rho(x_N)$ with the source $\rho(x)$ then taken to vanish:

$$\begin{aligned}\Big\{\mu\frac{\partial}{\partial\mu} + \beta(\lambda;n)\frac{\partial}{\partial\lambda} + \delta(\lambda)m^2\frac{\partial}{\partial m^2} + N\gamma(\lambda)\Big\}\\ G(x_1,\cdots,x_N;\lambda,m^2,\mu^2) = 0.\end{aligned} \tag{5.4.4}$$

One could make some fuss about solving this differential equation as is sometimes done. But there is no need for this. The solution follows immediately from the fact that the generating functional is independent of the arbitrary mass unit,

$$Z[\rho(x;\mu_2);\lambda(\mu_2),m^2(\mu_2),\mu_2^2] = Z[\rho(x;\mu_1);\lambda(\mu_1),m^2(\mu_1),\mu_1^2]. \tag{5.4.5}$$

Taking functional derivatives of this equality with respect to $\delta/\delta\rho(x;\mu_2)$ and using Eq. (5.3.48) to express $\delta/\delta\rho(x;\mu_2)$ in terms of $\delta/\delta\rho(x;\mu_1)$ when taking a derivative of the right-hand side yields

$$G(x_1,\cdots,x_N;\lambda(\mu_2),m^2(\mu_2),\mu_2^2) =$$
$$\exp\left\{-N\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda' \frac{\gamma(\lambda')}{\beta(\lambda';n)}\right\} G(x_1,\cdots,x_N;\lambda(\mu_1),m^2(\mu_1),\mu_1^2)\,. \tag{5.4.6}$$

This is the solution to the Green's function renormalization group equation (5.4.4). To see what this solution implies, we take $n = 4$ and pass to momentum space,

$$G(p_1,\cdots,p_N;\lambda(\mu_2),m^2(\mu_2),\mu_2^2) =$$
$$\exp\left\{-N\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda' \frac{\gamma(\lambda')}{\beta(\lambda')}\right\} G(p_1,\cdots,p_N;\lambda(\mu_1),m^2(\mu_1),\mu_1^2)\,. \tag{5.4.7}$$

Remembering [Eqs. (5.2.13), (5.2.14)] that $G(p_1,\cdots,p_N)$ has the dimension M^{4-3N}, we note that

$$G(p_1,\cdots,p_N;\lambda(\mu),m^2(\mu),\mu^2) = \mu^{4-3N} F\left(\frac{p_1}{\mu},\cdots,\frac{p_N}{\mu};\lambda(\mu),\frac{m^2(\mu)}{\mu^2}\right), \tag{5.4.8}$$

where F is a dimensionless function of the dimensionless arguments shown. We scale all the momenta by a common factor by writing

$$p_a^\mu = W q_a^\mu\,, \tag{5.4.9}$$

where W has the units of mass and the q_a^μ are dimensionless. We take $\mu_2 = \mu_0$ as some fixed mass scale but set $\mu_1 = W$. Then Eq. (5.4.7) in conjunction with Eq. (5.4.8) gives

$$G(Wq_1,\cdots,Wq_N;\lambda(\mu_0),m^2(\mu_0),\mu_0^2) = W^{4-3N}$$
$$\exp\left\{N\int_{\lambda(\mu_0)}^{\lambda(W)} d\lambda'\frac{\gamma(\lambda')}{\beta(\lambda')}\right\} F\left(q_1,\cdots,q_N;\lambda(W),\frac{m^2(W)}{W^2}\right). \tag{5.4.10}$$

The solution in this form shows that how the renormalization group controls the asymptotic behavior of the Green's functions. The high-energy limit in which all the momenta become large is equivalent to the limit $W \to \infty$. In this limit, the Green's function has the energy dependence displayed in the first two factors on the right-hand side of Eq. (5.4.10) times the energy dependence in the dimensionless function F brought about by the running coupling $\lambda(W)$ and running mass $m^2(W)$. The latter are determined by $\beta(\lambda)$ and $\delta(\lambda)$. We could have instead taken $W \to 0$ to investigate the infrared behavior of the theory. The renormalization group coefficient functions $\beta(\lambda), \gamma(\lambda)$, and $\delta(\lambda)$ control

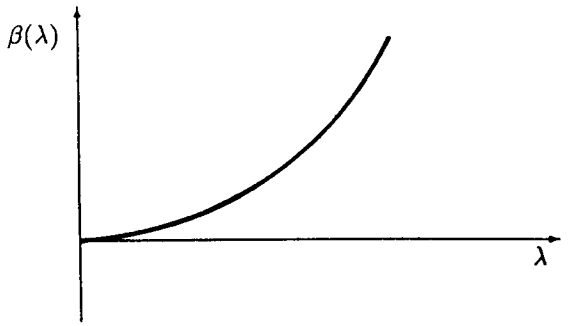


Fig. 5.21. Lowest-order β-function.

both asymptotic behaviors of the theory. Although we have been explicitly working with the $\lambda\phi^4$ theory, the results which have been presented are of a general character. We may therefore use these results to examine what general characteristics a field theory might have. This we shall do by listing various possibilities.

1 Perturbation theory

The β-function in the $\lambda\phi^4$ theory is given to lowest order by

$$\beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2}\,, \tag{5.4.11}$$

corresponding to the sketch in Fig. 5.21. Suppose for the moment that this is the exact β-function of the theory. Then, placing $n = 4$ in Eq. (5.3.19) and carrying out the integration produces

$$\ln\left(\frac{\mu}{\mu_0}\right) = \frac{(4\pi)^2}{3}\left[\frac{1}{\lambda(\mu_0)} - \frac{1}{\lambda(\mu)}\right], \tag{5.4.12}$$

or

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{(4\pi)^2}\ln\left(\frac{\mu}{\mu_0}\right)}\,. \tag{5.4.13}$$

As μ increases, the coupling strength $\lambda(\mu)$ increases: The theory becomes a strongly coupled theory at high energy. In fact, if the β-function were given by Eq. (5.4.11) over a wide range of coupling, the theory would diverge — it would blow up with $\lambda(\mu_1) = \infty$ at a *finite* energy scale $\mu = \mu_1$ where the denominator in Eq. (5.4.13) vanishes,

$$\mu_1 = \mu_0 \exp\left\{\frac{(4\pi)^2}{3\lambda(\mu_0)}\right\}. \tag{5.4.14}$$

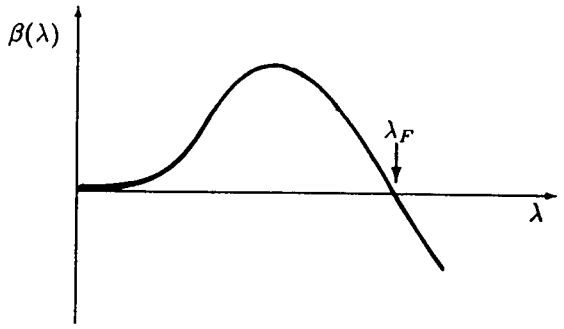


Fig. 5.22. β-function with an ultra-violet fixed point.

A theory with this behavior is, of course, physically unacceptable. At this point we might mention that non-perturbative calculations indicate that the $\lambda\phi^4$ theory does not exist as a complete theory. Some indication of the trouble follows from examining the relation (5.3.17) between the bare and renormalized couplings. Assuming that Eq. (5.4.11) holds for all λ and doing the integral in Eq. (5.3.16) gives

$$\lambda_0 = \lambda\mu^{4-n}\left[1+\frac{3\lambda}{(4\pi)^2}\frac{1}{n-4}\right]^{-1}. \tag{5.4.15}$$

The space-time dimensionality n appears as a regulator which yields a convergent theory for sufficiently small n. Thus we should use Eq. (5.4.15) for values of n less than 4, taking the limit $n \to 4$ from below. However, as n approaches 4 from below, the denominator in Eq. (5.4.15) vanishes at $n = 4 - 3\lambda/(4\pi)^2 < 4$ and the bare coupling blows up before the limit $n = 4$ is reached. The only out is to require that the renormalized coupling vanishes when $n \to 4$, making the $\lambda\phi^4$ theory a "trivial" theory. (One cannot take $\lambda < 0$ for this would give $\lambda_0 < 0$ and an unbounded energy.) Nonetheless, the renormalized $\lambda\phi^4$ theory does exist order by order in perturbation theory and many of its properties are a useful guide to the character of all field theories.

2 Fixed point

To continue with our discussion, suppose now that the β-function has a zero at $\lambda = \lambda_F$, $\beta(\lambda_F) = 0$, with the plot of $\beta(\lambda)$ of the general form shown in Fig. 5.22. The value $\lambda = \lambda_F$ is a "fixed point" because, if $\lambda = \lambda_F$,

$$\mu\frac{d\lambda}{d\mu}\bigg|_{\lambda=\lambda_F} = \beta(\lambda_F) = 0\,, \tag{5.4.16}$$

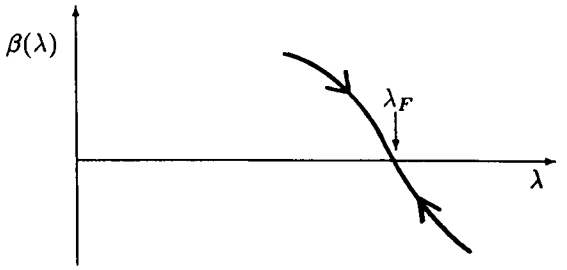


Fig. 5.23. Coupling flow near an ultra-violet fixed point.

and the coupling does not move away from λ_F as the mass unit μ changes. To analyze the behavior of λ near λ_F, we assume that the β-function has a simple zero at λ_F and write

$$\beta(\lambda) \simeq (\lambda - \lambda_F)\beta'_F\,, \tag{5.4.17}$$

so that for λ near λ_F

$$\mu\frac{d\lambda}{d\mu} = (\lambda - \lambda_F)\beta'_F\,. \tag{5.4.18}$$

The sign of β'_F is crucial. With $\beta'_F < 0$ as in Fig. 5.22, when $\lambda < \lambda_F$, $\mu d\lambda/d\mu > 0$ and λ is driven to *larger* values when μ increases. On the other side of the fixed point, when $\lambda > \lambda_F$, $\mu d\lambda/d\mu < 0$, and λ is driven to *smaller* values as μ increases. That is, any value of λ in the neighborhood of λ_F is driven to λ_F as $\mu \to \infty$. This point is an "ultraviolet stable fixed point" which can be illustrated by a "flow" as sketched in Fig. 5.23 or by a plot of two couplings starting from values above and below λ_F as shown in Fig. 5.24. The alternative $\beta'_F > 0$ gives rise to an "infrared stable fixed point". Following along the previous discussion, this case is illustrated in the sketch in Fig. 5.25. The coupling is now driven to the fixed point λ_F as $\mu \to 0$. The couplings starting from values above and below λ_F run as shown in Fig. 5.26. The behavior of the coupling near λ_F follows from Eq. (5.3.19). Taking $n = 4$,

$$\begin{aligned}\ln\left(\frac{\mu}{\mu_0}\right) &= \int_{\lambda(\mu_0)}^{\lambda(\mu)}\frac{d\lambda}{\beta(\lambda)}\\ &= \text{const.} + \int^{\lambda(\mu)}\frac{d\lambda}{\beta'_F(\lambda-\lambda_F)}\\ &= \text{const.} + \frac{1}{\beta'_F}\ln\left[\lambda(\mu)-\lambda_F\right].\end{aligned} \tag{5.4.19}$$

The additive numerical constant is, of course, a function of the coupling constant at the scale μ_0 , $\lambda(\mu_0)$. This constant may be removed by

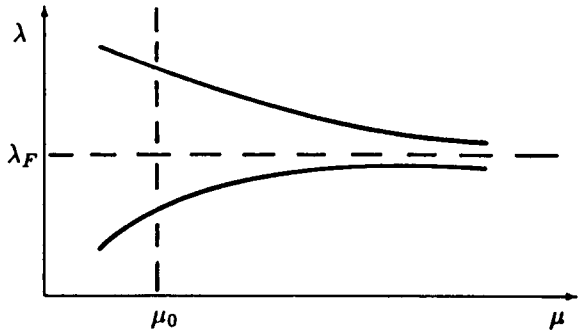


Fig. 5.24. Coupling behavior for an ultra-violet fixed point as a function of the mass unit.

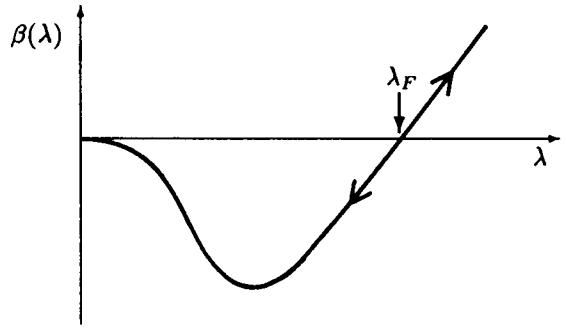


Fig. 5.25. β-function with an infrared fixed point.

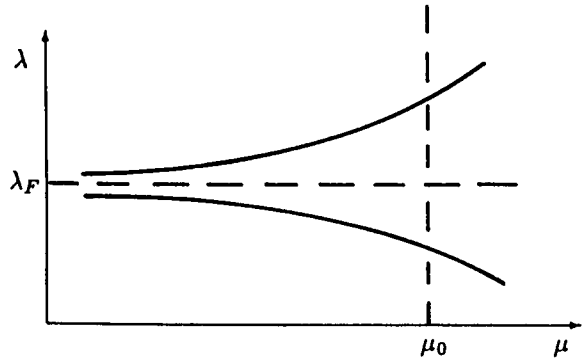


Fig. 5.26. Coupling behavior for an infra-red fixed point as a function of the mass unit.

replacing the mass unit μ_0 by an appropriate constant multiple $\mu_1 =$ const. μ_0. Thus for λ near λ_F

$$\lambda(\mu) = \lambda_F + \left(\frac{\mu}{\mu_1}\right)^{\beta'_F} . \tag{5.4.20}$$

The behavior of the Green's functions near the fixed point is given in Eq. (5.4.10). Near the fixed point

$$\int_{\lambda(\mu_0)}^{\lambda(\mu)} d\lambda \frac{\gamma(\lambda)}{\beta(\lambda)} \simeq \gamma(\lambda_F) \int_{\lambda(\mu_0)}^{\lambda(\mu)} \frac{d\lambda}{\beta(\lambda)} \simeq \gamma(\lambda_F) \ln\left(\frac{\mu}{\mu_2}\right), \tag{5.4.21}$$

since the integral is dominated by the region near the upper limit where $\beta(\lambda)$ becomes very small and $\lambda \simeq \lambda_F$. Again, an additive constant has been removed by replacing μ_0 by μ_2 which differs from μ_0 by a constant multiple. Similarly

$$\begin{aligned} m^2(\mu) &= m^2(\mu_0) \exp\left\{ \int_{\lambda(\mu_0)}^{\lambda(\mu)} d\lambda \frac{\delta(\lambda)}{\beta(\lambda)} \right\} \\ &\simeq m^2(\mu_0) \left(\frac{\mu}{\mu_3}\right)^{\delta(\lambda_F)} , \end{aligned} \tag{5.4.22}$$

where again $\mu_3 =$ const. μ_0. Using Eq. (5.4.10), we now have

$$\begin{aligned} G(\mu q_1, \cdots, \mu q_N; \lambda(\mu_0), m^2(\mu_0), \mu_0^2) \simeq \left(\frac{\mu}{\mu_2}\right)^{4+[\gamma(\lambda_F)-3]N} \mu_2^{4-3N} \\ F\left(q_1, \cdots, q_N; \lambda_F, \left(m^2(\mu_0)/\mu_3^2\right)(\mu/\mu_3)^{\delta(\lambda_F)-2}\right) . \end{aligned} \tag{5.4.23}$$

Several points need to be made. (i) The overall μ-dependence of the Green's function is determined by μ raised to the power $4+[\gamma(\lambda_F)-3]N$. If no renormalization were required, this dependence would be determined by simply counting the (mass) dimension of the Green's function. Thus, $\gamma(\lambda_F)$ plays the role of an *anomalous dimension.* (ii) The mass appears with the factor $\mu^{\delta(\lambda_F)-2}$, and thus the mass becomes negligible in the high-energy limit if $\delta(\lambda_F) < 2$. In this case, the entire energy dependence is given by the μ factor out in front. (iii) There are no free parameters in the asymptotic limit. The fixed point λ_F is, of course, a number determined as an intrinsic property of the theory. This "universal" behavior is famous in condensed matter physics where phase transitions appear as infrared fixed points. (iv) The way in which the fixed point limit is approached is governed by the mass units μ_1, μ_2, μ_3, which are given by dimensionless functions of $\lambda(\mu_0)$ multiplying μ_0. These mass units do not depend upon μ_0 in the sense that a change in μ_0 is compensated by a change in the coupling $\lambda(\mu_0)$ in accordance with the renormalization group. Their determination does, however, require the experimental measurement of the coupling $\lambda(\mu_0)$ at some value of the basic mass unit μ_0. The character of

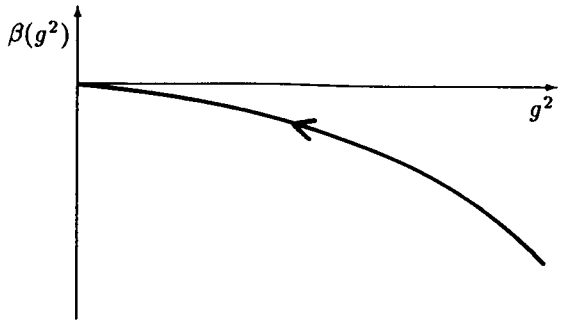


Fig. 5.27. Asymptotic freedom β-function.

the mass units μ_1, μ_2, μ_3 is an example of "dimensional transmutation" in which the original dimensionless coupling constant becomes replaced with a parameter bearing dimensions. The point is that although the fixed point dimensionless coupling λ_F is determined by the theory itself, experimental measurements must be made to fix the energy scale — or equivalently the length scale. Clearly the theory itself cannot determine what is the proper ruler to be used — how big an inch is.

3 Asymptotic freedom

Non-Abelian gauge theories are asymptotically free. To describe this, we first change notation and use g^2 to denote the coupling as is done in these theories, and write

$$\mu \frac{dg^2}{d\mu} = \beta(g^2)\,. \tag{5.4.24}$$

The β-function in non-Abelian theories has the form sketched in Fig. 5.27. In this case $g^2(\mu) \to 0$ as $\mu \to \infty$. As the energy increases the coupling is driven to the origin as indicated by the arrow on the curve in the sketch. Thus a plot of the coupling as a function of the mass unit appears as in Fig. 5.28. Since the coupling vanishes at large energy, the theory is said to be "asymptotically free". At finite but large energies, the coupling is small and perturbation theory may be used. Quantum chromodynamics (QCD), the fundamental theory of hadronic interactions, has this property. At low energies, the QCD coupling is large and the theory cannot be treated perturbatively. However, as the energy increases, the coupling weakens; at sufficiently high energies, hadronic processes can be calculated using perturbation theory.

The β-function in the non-Abelian gauge theories behaves for small g^2

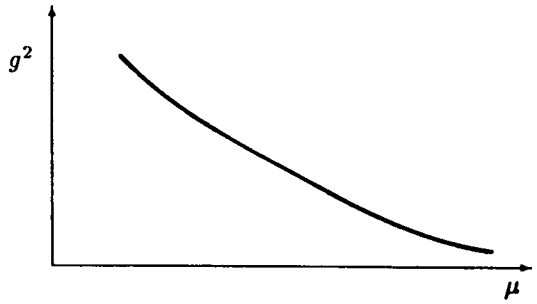


Fig. 5.28. Coupling decrease in asymptotic freedom.

as

$$\beta(g^2) = -bg^4\,, \tag{5.4.25}$$

with b positive, $b > 0$. This gives, for small coupling,

$$g^2(\mu) = \frac{g^2(\mu_0)}{1 + b\, g^2(\mu_0) \ln(\mu/\mu_0)}\,. \tag{5.4.26}$$

As μ becomes large, $\mu \to \infty$, $g^2(\mu)$ becomes small, $g^2(\mu) \to 0$, albeit in a very slow fashion, behaving as $1/\ln \mu$. The running coupling may be written as

$$\begin{aligned} g^2(\mu) &= \frac{1}{g(\mu_0)^{-2} + b\, \ln(\mu/\mu_0)} \\ &= \frac{1}{b\, \ln(\mu/\Lambda)}\,, \end{aligned} \tag{5.4.27}$$

where

$$\Lambda = \mu_0 \exp\left\{ -\frac{1}{bg(\mu_0)^2} \right\}. \tag{5.4.28}$$

Using the β-function Eq. (5.4.25), it is easy to check that Λ is independent of μ_0 to the order in which we are working,

$$\mu_0 \frac{d\Lambda}{d\mu_0} = 0\,. \tag{5.4.29}$$

We have again encountered the phenomenon of "dimensional transmutation". The dimensionless coupling $g(\mu_0)^2$ has been traded for the parameter Λ which has the dimensions of mass. High-energy experiments can measure $g^2(W)$ over a range of energies W. According to Eq. (5.4.27), what is in fact measured is the basic QCD scale Λ.

To investigate further the behavior of an asymptotically free theory, we take the small coupling limit

$$\gamma(g^2) \simeq cg^2 \tag{5.4.30}$$

to compute

$$\int_{g^2(\mu_0)}^{g^2(\mu)} dg^2 \frac{\gamma(g^2)}{\beta(g^2)} \simeq -\frac{c}{b} \int_{g^2(\mu_0)}^{g^2(\mu)} dg^2 \frac{g^2}{g^4}$$
$$\simeq -\frac{c}{b} \ln \left[\frac{g^2(\mu)}{g^2(\mu_0)} \right]. \tag{5.4.31}$$

Moreover, writing the small g^2 limit as

$$\delta(g^2) \simeq dg^2 , \tag{5.4.32}$$

gives

$$m^2(\mu) = m^2(\mu_0) \exp \left\{ \int_{g^2(\mu_0)}^{g^2(\mu)} dg^2 \frac{\delta(g^2)}{\beta(g^2)} \right\}$$
$$\simeq m^2(\mu_0) \left[\frac{g^2(\mu)}{g^2(\mu_0)} \right]^{-d/b} . \tag{5.4.33}$$

Since $g^2(\mu)$ vanishes like $1/\ln \mu$ for large μ, whatever the sign of d/b, this gives only a correction involving some power of the logarithm and so in any case

$$\mu \to \infty : \quad \frac{m^2(\mu)}{\mu^2} \to 0 . \tag{5.4.34}$$

Therefore, the high-energy behavior of a Green's function in an asymptotically free theory has the structure

$$G(Wq_1, \cdots, Wq_N; g^2(\mu_0), m^2(\mu_0), \mu_0^2)$$
$$\simeq \left[\frac{g^2(W)}{g^2(\mu_0)} \right]^{-Nc/b} W^{4-3N} F(q_1, \cdots, q_N; g^2(W)) , \tag{5.4.35}$$

where $g^2(W)$ is given by Eq. (5.4.27).

5.5 Composite operators

High energies and large momentum transfers probe the short-distance behavior of a theory. Thus it is natural to investigate the nature of composite operators such as $\phi^2(x)$, $\phi^4(x)$, $\partial_\mu \phi(x) \partial_\nu \phi(x)$ and so forth. Such composite operators already appear in the Lagrange function and, more to the point, in the stress-energy tensor whose matrix elements may be measured. Moreover, composite operators are a necessary ingredient for the operator product expansion which will be discussed in the next section. This expansion is an essential tool in quantum field theory. It is extensively used, for example, in the analysis of large momentum transfer inelastic scattering processes. Since relativistic field theory is singular

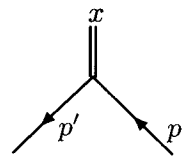


Fig. 5.29. Insertion of $\phi^2(x)$ in a tree graph.

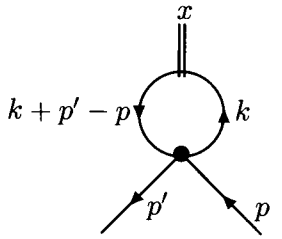


Fig. 5.30. One-loop correction to previous tree graph.

in the short-distance limit, the composite operators must be carefully defined in a regulated theory and divergent quantities must be subtracted to form renormalized operators. We shall sketch this process for some simple operators. Our purpose here is to illustrate the general character of composite operators but not to give a detailed and exhaustive account. We shall continue to work in Euclidean space-time, regulating the theory by working in n dimensions.

1 $[\phi^2]$

The simplest composite operator is ϕ^2. Neglecting disconnected graphs, the insertion of this operator in the two-point function gives, to lowest order,

$$\langle \phi(x_1) \phi(x_2) \phi^2(x) \rangle^{(0)} = 2\Delta_E(x_1 - x) \Delta_E(x_2 - x) . \tag{5.5.1}$$

In momentum space, this corresponds to the graph of Fig. 5.29. Note that in transforming to momentum space we do not Fourier transform the coordinate x of the composite operator – the composite operators are always kept in coordinate space. The first order correction is given by

$$\langle \phi(x_1) \phi(x_2) \phi^2(x) \rangle^{(1)} = -\frac{\lambda_0}{4!} \int (d_E^n y) \langle \phi(x_1) \phi(x_2) \phi^4(y) \phi^2(x) \rangle^{(0)} . \tag{5.5.2}$$

This correction corresponds to the momentum space graph shown in Fig. 5.30. Removing the external propagator factors and temporarily

setting $x = 0$, the graph represents the function

$$\Gamma^{(1)}(p', p; 0) = -\lambda \mu^{4-n} \int \frac{(d_E^n k)}{(2\pi)^n} \frac{1}{k^2 + m^2} \frac{1}{(k + p' - p)^2 + m^2} . \quad (5.5.3)$$

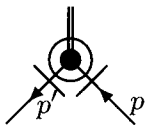
This integral is the simple loop integral (3.5.17) discussed at length in Section 5 of Chapter 3.

The integral has a logarithmic divergence in four dimensions. The divergence is contained in

$$\Gamma^{(1)}_{\text{div}}(p', p; 0) = -\lambda \mu^{4-n} \int \frac{(d_E^n k)}{(2\pi)^n} \frac{1}{(k^2 + m^2)^2} , \quad (5.5.4)$$

which is independent of the momenta p', p. This independence of the momenta means that the divergent part appears as a point interaction in coordinate space. It is a constant multiplying the lowest-order tree graph of Fig. 5.29. In graphical terms, one can say that the divergence "shrinks" the graph to a point:

$$\Gamma^{(1)}_{\text{div}}(p', p; 0) \quad \longleftrightarrow \quad$$



. (5.5.5)

As always, we remove the divergent part by minimal subtraction. This has already been done in the discussion in Chapter 3, Section 5. However, as we shall see, the renormalization of ϕ^2 is intimately connected with the mass renormalization. An indication of this relationship is brought out by writing

$$\Gamma^{(1)}_{\text{div}}(p', p; 0) = \lambda \mu^{4-n} \frac{\partial}{\partial m^2} \int \frac{(d_E^n k)}{(2\pi)^n} \frac{1}{k^2 + m^2} . \quad (5.5.6)$$

The integral which appears here is, except for a factor of 2, just that which gives the mass renormalization — see Eqs. (4.3.7) and (4.2.5). Thus we may infer from the mass renormalization (4.3.9) that the minimal divergence, the pole part, is given by

$$\Gamma^{(1)}_{\text{div,min}}(p', p; 0) = \frac{2\lambda}{(4\pi)^2} \frac{1}{n - 4} . \quad (5.5.7)$$

Remembering the combinatorial factor of 2 in the lowest order result (5.5.1), we see that, to order λ, the insertion of

$$[\phi^2](x) = \left[1 - \frac{\lambda}{(4\pi)^2} \frac{1}{n - 4}\right] \phi_0^2(x) \quad (5.5.8)$$

produces a finite result in the $n \to 4$ limit. To our first order in λ, this defines a composite operator renormalized by minimal subtraction. We denote such renormalized composite operators by enclosing them in square brackets as shown. It should be emphasized that ϕ_0^2 is the square of the bare field which we now make explicit with the zero subscript. The renormalization of the composite operator is an entirely different renormalization than the wave function renormalization which converts ϕ_0 to ϕ. One could write $[\phi^2]$ in terms of a renormalization constant times the square of the renormalized field, ϕ^2, but this would obscure the analysis since ϕ_0^2 is a renormalization group invariant $\left[\mu \frac{d}{d\mu} \phi_0^2 = 0\right]$ while ϕ^2 is not. Of course, this distinction appears only in the next order in λ since the wave function renormalization starts at order λ^2.

A summary is in order: The Fourier transform of

$$\langle \phi(x_1)\, \phi(x_2)\, [\phi^2](x) \rangle \quad (5.5.9)$$

is given, up to terms of order λ, by

$$\frac{1}{p'^2 + m_{\text{phy}}^2} \frac{1}{p^2 + m_{\text{phy}}^2} \{2 + \lambda F_{\text{ren}}(-(p' - p)^2)\} e^{-i(p'-p)x} , \quad (5.5.10)$$

where we have now restored the x-dependence. The external propagator corrections, which we previously neglected, replace the bare mass m_0^2 by the self-energy function $\Sigma(p)$. To order λ, this self-energy function is simply the square of the physical mass, $\Sigma(p) = m_{\text{phy}}^2$. The renormalization entailed in the definition of $[\phi^2]$ provides precisely the minimal subtraction needed to render the one-loop function $F_{\text{ren}}(-(p' - p)^2)$ finite. This function is computed explicitly in Chapter 3, Section 5.

To proceed to higher orders, we recall that the basic divergencies are contained in the graphs that have a positive degree of superficial divergence D. The insertion of the operator ϕ^2 adds an additional propagator factor to the graph, decreasing D by two. The insertion of ϕ^2 alters Eq.(5.2.21) to read $D = 2 - N$. Graphs that have two external lines, $N = 2$, give the basic divergence. These divergencies are independent of the external momenta. They correspond to a point interaction and they are removed by a multiplicative renormalization of ϕ^2. The vacuum graphs with $N = 0$ also diverge. By dimensional analysis, they must be proportional to m^2. These disconnected graphs are usually of no importance, and they, for the most part, may be neglected. To illustrate how they may be treated, we shall include them here in the renormalization of ϕ^2, but neglect such graphs later. Thus the full renormalization of ϕ^2 reads

$$\phi_0^2(x) = z_2(\lambda; n)\, [\phi^2](x) + m^2 v_2(\lambda; n) . \quad (5.5.11)$$

With the minimal subtraction which we employ, z_2 equals unity plus an ascending series of poles while $v_2(\lambda; n)$ only contains poles in $(n - 4)$.

The renormalization constants z_2 and v_2 may be obtained by sharpening the previous indication that the insertion of ϕ^2 is related to a mass

derivative. Thus we consider the derivative of the generating functional $Z[\rho]$ with respect to the renormalized mass m. To do this, we first recall that the Lagrange function is given by

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda_0\phi^4 - \Lambda_0 - \rho_0\phi\,. \tag{5.5.12}$$

Keeping the finite, renormalized parameters λ and Λ fixed, the dependence of $\mathcal{L}_E$ on the renormalized mass m is given by

$$m_0^2 = m^2 F_m(\lambda;n)\,, \tag{5.5.13}$$

and

$$\Lambda_0 = \mu^{n-4}\{\Lambda + m^4 F_m(\lambda;n)^2 F_\Lambda(\lambda;n)\}\,. \tag{5.5.14}$$

Hence the functional integral representation of the generating functional presents the mass derivative in the form

$$\begin{aligned}&-m^2\frac{\partial}{\partial m^2}Z[\rho]\\ &\quad=\int[d\phi]e^{-\int\mathcal{L}_E}\int(d_E^n x)\left\{\frac{1}{2}m_0^2\phi^2(x) - 2\mu^{n-4}m_0^4 F_\Lambda(\lambda;n)\right\}.\end{aligned} \tag{5.5.15}$$

The functional integral with an extra factor of $\phi^2(x)$ in the integrand corresponds to the insertion of the bare field $\phi_0^2(x)$ in a Green's function. Since the left-hand side of this equation is finite (the derivative of a finite functional with respect to a finite parameter), we conclude that

$$m^2[\phi^2] = m_0^2\phi_0^2(x) - 4\mu^{n-4}m_0^4 F_\Lambda(\lambda;n) \tag{5.5.16}$$

is a finite operator. Some explanation of this conclusion needs to be made.

We proved that the space-time volume integral of this expression is finite. The renormalization of a general composite operator involves a mixing of all possible operators of the same and lower (mass) dimension with factors of the mass or derivatives making up the missing dimension for the lower dimensional operators. By dimensional counting, no derivatives can enter here in the renormalization of ϕ^2, and so the finiteness of the integral ensures that the integrand is finite. The operator ϕ^2 does mix with the identity operator 1 with a coefficient of $m^2\mu^{n-4}$ to make up the extra dimensions. Note, as was discussed before, that only integer powers of m^2 can appear with the scale mass μ taking up the non-integer dimensions. The identification of the finite expression with $m^2[\phi^2]$ follows from the minimal subtraction scheme: Bare and renormalized quantities differ only by the appearance of extra poles in $(n-4)$; deleting the extra pole terms in Eq. (5.5.16) converts m_0^2 to m^2 and removes $F_\Lambda(\lambda;n)$ since it contains only poles. In view of Eq. (5.5.16), the renormalization

constants in Eq. (5.5.11) are given by

$$z_2(\lambda;n) = F_m(\lambda;n)^{-1}\,, \tag{5.5.17}$$

and

$$v_2(\lambda;n) = 4\mu^{n-4}F_m(\lambda;n)F_\Lambda(\lambda;n)\,. \tag{5.5.18}$$

As a check on these general results, we note the first order renormalization of ϕ^2 given in Eq. (5.5.8) does coincide with the lowest order mass renormalization of Eq. (4.3.9). Moreover, to zeroth order in λ, we also see that the vacuum energy renormalization $F_\Lambda(\lambda;n)$ is given by the divergent pole part of $\langle\phi^2\rangle/4m^2 = \Delta_E(0)/4m^2$ in agreement with the result of Eq. (4.2.16).

To conclude the discussion of $[\phi^2]$ we note that Eqs. (5.5.15) and (5.5.16) show that the mass derivative of a Green's function corresponds to an insertion of $[\phi^2]$ in the Green's function. We express this by writing the mass derivative of the generating functional as

$$\frac{\partial}{\partial m^2}Z[\rho] = -\frac{1}{2}\int(d_E^n x)Z[\rho;[\phi^2](x)]\,. \tag{5.5.19}$$

We also note that since $m_0^2\phi_0^2$ is independent of the unit of mass μ, Eqs. (5.5.16) and (5.3.33) inform us that

$$\begin{aligned}\mu\frac{d}{d\mu}\left\{m^2[\phi^2]\right\} &= -4\mu\frac{d}{d\mu}\left\{\mu^{n-4}m_0^4F_\Lambda(\lambda;n)\right\}\\ &= 4\mu\frac{d}{d\mu}\{\mu^{n-4}\Lambda\}\,.\end{aligned} \tag{5.5.20}$$

Remembering Eqs. (5.3.23) and (5.3.35), this gives the renormalization group equation

$$\mu\frac{d}{d\mu}[\phi^2](x) = -\delta(\lambda)[\phi^2](x) + 4\mu^{n-4}m^2\beta_\Lambda(\lambda)\,. \tag{5.5.21}$$

2 $\langle[\phi^2](x)[\phi^2](x')\rangle$

The insertion of a single renormalized composite operator such as $[\phi^2]$ into an N-point Green's function produces a finite quantity. If, however, a set of coordinates of the remaining fields are coalesced to form a second insertion of a renormalized composite operator, the result is not finite; another additive renormalization must be performed. The simplest example is provided by the connected two-point function

$$K(x-x') = \langle[\phi^2](x)\,[\phi^2](x')\rangle - \langle[\phi^2](x)\rangle\langle[\phi^2](x')\rangle\,. \tag{5.5.22}$$

Here we shall show that this additive renormalization can be obtained from the disconnected vacuum graph counter term which appears in the renormalization of ϕ_0^2.

In the limit $n = 4$, $K(x - x')$ is of dimension four, the dimensionality of $\delta^{(4)}(x - x')$. Hence this function is renormalized by writing

$$K_R(x - x') = K(x - x') + \mu^{n-4} A(\lambda; n)\delta^{(n)}(x - x') . \tag{5.5.23}$$

The dimensionless function $A(\lambda; n)$ is taken to have only an ascending series of poles in $(n-4)$ so as to keep to the minimal subtraction scheme. The counter term $A(\lambda; n)$ is easily determined from the remark that

$$\frac{\partial}{\partial m^2}\langle [\phi^2](x)\rangle = -\frac{1}{2}\int (d_E^n x')\, K(x - x') + \langle \frac{\partial}{\partial m^2}[\phi^2](x)\rangle \tag{5.5.24}$$

is finite. The right-hand side of this formula is produced by the renormalized action principle stated in Eq. (5.5.19) and the fact that the renormalized composite operator $[\phi^2]$ depends explicitly upon m^2. [The *connected* Green's function defined by Eq. (5.5.22) appears here because the expectation value involves the normalizing denominator $\langle 0+|0-\rangle$.] The explicit renormalization (5.5.16) of ϕ_0^2 and the fact that ϕ_0^2 is independent of m^2 yield

$$\frac{\partial}{\partial m^2}[\phi^2] = -\mu^{n-4}\, 4F_m(\lambda; n)^2\, F_\Lambda(\lambda; n) . \tag{5.5.25}$$

Hence, comparing Eq. (5.5.24) with Eq. (5.5.23), we conclude that

$$A(\lambda; n) = 8F_m(\lambda; n)^2 F_\Lambda(\lambda; n) . \tag{5.5.26}$$

3 $[E]$

The renormalization of ϕ^4 entails an "equation of motion operator". We pause here to discuss this operator. On the one hand the functional integral of a total functional derivative vanishes while on the other hand the variation of the action produces the equation of motion. Hence

$$\begin{aligned} 0 &= -\int [d\phi]\frac{\delta}{\delta\phi(x)}\phi(x)e^{-\int \mathcal{L}_E} \\ &= \int [d\phi]e^{-\int \mathcal{L}_E}\left\{\phi(x)(-\partial^2 + m_0^2)\phi(x) + \frac{\lambda_0}{3!}\phi^4(x) - \rho_0(x)\phi(x)\right\} , \end{aligned} \tag{5.5.27}$$

since $\delta\phi(x)/\delta\phi(x) = \delta^{(n)}(0)$ vanishes in dimensional continuation.* The first set of field variables correspond to the insertion of the equation of motion operator

$$E_0(x) = \phi_0(x)(-\partial^2 + m_0^2)\phi_0(x) + \frac{\lambda_0}{3!}\phi_0^4(x) . \tag{5.5.28}$$

* We take it as a postulate that $\delta^{(n)}(0) = 0$ because there is no dimensional quantity available to make up this number.

The field in the source term is produced by the action of $\delta/\delta\rho_0(x)$ on the exponential of the action. Since the functional integral produces the generating functional of the Green's functions, the identity (5.5.27) may be expressed as

$$Z[\rho; E_0(x)] = \left\langle e^{\int \rho_0\phi_0}\, E_0(x)\right\rangle = \rho_0(x)\frac{\delta}{\delta\rho_0(x)}\left\langle e^{\int \rho_0\phi_0}\right\rangle . \tag{5.5.29}$$

Functional differentiating with respect to $\rho_0(x_1), \ldots, \rho_0(x_N)$ and then taking the source to vanish produces the Green's function identity

$$\langle \phi_0(x_1)\cdots\phi_0(x_N)E_0(x)\rangle = \sum_{a=1}^{N}\delta(x_a - x)\langle \phi_0(x_1)\cdots\phi_0(x_N)\rangle . \tag{5.5.30}$$

To explain the structure of this identity, we note that transforming back to Minkowski space-time produces the vacuum expectation values of the time-ordered product. The identity above follows from the vanishing of the field equation for $\phi_0(x)$ inside the time ordering. The derivative operator in $-\partial^2\phi_0$ produced by the functional integral appears outside the time-ordering process. Bringing these derivatives inside the time ordering involves the time derivative of the step functions which appear in the definition of the time-ordering. Thus, in addition to the vanishing field equation term, there are $\delta(t_a - t)$ terms that multiply equal-time, canonical field commutators which produce spatial $\delta(\mathbf{x}_a - \mathbf{x})$. These combine to form the $\delta(x_a - x)$ terms on the right-hand side of the identity.

Since both sides of the identity (5.5.30) are made finite by the same wave function renormalization factors, we conclude that $E_0(x)$ itself is finite. Hence, with minimal subtraction, the set of (infinite) bare operators and bare parameters in $E_0(x)$ combine to form a finite composite operator,

$$E_0(x) = [E](x) . \tag{5.5.31}$$

Since $[E]$ is a finite operator which is composed only of bare fields, it is a renormalization group invariant,

$$\mu\frac{d}{d\mu}[E](x) = 0 . \tag{5.5.32}$$

4 $[\phi^4]$

A graph with an insertion of the operator ϕ^4 has the superficial degree of divergence $D = 4 - N$ which applies to the graph without this insertion. This follows from the remark that the degree of divergence for the insertion of ϕ^4 does not depend on the external momentum carried off by the insertion. The limit of zero momentum corresponds to the insertion of an additional interaction vertex — to a graph describing one of the corrections of the next order in the couping λ to the original graph.

Primitively divergent graphs with $N = 4$ external legs are made finite by a multiplicative renormalization of ϕ_0^4. The divergencies in graphs with $N = 2$ external legs require a counter term involving ϕ_0^2; to make up the dimension, ϕ_0^2 must be multiplied by m^2 or two derivatives of ϕ_0^2 must appear. Since $\partial^2\phi_0^2 = 2(\partial_\mu\phi_0)^2 + 2\phi_0\partial^2\phi_0$, there are only two independent terms involving derivatives which we shall take to be $\partial^2\phi_0^2$ and $\phi_0\partial^2\phi_0$. It is convenient to replace $\phi_0\partial^2\phi_0$ by the equation of motion operator E_0. This may be done because the two differ only by other operators already involved in the renormalization. This replacement is convenient because E_0 is by itself a finite composite operator which needs no renormalization. Thus a complete set of the dimension four operators which mix under renormalization may be taken to be ϕ_0^4, $m^2\phi_0^2$, $\partial^2\phi_0^2$, and E_0. We shall not consider disconnected vacuum graphs with $N = 0$ which require an additional constant, $m^4 1$ counter term. Previously, we found that the renormalization of $m_0^2\phi_0^2$ was simple. For similar reasons, it is simplier to consider the renormalization of $\lambda_0\phi_0^4$ rather than just the operator ϕ_0^4 alone. With minimal subtraction, this renormalization has the structure

$$\frac{1}{4!}\lambda_0\phi_0^4(x) = [1 + \text{poles}]\frac{1}{4!}\lambda\mu^{4-n}[\phi^4](x) + [\text{poles}]\frac{1}{2}m^2[\phi^2](x) + [\text{poles}][E](x) + [\text{poles}](-\partial^2)[\phi^2](x)\,, \tag{5.5.33}$$

which defines the renormalized composite operator $[\phi^4](x)$.

The renormalized operator $[\phi^4]$ can be constructed with a technique that parallels the previous construction of $[\phi^2]$. The structure of $[\phi^2]$ was dictated by the requirement that the mass derivative of the generating functional, $\partial Z/\partial m^2$, be finite. In a similar manner, the structure of $[\phi^4]$ follows from the condition that $\partial Z/\partial\lambda$ be finite. Since we are already assured that $\partial Z/\partial m^2$ is finite, it is convenient not to consider $\partial Z/\partial\lambda$ with all the other renormalized parameters held fixed, but rather a variation of the renormalized coupling λ and the renormalized mass m^2 that keeps m_0^2 fixed. This is accomplished by the scale variation

$$d\lambda = \mu\frac{d\lambda}{d\mu}d\epsilon = [(n-4)\lambda + \beta(\lambda)]\,d\epsilon\,, \tag{5.5.34}$$

and

$$dm^2 = \mu\frac{dm^2}{d\mu}d\epsilon = \delta(\lambda)m^2 d\epsilon\,. \tag{5.5.35}$$

Since

$$\lambda_0 = \mu^{4-n}\,\lambda\,F_\lambda(\lambda; n) \tag{5.5.36}$$

is independent of the mass unit μ, the variation $\delta\lambda_0$ induced by the scale variation $\delta\lambda$ is canceled by the explicit change of μ. Hence

$$d\lambda_0 = (n-4)\lambda_0 d\epsilon\,. \tag{5.5.37}$$

On the other hand, since the mass unit μ does not explicitly occur in the renormalization of m_0^2, we conclude that

$$dm_0^2 = 0\,. \tag{5.5.38}$$

It proves convenient to keep the renormalized source fixed,

$$d\rho(x) = 0\,. \tag{5.5.39}$$

If instead we would have taken the source to undergo a scale variation $d\rho = \gamma(\lambda)\rho d\epsilon$, the bare source ρ_0 would have been kept fixed. Since the relationship between ρ_0 and ρ does not explicitly involve μ, we conclude that with ρ held fixed

$$d\rho_0(x) = -\gamma(\lambda)\rho_0(x)d\epsilon\,. \tag{5.5.40}$$

The structure of the renormalized composite operator emerges from the comparison of the two forms of the variation of the generating functional which result when it is expressed in terms of renormalized and bare parameters. In terms of the renormalized parameters, this variation is given by

$$\begin{aligned}\frac{d}{d\epsilon}Z[\rho] &= \left\{[(n-4)\lambda + \beta(\lambda)]\frac{\partial}{\partial\lambda} + \delta(\lambda)m^2\frac{\partial}{\partial m^2}\right\}Z[\rho] \\ &= [(n-4)\lambda + \beta(\lambda)]\frac{\partial}{\partial\lambda}Z[\rho] \\ &\quad -\frac{1}{2}\delta(\lambda)m^2\int(d_E^n x)Z\left[\rho; [\phi^2](x)\right]\,,\end{aligned} \tag{5.5.41}$$

which must be a finite quantity. On the other hand, we may use the functional integral representation and the bare parameter variations to compute

$$\begin{aligned}\frac{d}{d\epsilon}Z[\rho] &= -\int[d\phi]\,e^{-\int\mathcal{L}_E}\int(d_E^n x)\left\{\frac{1}{4!}(n-4)\lambda_0\phi^4(x) + \gamma(\lambda)\rho_0(x)\phi(x)\right\} \\ &= -\int(d_E^n x)\left\{(n-4)\frac{1}{4!}\lambda_0 Z[\rho; \phi_0^4(x)] + \gamma(\lambda)Z[\rho; [E](x)]\right\}\,.\end{aligned} \tag{5.5.42}$$

The bare operator ϕ_0^4 and the renormalized composite operator $[\phi^4]$ differ only by an ascending series of poles in $(n-4)$ as shown in Eq. (5.5.33). Hence comparing the two expressions for $dZ/d\epsilon$, we conclude that

$$\frac{\partial}{\partial\lambda}Z[\rho] = -\frac{\mu^{4-n}}{4!}\int(d_E^n x)Z[\rho; [\phi^4](x)]\,, \tag{5.5.43}$$

and that

$$\frac{1}{4!}\lambda_0\phi_0^4(x) = \left[1+\frac{\beta(\lambda)/\lambda}{n-4}\right]\frac{\mu^{4-n}}{4!}\lambda[\phi^4](x) - \frac{\gamma(\lambda)}{n-4}[E](x) + \frac{\delta(\lambda)}{n-4}\frac{1}{2}m^2[\phi^2](x) + \frac{d(\lambda;n)}{n-4}(-\partial^2)[\phi^2](x)\,. \tag{5.5.44}$$

The last term in Eq. (5.5.44) contains a total derivative, and so the coefficient $d(\lambda;n)$ cannot be determined from the integral constraint. Minimal subtraction requires that $d(\lambda;n)$ is a function of the coupling λ plus other functions of λ multiplying an ascending series of poles in $(n-4)$.

5 $[T^\mu{}_\mu]$

The trace of the stress-energy tensor is related to the renormalization group. To bring out this relation, we first need to examine the construction of the stress tensor $T^{\mu\nu}$. The structure of $T^{\mu\nu}$ for a free field theory was worked out in Problem 13 of Chapter 3. The generalization of this work to the interacting theory produces

$$T^{\mu\nu} = \partial^\mu\phi_0\,\partial^\nu\phi_0 - g^{\mu\nu}\left\{\frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{4!}\lambda_0\phi_0^4\right\} + \left[\eta_0 + \frac{n-2}{4(n-1)}\right](g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\phi_0^2\,. \tag{5.5.45}$$

Here η_0 is an undetermined, free parameter. The addition of $\frac{1}{4}(n-2)(n-1)^{-1}$ is chosen so as to simplify the stress tensor trace.† The other parameters appearing in the stress tensor are determined for the following reasons. The divergence of $T^{\mu\nu}$ is easily calculated to be

$$\partial_\nu T^{\mu\nu} = -E_0^\mu\,, \tag{5.5.46}$$

where

$$E_0^\mu = (\partial^\mu\phi_0)\left(-\partial^2\phi_0 + m_0^2\phi_0 + \frac{1}{3!}\lambda_0\phi_0^3\right). \tag{5.5.47}$$

Thus, $\partial_\nu T^{\mu\nu} = 0$ when the field equation is obeyed. This condition fixes the numerical coefficients displayed in Eq. (5.5.45) except for an overall multiplicative factor.

To justify this overall factor we note, just as in the proof of Eq. (5.5.30) for the insertion of the equation of motion operator E_0, that it is not difficult to establish the divergence relation

$$-\partial_\nu\langle\phi(x_1)\cdots\phi(x_N)\,T^{\mu\nu}(x)\rangle = \langle\phi(x_1)\cdots\phi(x_N)E_0^\mu(x)\rangle = \sum_{a=1}^{N}\delta(x-x_a)\frac{\partial}{\partial x_{\mu a}}\langle\phi(x_1)\cdots\phi(x_N)\rangle\,. \tag{5.5.48}$$

For the moment, let us transform back to Minkowski space-time where the products of fields appearing in Eq. (5.5.48) become time-ordered products. Integrating the resulting identity over a small time slice (with $x_4 = ix^0$) about $x^0 = x_b^0$ gives a finite result when the width of the slice is taken to vanish. Thus the time-ordered product has discontinuities involving the equal-time commutators of the stress tensor with the field. Since the stress tensor $T^{\mu\nu}$ and coordinate x^μ transform in the same way, $\partial_4 T^{\mu 4} = \partial_0 T^{\mu 0}$, and these discontinuities imply that the equal-time commutators are given by

$$i\left[\phi(y), T^{\mu 0}(x)\right]\Big|_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial y_\mu}\phi(y)\,. \tag{5.5.49}$$

This is the normalization needed to ensure that the generators of space-time translations and rotations,

$$P^\mu = \int(d\mathbf{r})T^{0\mu}\,, \tag{5.5.50}$$

and

$$J^{\mu\nu} = \int(d\mathbf{r})(x^\mu T^{0\nu} - x^\nu T^{0\mu})\,, \tag{5.5.51}$$

have the correct commutation relations with the field $\phi(x)$.

As in the discussion of E_0, it is clear that E_0^μ is a finite operator,

$$E_0^\mu = [E^\mu]\,. \tag{5.5.52}$$

Hence the stress tensor $T^{\mu\nu}$ must also be a finite operator if the only free, bare parameter η_0 is suitably renormalized,

$$T^{\mu\nu} = [T^{\mu\nu}]\,. \tag{5.5.53}$$

Note, incidentally, that since the stress tensor is composed only of bare quantities, it is independent of the mass unit,

$$\mu\frac{d}{d\mu}T^{\alpha\beta} = 0\,. \tag{5.5.54}$$

A little algebra shows that the stress tensor trace may be written as

$$T^\mu{}_\mu = -\frac{1}{2}(n-2)E_0 - m_0^2\phi_0^2 + (n-4)\frac{1}{4!}\lambda_0\phi_0^4 + (n-1)\eta_0\partial^2\phi_0^2\,. \tag{5.5.55}$$

† The stress tensor can be derived by making the theory invariant under general coordinate transformations with $T^{\mu\nu}$ appearing as the response to an arbitrary metric variation $\delta g_{\mu\nu}(x)$. The arbitrary constant η_0 corresponds to the coupling chosen for the undetermined $R\phi^2$ term, where R is the scalar curvature.

Expressing this in terms of the renormalized composite operators produces (neglecting unit operator terms)

$$T^{\mu}{}_{\mu} = -\left[\frac{1}{2}(n-2)+\gamma(\lambda)\right][E] - \left[1-\frac{1}{2}\delta(\lambda)\right]m^2[\phi^2] + [(n-4)\lambda+\beta(\lambda)]\frac{1}{4!}\mu^{4-n}[\phi^4] + (n-1)\eta\partial^2[\phi^2]\,, \quad (5.5.56)$$

where

$$\eta = \eta_0 F_m(\lambda;n)^{-1} - \frac{d(\lambda;n)}{n-1}\,. \quad (5.5.57)$$

The condition that $T^{\mu}{}_{\mu}$ be finite requires that η_0 be chosen such that the parameter η is finite.

6 Renormalization group and $T^{\mu}{}_{\mu}$

The connection between the stress tensor trace and the renormalization group is brought out by considering the dilation current defined by

$$D_\mu = T_{\mu\nu}\,x^\nu\,. \quad (5.5.58)$$

In view of the stress tensor divergence condition (5.5.46), this current obeys the conservation law

$$\partial_\mu D^\mu = -x^\mu[E_\mu] + T^{\mu}{}_{\mu}\,. \quad (5.5.59)$$

Inserting this in an N-point Green's function and integrating over space time gives

$$0 = -\sum_{a=1}^{N} x_a^\mu \frac{\partial}{\partial x_a^\mu}\langle\phi(x_1)\cdots\phi(x_N)\rangle + \int (d_E^n x)\,\langle\phi(x_1)\cdots\phi(x_N)\,T^{\mu}{}_{\mu}(x)\rangle\,. \quad (5.5.60)$$

The left-hand side of this equation vanishes since it is the integral of a total divergence. The first set of terms on the right-hand side follow from the use of Eq. (5.5.48). The dimensional structure of the Green's function is displayed by writing

$$\langle\phi(x_1)\cdots\phi(x_N)\rangle = \mu^{N(\frac{n}{2}-1)}\,F(\mu x_1,\ldots,\mu x_N\,;\,m/\mu)\,. \quad (5.5.61)$$

Hence

$$\sum_{a=1}^{N} x_a^\mu \frac{\partial}{\partial x_a^\mu}\langle\phi(x_1)\cdots\phi(x_N)\rangle = \left\{\mu\frac{\partial}{\partial\mu} + m\frac{\partial}{\partial m} - N\left(\frac{n}{2}-1\right)\right\}\langle\phi(x_1)\cdots\phi(x_N)\rangle\,. \quad (5.5.62)$$

The other ingredient in Eq. (5.5.60), the insertion of $T^{\mu}{}_{\mu}$, contains the terms shown in Eq. (5.5.56). From Eq. (5.5.30) it follows that

$$\int (d_E^\mu x)\langle\phi(x_1)\cdots\phi(x_N)[E](x)\rangle = N\langle\phi(x_1)\cdots\phi(x_N)\rangle\,. \quad (5.5.63)$$

According to Eqs. (5.5.19) and (5.5.43) the insertion of the space-time integrals of $[\phi^2]$ and $[\phi^4]$ are equivalent to derivatives with respect to m^2 and λ. Putting the various terms together, Eq. (5.5.60) may be written as

$$\left\{\mu\frac{\partial}{\partial\mu} + \delta(\lambda)m^2\frac{\partial}{\partial m^2} + [(n-4)\lambda+\beta(\lambda)]\frac{\partial}{\partial\lambda} + N\gamma(\lambda)\right\}\langle\phi(x_1)\cdots\phi(x_N)\rangle = 0\,. \quad (5.5.64)$$

This is the renormalization group equation derived 22 pages ago. We have now gone through a full circle just as Richard Wagner did in his composition of the Ring cycle.

5.6 Operator products

High energies with large momentum transfer correspond to small space-time intervals. This remark motivated the study in the proceeding section on composite operators. But we also need to study products of field operators at short distances that are not in the previous coincident limit. We shall sketch this behavior here by examining $\phi(x+\epsilon/2)\phi(x-\epsilon/2)$ in Euclidean space. In the dimensionally regulated theory, we may simply expand the product of bare field operators in powers of ϵ^μ,

$$\phi_0\Big(x+\frac{1}{2}\epsilon\Big)\,\phi_0\Big(x-\frac{1}{2}\epsilon\Big) = \phi_0(x)^2 + \frac{1}{8}\epsilon^\mu\epsilon^\nu\,[\partial_\mu\partial_\nu\phi_0(x)\phi_0(x) + \phi_0(x)\partial_\mu\partial_\nu\phi_0(x) - 2\partial_\mu\phi_0(x)\partial_\nu\phi_0(x)] + \ldots\,. \quad (5.6.1)$$

This expansion, however, breaks down in the limit $n\to 4$. First we note that, to obtain a well defined $n\to 4$ limit, the bare operators $\phi_0(x+\epsilon/2)$ and $\phi_0(x-\epsilon/2)$ must be replaced by the renormalized fields $\phi(x+\epsilon/2)$ and $\phi(x-\epsilon/2)$. Moreover, the coincident limits $\phi_0(x)^2\,,\ldots$, must be replaced by the appropriate renormalized composite operators $[\phi^2](x)\,,\ldots$. To illustrate the basic character of this operator product expansion, we shall consider only the leading term in the $\epsilon^\mu\to 0$ limit, the term involving the composite operator $[\phi^2](x)$. The simple limit $\phi(x-\epsilon/2)\phi(x+\epsilon/2)\to[\phi^2](x)$ cannot hold since the operators have different renormalization factors. As we shall see in an explicit example, this mismatch in the divergence structure is reflected in a numerical coefficient which becomes singular in the $\epsilon^\mu\to 0$ limit. The proper form of the operator product

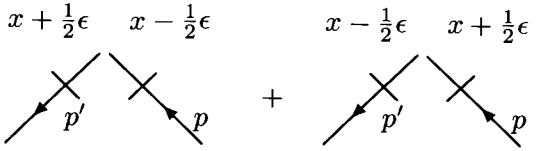


Fig. 5.31. Tree graphs for the simple operator product.

expansion, the form which has a well-defined $n \to 4$ limit, reads

$$\phi\Big(x+\frac{1}{2}\epsilon\Big)\phi\Big(x-\frac{1}{2}\epsilon\Big) = C(\mu\epsilon;\, \lambda(\mu);\, n)[\phi^2](x) + \dots . \tag{5.6.2}$$

Several comments need to be made about this expansion. We consider only the leading terms in the $\epsilon^\mu \to 0$ limit. Thus the numerical coefficient $C(\mu\epsilon;\, \lambda(\mu);\, n)$ contains only the singular and constant terms which appear in the limit. The coefficient function is dimensionless, and so it is a function only of the dimensionless variables that have been displayed. The mass parameter m^2 does not appear because, by the dimensional analysis which holds in the minimal subtraction scheme, the mass parameter occurs in the form $m^2\epsilon^2$, and this contribution vanishes in the $\epsilon^\mu \to 0$ limit. As was shown in the preceding section, the renormalized composite operator $[\phi^2]$ mixes with the unit operator 1. Numerical terms of the form $(\epsilon^2)^{1-n/2}E_0(\mu\epsilon;\, \lambda;\, n)$ and $m^2(\epsilon^2)^{2-n/2}E_2(\mu\epsilon;\, \lambda;\, n)$, where E_0 and E_2 are dimensionless functions, also appear in leading order in the operator product expansion. To simplify the exposition, we shall omit these purely numerical terms which contribute only to vacuum expectation values (and to disconnected parts). Finally we note that since

$$\begin{aligned}\phi(x)^2 &= F_\rho(\lambda;\, n)^{-2}\phi_0(x)^2 \\ &= F_\rho(\lambda;\, n)^{-2}F_m(\lambda;\, n)^{-1}[\phi^2](x)\,,\end{aligned} \tag{5.6.3}$$

the coefficient functions must have the limiting value when $\epsilon^\mu \to 0$ with $n \neq 4$ given by

$$C(0;\, \lambda;\, n) = F_\rho(\lambda;\, n)^{-2}F_m(\lambda;\, n)^{-1}\,. \tag{5.6.4}$$

To compute the coefficient function $C(\mu\epsilon;\, \lambda;\, n)$ through first order in the coupling λ,

$$C(\mu\epsilon;\, \lambda;\, n) \simeq 1 + \lambda C_1(\mu\epsilon;\, n)\,, \tag{5.6.5}$$

we consider the two-field matrix element of the operator product. Removing the external propagator factors, this corresponds to the zeroth- and first-order graphs shown in Figs. 5.31 and 5.32. These are just the graphs for the composite operator $\phi^2(x)$ evaluated in the preceding section except that now the two coordinate points are spread apart. This generalization of the previous amplitude gives, for the first two terms of

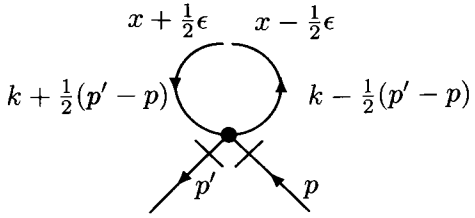


Fig. 5.32. One loop correction to previous tree graphs.

the perturbative expansion in powers of the coupling constant,

$$\Gamma(p',\, p;\, x,\, \epsilon) = e^{-i(p'-p)x}\left\{2 - \lambda\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\,\frac{1}{[(k-\frac{1}{2}(p'-p))^2+m^2]}\,\frac{1}{[(k+\frac{1}{2}(p'-p))^2+m^2]}e^{ik\epsilon}\right\}. \tag{5.6.6}$$

Here we neglect exponential terms such as $\exp\{-ip\epsilon/2\}$ in the tree graph contribution since we are not interested in quantities that vanish in the $\epsilon^\mu \to 0$ limit. By the evaluation of the preceding section, the amplitude for the corresponding renormalized composite operator $[\phi^2](x)$ reads

$$\Gamma(p',\, p;\, x) = e^{-i(p'-p)x}\left\{2 - \lambda\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\,\frac{1}{[(k-\frac{1}{2}(p'-p))^2+m^2]}\,\frac{1}{[(k+\frac{1}{2}(p'-p))^2+m^2]} + \frac{2\lambda}{(4\pi)^2}\,\frac{1}{n-4}\right\}. \tag{5.6.7}$$

The operator product expansion (5.6.2) with the first-order approximation (5.6.5) implies that

$$\Gamma(p',\, p;\, x,\, \epsilon) \simeq [1 + \lambda C_1(\mu\epsilon;\, n)]\,\Gamma(p',\, p;\, x) \tag{5.6.8}$$

which, in comparison with Eqs. (5.6.6) and (5.6.7), yields

$$C_1(\mu\epsilon;\, n) = -\frac{1}{2}\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\left(\frac{1}{k^2}\right)^2[e^{ik\epsilon}-1] + \frac{1}{(4\pi)^2}\,\frac{1}{n-4}\,. \tag{5.6.9}$$

In obtaining this result, we have set $p' = p = 0$ and $m^2 = 0$. We have done this because the inclusion of these terms gives contributions that vanish in the $\epsilon^\mu \to 0$ limit, terms that are not included in the definition of the coefficient function.

The remaining computation of $C_1(\mu\epsilon\,; n)$ may now be done with the standard method. One exponentiates the denominators with

$$\left(\frac{1}{k^2}\right)^2 = \int_0^\infty ds\, s e^{-sk^2}\,, \tag{5.6.10}$$

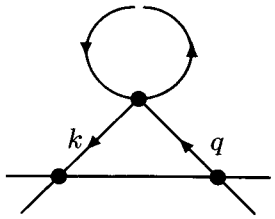


Fig. 5.33. Operator product in a higher order graph.

and performs the resulting Gaussian k-integral. Changing integration variables to $t = \epsilon^2/4s$ puts the result in the form

$$C_1(\mu\epsilon;\, n) = \frac{1}{(4\pi)^2}\,\frac{1}{n-4} - \frac{1}{2}\,\frac{1}{(4\pi)^{n/2}}\left(\frac{\mu^2\epsilon^2}{4}\right)^{2-n/2}\int_0^\infty dt\, t^{\frac{n}{2}-3}[e^{-t}-1]\,. \tag{5.6.11}$$

The t-integration which appears here converges if $2 < n < 4$. It may be done by writing

$$t^{\frac{n}{2}-3} = \frac{1}{\frac{n}{2}-2}\,\frac{d}{dt}\,t^{\frac{n}{2}-2}\,, \tag{5.6.12}$$

and integrating by parts. This gives the standard integral representation of the Γ-function and so

$$C_1(\mu\epsilon;\, n) = \frac{1}{(4\pi)^2}\,\frac{1}{n-4}\left[1-(\pi\mu^2\epsilon^2)^{2-\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)\right]. \tag{5.6.13}$$

The limit $\epsilon^\mu \to 0$ may be taken if $n < 4$. This limit yields

$$C(0;\,\lambda;\, n) \simeq \left[1+\frac{\lambda}{(4\pi)^2}\,\frac{1}{n-4}\right], \tag{5.6.14}$$

in agreement with the terms to first order in Eq. (5.6.4). The physical limit $n \to 4$ produces

$$C(\mu\epsilon;\,\lambda;\, n=4) \simeq \left\{1+\frac{1}{2}\,\frac{\lambda}{(4\pi)^2}\left[\ln(\pi\mu^2\epsilon^2)+\gamma\right]\right\}. \tag{5.6.15}$$

The order λ term entails a logarithmic correction at short distances.

It can be proven that the operator product expansion is valid in all orders of perturbation theory. We shall not attempt such a proof here, but it is worthwhile to briefly consider a graph which illustrates how the operator product expansion works out in higher orders. This is the graph shown in Fig. 5.33. The bottom, "triangle", part of this graph corresponds to a convergent integral at $n = 4$. Thus, effectively, the momenta labeled by k and q are finite. The top part of the graph describes the amplitude which was just computed. It gives a short-distance limit

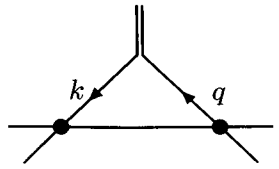


Fig. 5.34. Composite operator factor of the short-distance limit of the previous graph.

which is independent of the momenta k and q flowing into it. The short-distance limit of the complete graph factors into two parts, the bottom triangle part corresponding to the insertion of the composite operator $\phi^2(x)$ as illustrated in Fig. 5.34. The top part corresponds to the first-order correction to the coefficient function $C_1(\mu\epsilon)$. The top part of the graph is renormalized just as was done previously, with a counter term added to the lower order graph for the composite operator insertion.

The coefficient function obeys a renormalization group equation. Operating on Eq. (5.6.2) with $\mu d/d\mu$ and using

$$\mu\frac{d}{d\mu}\phi = -\gamma\phi\,, \qquad \mu\frac{d}{d\mu}[\phi^2] = -\delta[\phi^2]\,, \tag{5.6.16}$$

shows that

$$\begin{aligned}\mu\frac{d}{d\mu}C(\mu\epsilon;\,\lambda;\, n) &= \left\{\mu\frac{\partial}{\partial\mu}+[(n-4)\lambda+\beta(\lambda)]\frac{\partial}{\partial\lambda}\right\}C(\mu\epsilon;\,\lambda;\, n)\\ &= [\delta(\lambda)-2\gamma(\lambda)]\,C(\mu\epsilon;\,\lambda;\, n)\,.\end{aligned} \tag{5.6.17}$$

To solve this renormalization group equation, we define a distance-dependent coupling $\lambda(\epsilon)$ by

$$\int_{\lambda(\epsilon)}^{\lambda}\frac{d\lambda'}{(n-4)\lambda'+\beta(\lambda')} = \frac{1}{2}\ln(\mu^2\epsilon^2)\,. \tag{5.6.18}$$

Since the right-hand side vanishes when $\epsilon^2 = 1/\mu^2$, $\lambda(\epsilon) = \lambda$ in this case. On the other hand, keeping $n < 4$, $\lambda(\epsilon) \to 0$ when $\epsilon^\mu \to 0$: In this limit, the right-hand side of Eq. (5.6.18) diverges; this divergence is reproduced on the left-hand side by the vanishing of the lower limit of the integration since $\beta(\lambda') \simeq (n-4)\lambda'$ for small λ' and so the integral also diverges. Acting upon Eq. (5.6.18) with $\mu d/d\mu$, with the upper limit of integration varying in accordance with $\mu d\lambda/d\mu = \beta(\lambda;\, n)$, shows that $\lambda(\epsilon)$ is an invariant of the renormalization group,

$$\mu\frac{d}{d\mu}\lambda(\epsilon) = 0\,. \tag{5.6.19}$$

With this discussion in mind, it is easy to see that the solution of the

renormalization group equation (5.6.17) is given by

$$C(\mu\epsilon\,;\lambda;\,n) = \exp\left\{\int_{\lambda(\epsilon)}^{\lambda} d\lambda' \,\frac{\delta(\lambda') - 2\gamma(\lambda')}{(n-4)\lambda' + \beta(\lambda')}\right\} C(1;\,\lambda(\epsilon);\,n)\,. \quad (5.6.20)$$

Since $\lambda(\epsilon) = \lambda$ when $\epsilon^2 = 1/\mu^2$, this solution has the proper boundary condition. As a further check, we note that the limit $\epsilon^\mu \to 0$ gives

$$C(0;\,\lambda;\,n) = \exp\left\{\int_0^{\lambda} d\lambda' \,\frac{\delta(\lambda') - 2\gamma(\lambda')}{(n-4)\lambda' + \beta(\lambda')}\right\}, \quad (5.6.21)$$

since in this limit $\lambda(\epsilon) \to 0$ and $C(1;\,0;\,n) = 1$. The exponential of integrals which appears here is precisely the product of renormalization factors shown in Eq. (5.6.4).

It is of some interest to show how the general solution (5.6.20) conforms to the first-order explicit result which we obtained. We leave this exercise to the reader, remarking only that to lowest order, the distance-dependent coupling defined in Eq. (5.6.18) is given by

$$\frac{1}{n-4}\ln\left[\frac{\lambda}{\lambda(\epsilon)}\right] = \frac{1}{2}\ln(\mu^2\epsilon^2)\,, \quad (5.6.22)$$

or

$$\lambda(\epsilon) = (\mu^2\epsilon^2)^{2-\frac{n}{2}}\lambda\,. \quad (5.6.23)$$

Problems

"What a curious feeling!!" said Alice. "I must be shutting up like a telescope!"

And so it was indeed: she was now only ten inches high, and her face brightened up at the thought that she was now the right size for going through the little door into that lovely garden. First, however, she waited for a few minutes to see if she was going to shrink any further.

After a while, finding that nothing more happened, she decided on going into the garden at once; but, alas poor Alice! when she got to the door, she found she had forgotten the little golden key, and when she went back to the table for it, she found she could not possibly reach it: she could see it quite plainly through the glass, and she tried her best to climb up one of the legs of the table, but it was too slippery; and when she had tired herself out with trying, the poor little thing sat down and cried. "Come, there's no use in crying like that!" said Alice to herself rather sharply. "I'll advise you to leave off this minute!" She generally gave herself very good advice (though she very seldom followed it).

Soon her eye fell on a little glass box that was lying under the table: she opened it, and found in it a very small cake, on which the words "EAT ME" were beautifully marked in currants. "Well, I'll eat it," said Alice "and if it makes me grow larger, I can reach the key; and if it makes me grow smaller, I can creep under the door: so either way I'll get into the garden, and I don't care which happens!"

She ate a little bit, and said anxiously to herself "Which way? Which way?", holding her hand on the top of her head to feel which way it was growing; and she was quite surprised to find that she remained the same size."

Lewis Carroll

With this quotation having explained the renormalization group one can now turn to work out some problems.

1. Pauli-Villars Regularization

The text uses dimensional continuation to regulate quantum field theory because this is a very convenient method. However, any method of regularization which respects the basic symmetries of the theory such as

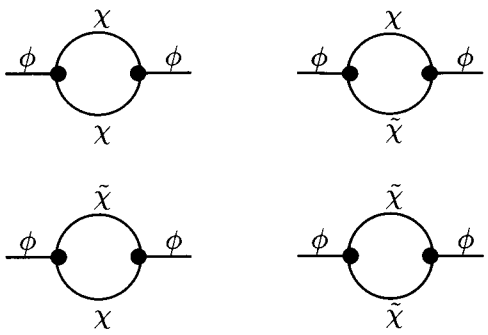


Fig. 5.35.

Poincaré invariance must give the same final result for physically measurable amplitudes if the theory is to make sense. An alternative method of regularization is that of Pauli and Villars. The purpose of this problem is to explain this method by applying it to the theory described by

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2} g\phi\chi^2 ,$$

where

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}M_0^2\chi^2 ,$$

a theory already considered in Problem 2 of Chapter 4.

The method is to add fields with negative norm so as to cancel the infinities, with these fields effectively removed after this cancellation by sending their masses to infinity. In our case this is done by extending

$$\mathcal{L} \to \mathcal{L}_0 - \frac{1}{2}(\partial_\mu\tilde{\chi})^2 - \frac{1}{2}\tilde{M}_0^2\tilde{\chi}^2 + \frac{1}{2}g\phi(\chi + i\tilde{\chi})^2 .$$

a) Compute the one-loop self energy function $\Sigma(p)$ corresponding to the graphs in Fig. 5.35. Take the limit $\tilde{M}_0^2 \to \infty$ keeping, of course, the divergent $\ln(\tilde{M}_0^2/\mu^2)$ and the finite remainder. Here μ is a unit of mass introduced to keep the dimensions straight. Absorb the $\ln(\tilde{M}_0^2/\mu^2)$ by a mass renormalization and show that, up to an additive constant, the finite remainder is identical to that found in Problem 2 of Chapter 4 by using dimensional regularization. Note that the pole $(n-4)^{-1}$ in dimensional regularization is now replaced by $\ln(\tilde{M}_0^2/\mu^2)$; compare the factors multiplying $(n-4)^{-1}$ and $\ln(\tilde{M}_0^2/\mu^2)$.

b) Show that the four graphs in part a) above combine in such a way as

to effectively make the replacement

$$\frac{1}{k^2 + M_0^2} \to \frac{\tilde{M}_0^2 - M_0^2}{[k^2 + M_0^2][k^2 + \tilde{M}_0^2]}$$

for the propagator in the original, unregulated theory. Use the functional integral representation with source terms $\rho\phi$ and $\sigma\chi$ to prove that this always happens for any set of graphs. Do this by first changing integration variables from χ to $\chi' = \chi + i\tilde{\chi}$ and then integrate out the $\tilde{\chi}$ field.

2. Renormalization in Two Dimensions

Consider the scalar field theory in one-space, one-time dimension defined by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_{2,0}^2\phi^2 - \frac{1}{4!}m_{4,0}^2\phi^4 - \frac{1}{6!}m_{6,0}^2\phi^6 . \qquad (5.6.24)$$

a) Explain why the only divergencies arise from closed loops which start and end on the same vertex.

In working out this problem be sure to draw graphs to explain what you are doing.

b) First work out the renormalization associated with the ϕ^6 interaction. Suppose four external lines or four lines from other vertices or some combination of external lines and lines from other vertices are connected to this interaction. They can do so in $6 \cdot 5 \cdot 4 \cdot 3$ ways. The remaining ϕ^2 can form a single loop. This gives a divergent term

$$\frac{1}{2}m_{6,0}^2\langle\phi(0)^2\rangle .$$

Continue to n dimensions and write

$$m_{6,0}^2 = m_6^2\mu^{4-2n} ,$$

where m_6 is a finite six-leg coupling constant with the dimensions of mass. Why is m_6 finite? Show that

$$\langle\phi(0)^2\rangle = m^{n-2}\frac{\Gamma(1-n/2)}{(4\pi)^{n/2}} ,$$

where m is the mass used for the propagator. We shall see that the particular value of this mass is irrelevant. Thus the divergent term is given by

$$-\frac{1}{2}m_6^2\mu^{2-n}\frac{1}{4\pi}\,\frac{1}{1-n/2} ,$$

and this divergence is removed by the renormalization

$$m_{4,0}^2 = \mu^{2-n}\left[m_4^2 + \frac{m_6^2}{4\pi}\,\frac{1}{n-2}\right].$$

Suppose now that two "external" lines are connected to the ϕ^6 vertex. They can do so in $6\cdot 5$ ways. The remaining ϕ^4 can form two closed loops. This gives a divergent term

$$-\frac{1}{4!}m_{6,0}^2 3\langle\phi(0)^2\rangle^2.$$

But the two "external" lines can also connect to the ϕ^4 interaction in $4\cdot 3$ ways and the remaining ϕ^2 can form a closed loop. This gives a divergent term

$$-\frac{1}{2}m_{4,0}^2\langle\phi^2(0)\rangle.$$

Show that these divergencies are removed by the renormalization

$$m_{2,0}^2 = m_2^2 + \frac{m_4^2}{4\pi}\,\frac{1}{n-2} + \frac{1}{2}\frac{m_6^2}{(4\pi)^2}\,\frac{1}{(n-2)^2}.$$

Note that this is independent of m as advertised; there are no terms of the form

$$\ln(m/\mu)\frac{1}{n-2}.$$

Note also the "triangular" counter term structure.

c) Check this result by computing

$$\mu\frac{dm_6^2}{d\mu}, \quad \mu\frac{dm_4^2}{d\mu}, \quad \mu\frac{dm_2^2}{d\mu},$$

and showing that they are finite.

d) There is a general method which gives the renormalization of any power of ϕ^2. Consider an interaction of the general form

$$\mathcal{L}_1 = -M_0^2\cosh(\lambda_0\phi),$$

which serves as a generating function for any power of ϕ^2. Although ϕ is an interacting field, at any given time it can be expressed in terms of creation and annihilation operator for a free field of mass m. That is one can write

$$\phi = \phi^{(+)} + \phi^{(-)},$$

where

$$\phi^{(+)} = \phi^{(-)\dagger},$$

and

$$\phi^{(+)}|0\rangle = 0,$$

with $|0\rangle$ the bare vacuum state. Moreover

$$[\phi^{(+)}(0), \phi^{(-)}(0)]$$

is a number, not an operator. Prove that

$$e^{(A+B)} = e^A e^B e^{-\frac{1}{2}[A,B]},$$

if $[A, B]$ commutes with A and B. [Hint: Replace $A \to \alpha A$, $B \to \alpha B$ and consider the derivative with respect to α.] Hence

$$\cosh(\kappa\phi) =: \cosh(\kappa\phi) : \exp\left\{-\frac{1}{2}\kappa^2[\phi^{(-)}(0), \phi^{(+)}(0)]\right\},$$

where $: \cdots :$ denotes the "normal ordered" product in which any power of ϕ is expressed in terms of $\phi^{(+)}$ and $\phi^{(-)}$ with all the $\phi^{(-)}$ written on the left and all the $\phi^{(+)}$ written on the right. Now

$$\langle\phi(0)^2\rangle = \langle\phi^{(+)}(0)\phi^{(-)}(0)\rangle = [\phi^{(+)}(0), \phi^{(-)}(0)].$$

Therefore

$$: \cosh(\kappa\phi) : = \cosh(\kappa\phi)\exp\left\{-\frac{1}{2}\kappa^2\langle\phi(0)^2\rangle\right\}.$$

With the interaction terms normal ordered, there are no loops which begin and end at the same vertex. Hence

$$\mathcal{L}_1 = -M^2\mu^{n-2}\exp\left\{\frac{\lambda^2}{4\pi}\,\frac{1}{n-2}\right\}\cosh(\lambda\mu^{1-n/2}\phi)$$

expresses the interaction in terms of a finite, dimensionless parameter λ. Since $\mathcal{L}_1$ is now finite, the finite terms in $\langle\phi(0)^2\rangle$ which involve the mass parameter m may be neglected. This defines the minimal subtraction.

Since $\mathcal{L}_1$ is finite, each of the independent terms in its expansion in powers of λ is finite. Thus if the terms are relabeled by making the replacements

$$M^2\lambda^2 \to m_2^2, \quad M^2\lambda^4 \to m_4^2,$$

and so forth, the renormalization of the previous form of the Lagrange function is obtained. Work out the details to obtain the renormalization of $m_{2,0}^2$, $m_{4,0}^2$ and $m_{6,0}^2$ found previously.

3. Asymptotic Freedom

Use dimensional regularization to compute the β-function to one loop order for the theory described by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2 - \frac{1}{3!}g\phi^3$$

in a space-time of six dimensions.

4. General Properties of the Renormalization Group

a) Suppose that one makes a general redefinition of the coupling constant by replacing λ by

$$\bar{\lambda} = \bar{\lambda}(\lambda)\,.$$

In order to make the perturbation theory have the same structure we require that for small λ

$$\bar{\lambda}(\lambda) = \lambda + 0(\lambda^2)\,.$$

How is $\bar{\beta}(\bar{\lambda})$ related to $\beta(\lambda)$? Expand

$$\begin{aligned}\beta(\lambda) &= b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + \dots\,,\\ \bar{\beta}(\bar{\lambda}) &= \bar{b}_2\bar{\lambda}^2 + \bar{b}_3\bar{\lambda}^3 + \bar{b}_4\bar{\lambda}^4 + \dots\,,\end{aligned}$$

and

$$\bar{\lambda} = \lambda + a_2\lambda^2 + a_3\lambda^3 + \dots\,.$$

How are $\bar{b}_2$, $\bar{b}_3$, $\bar{b}_4$ related to b_2, b_3, b_4? Suppose that $\lambda = \lambda_F$ is a fixed point. Show that $\bar{\lambda} = \bar{\lambda}_F$ is also a fixed point. Relate the slope at the fixed point $\bar{\beta}'_F$ to β'_F.

b) Consider an asymptotically free theory in which

$$\mu\frac{dg^2}{d\mu} = \beta(g^2)\,,$$

where

$$\beta(g^2) = -bg^4 - cg^6 + \dots$$

with $b > 0$. Compute the first two terms in the asymptotic expansion of $1/g(\mu)^2$ at large μ. Explain why these first two terms are unique and unchanged by a redefinition of the coupling constant. Explain why the first two terms are needed to determine the scale Λ.

5. The Decoupling Theorem

The decoupling theorem is illustrated by considering a simple theory of two scalar fields described by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}(\partial_\mu\phi_1) - \frac{1}{2}(\partial_\mu\phi_2)^2\\ &-\frac{1}{2}m_{1,0}^2\phi_1^2 - \frac{1}{2}m_{2,0}^2\phi_2^2 - \frac{\lambda_0}{4!}(\phi_1^2+\phi_2^2)^2\,.\end{aligned}$$

Use dimensional regularization and minimal subtraction.

a) Compute the β-function to one-loop order. Use this β-function to express $\lambda(\mu)$ in terms of $\lambda(M)$ including terms of order λ^2.

b) Now to the decoupling theorem: In the limit where the mass m_2 becomes large with the external momenta and mass unit μ fixed, the field ϕ_2 decouples from the theory leaving an effective theory described by

$$\bar{\mathcal{L}} = -\frac{1}{2}(\partial_\mu\phi_1)^2 - \frac{1}{2}\bar{m}_0^2\phi_1^2 - \frac{\bar{\lambda}_0}{4!}\phi_1^4\,.$$

The renormalized running coupling constant $\bar{\lambda}(\mu)$, however, differs from that of the original theory $\lambda(\mu)$ [and $\bar{m}^2(\mu)$ also differs from $m_1^2(\mu)$]. Demonstrate this decoupling by computing the $\phi_1 - \phi_1$ elastic scattering amplitude in the original theory through one-loop order. Take the limit of large m_2^2 (with the external momenta and μ fixed) and show explicitly that the scattering amplitude described by $\bar{\mathcal{L}}$ is obtained. Relate $\bar{\lambda}(\mu)$ to $\lambda(\mu)$ through order λ^2.

c) Compute the β-function $\bar{\beta}(\bar{\lambda})$ of $\bar{\mathcal{L}}$ in one-loop order (or look up the answer in the text). Use this $\bar{\beta}(\bar{\lambda})$ to compute $\bar{\lambda}(\mu)$ in terms of $\bar{\lambda}(M)$ through order λ^2.

d) Explain why

$$\bar{\lambda}(m_2) = \lambda(m_2) + 0(\lambda^2)\,,$$

where the $0(\lambda^2)$ has no large logarithms. Use the result of part c) to compute $\bar{\lambda}(\mu)$ in terms of $\bar{\lambda}(m_2)$. Use the result of part a) to compute $\lambda(m_2)$ in terms of $\lambda(\mu)$. We now have an expression for $\bar{\lambda}(\mu)$ in terms of $\lambda(\mu)$ which contains the large logarithm of the difference, but may have an error of order λ^2 with no large logarithm. Compare this result with the result found above at the end of part b).

6

Structural aspects

We turn now to describe some important general features of quantum field theory so that one can obtain a better overall view of the theory. This will be done by continuing to use the scalar theory as an illustration. We shall first investigate the structure of the interacting propagator — the two-point Green's function. This structure will tell us how to construct single-particle states for an interacting theory. The single-particle state construction will then be extended to multiparticle states, and the relationship of n-point Green's functions to scattering amplitudes will be obtained. This relationship is called the "reduction formula". The interacting propagator can also describe unstable particles, and this will be explained including an outline of how such particles are produced and detected and how very short-lived particles appear as resonances in scattering amplitudes. The effective action will then be introduced, and it will be shown to be the generating functional of connected amplitudes. As a by-product, the cluster decomposition theorem will be described and the fact that the power of Planck's constant $\hbar$ counts the number of closed loops in a graph will be explained. Finally, the Legendre transform of the effective action will be examined. It is the generating function of single-particle irreducible, connected graphs. Its restriction to constant fields defines the effective potential which is a useful instrument for describing spontaneously broken symmetry. We already have a precursor to these topics in the discussion of spontaneous symmetry breakdown in many-particle systems in Chapter 2, Section 7, and that work should provide additional motivation for this later exposition.

6.1 Lehmann representation

We return to Minkowski space time and turn to discuss the structure of the Green's function of two interacting Heisenberg field operators which

follows from simple assumptions — axioms — of a basic physical character. With sufficient work one could, in fact, derive the assumed axioms from the functional integral representation. However, since any physically sensible theory must have these properties, we shall follow the easy path of simply assuming the axioms.

The Green's function is a vacuum matrix element, and so we start with a consideration of the nature of the vacuum state. Since the energy-momentum operators P^μ commute, $[P^\mu, P^\nu] = 0$, they possess simultaneous eigenvalues. We assume that a *unique* vacuum state $|0\rangle$ exists* which is an eigenstate of these operators with zero eigenvalues,

$$P^\mu|0\rangle = 0\,. \tag{6.1.1}$$

It is a deep bit of physics that the momentum operator plays the dual role as the generator of space-time translations. This fact is reflected in its commutator with the generators of Lorentz transformations $J_{\mu\nu}$,

$$-i[J_{\mu\nu}\,,\,P_\lambda] = g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu\,. \tag{6.1.2}$$

Hence, in view of Eq. (6.1.1),

$$0 = [J_{\mu\nu}\,,\,P_\lambda]\,|0\rangle = -P_\lambda J_{\mu\nu}|0\rangle\,. \tag{6.1.3}$$

Since $|0\rangle$ is the unique state which is annihilated by P_λ, this implies that

$$J_{\mu\nu}|0\rangle = |0\rangle j_{\mu\nu}\,, \tag{6.1.4}$$

where the $j_{\mu\nu}$ are some set of numbers. These numbers must form a representation of the Lorentz group, they must obey the Lorentz group Lie algebra that is satisfied by the $J_{\mu\nu}$ operators,

$$i[J_{\mu\nu}\,,\,J_{\alpha\beta}] = g_{\nu\alpha}J_{\mu\beta} - \ldots\,. \tag{6.1.5}$$

This requirement follows by multiplying Eq. (6.1.4) by $J_{\alpha\beta}$ and subtracting the result where $J_{\alpha\beta}$ is interchanged with $J_{\mu\nu}$. It is impossible for the numbers $j_{\mu\nu}$ to satisfy this requirement except for the trivial representation $j_{\mu\nu} = 0$. Hence

$$J_{\mu\nu}|0\rangle = 0\,, \tag{6.1.6}$$

and we see that the vacuum state is necessarily a Lorentz invariant state. The Lorentz invariance of the vacuum state is *not* a separate postulate; it follows from the assumption that this state is the *unique* state of zero energy and momentum.

For simplicity, we still consider an Hermitian scalar field, $\phi(x) = \phi(x)^\dagger$, which we take to be the renormalized field. It should be emphasized,

* A theory with spontaneous symmetry breaking has a whole family of equivalent vacuum states. In such a case we assume that a particular vacuum state has been singled out.

however, that this is a scalar field in some arbitrary (but renormalizable) theory, not necessarily the simple $\lambda\phi^4$ which we have been considering. The unordered two-point function is given by

$$G^{(+)}(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle\,. \tag{6.1.7}$$

Writing

$$\phi(x) = e^{-iPx}\phi(0)e^{iPx}\,, \tag{6.1.8}$$

and using the fact that the vacuum state has no energy or momentum, we have

$$G^{(+)}(x-x') = \langle 0|\phi(0)e^{iP(x-x')}\phi(0)|0\rangle\,. \tag{6.1.9}$$

Let us now use the resolution of the identity provided by some complete set of orthonormal states

$$1 = \sum\nolimits_n |n\rangle\langle n|\,. \tag{6.1.10}$$

Here the summation is a complicated one which involves a summation over states with an arbitrary number of particles with an integration over all the particle momenta in each of these multiparticle states. We need not specify the precise nature of the states $|n\rangle$ save that they shall be required to be eigenstates of the (commuting) energy-momentum operators P^μ,

$$P^\mu|n\rangle = |n\rangle\, p_n^\mu\,. \tag{6.1.11}$$

Accordingly,

$$\begin{aligned} G^{(+)}(x-x') &= \sum\nolimits_n e^{ip_n(x-x')}\langle 0|\phi(0)|n\rangle\langle n|\phi(0)|0\rangle \\ &= \sum\nolimits_n e^{ip_n(x-x')}|\langle 0|\phi(0)|n\rangle|^2\,. \end{aligned} \tag{6.1.12}$$

A constant can always be added to $\phi(x)$ to ensure that $\langle 0|\phi(x)|0\rangle = 0$, which we shall assume has been done. Thus there is no loss of generality in assuming that the sum in Eq. (6.1.12) excludes the vacuum state $n = 0$. The energies of all the intermediate states must be positive, $p_n^0 > 0$, for otherwise the vacuum state of zero energy would not be the ground state — it would not be stable. The invariant total mass squared, $-p_n^2$, of the n-th intermediate state is just the total squared energy of this state when a boost is made to put the total spatial momentum $\mathbf{p}_n$ of the state to zero. This total mass squared cannot be negative, and it can vanish only if the theory contains a massless particle. We shall assume that the latter is not the case and require that p_n^μ is a time-like vector, $-p_n^2 \geq M^2$, with $M > 0$. Within these constraints on p_n^μ,

$$\int_{M^2}^\infty ds \int (d^4p)\,\theta(p^0)\delta(p^2+s)\delta^{(4)}(p-p_n) = 1\,. \tag{6.1.13}$$

Using this effective representation of the identity, we may make a "change of variables" by inserting it inside the sum and interchange sum and integral to write

$$\begin{aligned} G^{(+)}(x-x') = \int_{M^2}^\infty ds \int (d^4p)\,\theta(p^0)\delta(p^2+s)e^{ip(x-x')} \\ \times \sum\nolimits_n \delta^{(4)}(p-p_n)|\langle 0|\phi(0)|n\rangle|^2\,. \end{aligned} \tag{6.1.14}$$

In virtue of the Lorentz invariance of the theory,

$$\sum\nolimits_n \delta^{(4)}(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 = \rho(-p^2)/(2\pi)^3 \tag{6.1.15}$$

defines a Lorentz scalar, a function $\rho(-p^2)$ which depends only upon the invariant $-p^2$. This function appears as the sum of squared matrix elements and so it is real and non-negative,

$$\rho(-p^2)^* = \rho(-p^2) \geq 0\,. \tag{6.1.16}$$

We have now secured

$$\begin{aligned} G^{(+)}(x-x') &= \langle 0|\phi(x)\phi(x')|0\rangle \\ &= \int_{M^2}^\infty ds\,\rho(s)\,\Delta^{(+)}(x-x';\,s)\,, \end{aligned} \tag{6.1.17}$$

in which

$$\Delta^{(+)}(x-x';\,s) = \int \frac{(d^4p)}{(2\pi)^3}\,\theta(p^0)\delta(p^2+s)e^{ip(x-x')}\,. \tag{6.1.18}$$

This is the Lehmann representation.

Performing the p^0 integral in Eq. (6.1.18) with the aid of the δ-function gives

$$\Delta^{(+)}(x-x';\,s) = \int \frac{(d^3\mathbf{p})}{(2\pi)^3}\,\frac{1}{2p^0}\,e^{ip(x-x')}\,, \tag{6.1.19}$$

where

$$p^0 = E_s(\mathbf{p}) = \sqrt{\mathbf{p}^2+s} \tag{6.1.20}$$

is the energy of a particle of momentum $\mathbf{p}$ and mass $\sqrt{s}$. Recalling Eq. (3.2.29), this identifies $\Delta^{(+)}(x-x';\,s)$ as the vacuum matrix element

$$\Delta^{(+)}(x-x';\,s) = \langle 0|\phi(x)\phi(x')|0\rangle_s^{(0)} \tag{6.1.21}$$

for a free field of mass s. Indeed, the single-particle part of the resolution of the identity placed between the two free ϕ fields in Eq. (6.1.21) produces the integral representation given in Eq. (6.1.19). We thus find that the two-field function of a general interacting scalar field can be expressed as a superposition of the corresponding free field functions of variable mass. If the theory were that of a free field of mass m, one would have $\rho(s) = \delta(s-m^2)$.

1 Causality

Interchanging $x \leftrightarrow x'$ in Eq. (6.1.17) (or equivalently taking the complex conjugate) gives the vacuum matrix of the two fields in the reverse order. Subtracting this from the original form yields the commutator representation

$$\langle 0|i[\phi(x),\, \phi(x')]|0\rangle = \int_{M^2}^{\infty} ds\rho(s)\Delta_c(x-x'\,;\, s)\,, \tag{6.1.22}$$

where

$$\Delta_c(x-x'\,;\, s) = i[\Delta^{(+)}(x-x'\,;\, s) - \Delta^{(+)}(x'-x\,;\, s)]\,. \tag{6.1.23}$$

The Fourier representation (6.1.18) shows that $\Delta^{(+)}(x-x'\,;\, s)$ is not changed by the interchange of the spatial coordinates, $\mathbf{x} \leftrightarrow \mathbf{x}'$, for this is equivalent to reversing the sign of the spatial momentum, $\mathbf{p} \to -\mathbf{p}$, which leaves the remainder of the integral invariant. Hence

$$\Delta_c(x-x'\,;\, s) = \int \frac{(d^4p)}{(2\pi)^3} i\epsilon(p^0)\delta(p^2+s)e^{ip(x-x')}\,, \tag{6.1.24}$$

in which

$$\epsilon(p^0) = \theta(p^0) - \theta(-p^0) = \begin{cases} 1 & ,\quad p^0 > 0 \\ -1 & ,\quad p^0 < 0 \end{cases}\,. \tag{6.1.25}$$

The same procedure applied to the free field matrix element gives

$$\Delta_c(x-x'\,;\, s) = \langle 0|i[\phi(x),\, \phi(x')]|0\rangle_s^{(0)}\,. \tag{6.1.26}$$

The equal-time commutator vanishes,

$$\begin{aligned}\Delta_c(x-x'\,;\, s)|_{t=t'} &= \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \int dp^0 i\epsilon(p^0)\delta(p^2+s)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\ &= 0\,,\end{aligned} \tag{6.1.27}$$

since the integrand is odd in p^0. Now $\Delta_c(x-x'\,;\, s)$ is a Lorentz invariant. Any space-like interval $(\mathbf{x}-\mathbf{x}')^2 - (t-t')^2 > 0$ can be sent by a Lorentz transformation into the purely spatial interval $\bar{\mathbf{x}} - \bar{\mathbf{x}}' \neq 0\,,\ \bar{t} - \bar{t}' = 0\,.$ Hence, Lorentz invariance together with Eq. (6.1.27) implies that the commutator function vanishes for any space-like separation of its coordinates,

$$(x-x')^2 > 0\ : \qquad \Delta_c(x-x'\,;\, s) = 0\,. \tag{6.1.28}$$

We conclude from the commutator representation (6.1.22) that for any space-like separation $(x-x')^2 > 0$,

$$\langle 0|[\phi(x),\, \phi(x')]|0\rangle = 0\,. \tag{6.1.29}$$

This is the quantum field theory expression of *causality*. The commutator of two fields in space-like relation, an interval that cannot be bridged by

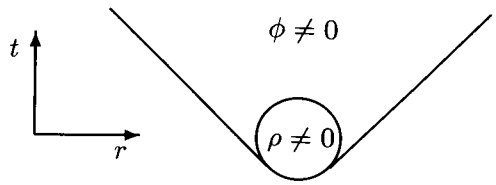


Fig. 6.1. Causality: A source produces an initial state expectation value of a field which is non-vanishing only in the future light cone of the source.

communication, even with a light signal, must vanish. Note that *scalar* field ϕ carries spin *zero*. Spin zero fields *commute* for space-like separation.

To make this connection with causality explicit, we consider the vacuum expectation value of the field in the presence of an external source function $\rho(x)$,

$$\langle 0-|\phi(x)|0-\rangle^{\rho}\,.$$

Note that this is *not* the "remote past–distant future" matrix element which is produced by the functional integral, the matrix element which enters into the description of a scattering experiment. Rather, it is the "diagonal" matrix element, the expectation value in the initial vacuum state of the field operator ϕ at the space-time point x. This matrix element gives the *average value* which the field ϕ will be observed to have at the spatial position $\mathbf{x}$ at the time x^0 if the initial state is the vacuum state. The expansion to first order in the external source involves the retarded commutator according to the solution of Problem 1:

$$\langle 0-|\phi(x)|0-\rangle^{\rho} = \int (d^4x')\,\theta(x^0-x'^0)\langle 0|i[\phi(x),\, \phi(x')]|0\rangle\rho(x') + \cdots\,. \tag{6.1.30}$$

This result makes the relationship of the commutator function with causality clear: Since the commutator vanishes for space-like separations, the source produces a field response only within its future light cone. That is, if the source is restricted to be non-vanishing only in some finite space-time region – if it has a finite space-time "support", then the expectation value of ϕ will be non-vanishing only in the future light cone of the points of this support as shown in the sketch in Fig. 6.1.

Spin zero fields *cannot anti-commute* at space-like separation. To see this, we note that, just as in the commutator construction, the vacuum value of the anti-commutator has the representation

$$\langle 0|\{\phi(x)\,,\, \phi(x')\}|0\rangle = \int_{M^2}^{\infty} ds\,\rho(s)\Delta^{(1)}(x-x'\,;\, s)\,, \tag{6.1.31}$$

in which

$$\Delta^{(1)}(x\,;\, s) = \Delta^{(+)}(x\,;\, s) + \Delta^{(+)}(-x\,;\, s)\,. \tag{6.1.32}$$

It follows directly from Eq. (6.1.18) that $\Delta^{(+)}(x\,;\,s) = \Delta^{(+)}(-x\,;\,s)$ for $t = 0$, and Lorentz invariance extends this equality to all space-like x. Moreover, at space-like separation, the free-field unordered and time-ordered functions are essentially equal, $\Delta^{(+)} = -i\Delta_+$. Hence we may use the result (4.5.17) to conclude that for space-like separations

$$x^2 > 0\ : \qquad \Delta^{(1)}(x\,;\,s) = \frac{1}{2\pi^2}\int_0^\infty d\zeta\, e^{-\zeta x^2 - s/4\zeta} > 0\,. \tag{6.1.33}$$

Hence for any space-like separation, $(x-x')^2 > 0$,

$$\langle 0|\{\phi(x)\,,\,\phi(x')\}|0\rangle = \int_{M^2}^\infty ds\,\rho(s)\Delta^{(1)}(x-x'\,;\,s) > 0\,. \tag{6.1.34}$$

Therefore, if the anti-commutator of two spin zero fields were to vanish for space-like separation, the integrand in Eq. (6.1.34) must vanish since $\rho(s)$ is non-negative. That is, this spectral weight must vanish identically,

$$\rho(s) = 0\,. \tag{6.1.35}$$

But Eq. (6.1.15) then implies that $\langle 0|\phi(0)|n\rangle = 0$, for all the states n, which further implies that the field ϕ annihilates the vacuum state,

$$\phi(x)|0\rangle = 0\,. \tag{6.1.36}$$

At this point, we need to quote a general theorem which we shall simply state without proof since this proof is lengthy and would take us too far astray: If $\phi(x)|0\rangle = 0$ then $\phi(x) = 0$. No field theory exists in which a local field annihilates the vacuum state. One concludes that a spin zero field *cannot* anticommute for any space-like separation.

What has been described is an elementary example of the general Spin - Statistics Theorem: At space-like separations integer spin fields commute while $\frac{1}{2}$-integer spin fields anticommute.

The Lehmann representation (6.1.17) expresses the vacuum matrix element of two field operators in the fully interacting theory in terms of a superposition of the corresponding matrix elements of free fields of mass s. We have just seen how this basic representation can be used to form representations of the commutator or anti-commutator functions — they are the same superposition of the corresponding free-field functions of mass s. This also carries over to the time-ordered product

$$\begin{aligned} G_+(x-x') &= \langle 0|iT\phi(x)\phi(x')|0\rangle \\ &= i\theta(x^0-x'^0)\langle 0|\phi(x)\phi(x')|0\rangle + i\theta(x'^0-x^0)\langle 0|\phi(x')\phi(x)|0\rangle\,, \end{aligned} \tag{6.1.37}$$

which appears as

$$G_+(x-x') = \int_{M^2}^\infty ds\,\rho(s)\Delta_+(x-x'\,;\,s)\,. \tag{6.1.38}$$

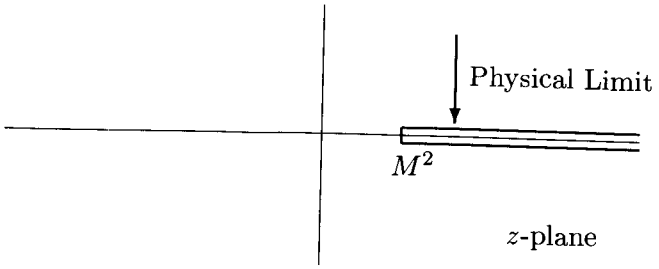


Fig. 6.2. Analytic structure of the Green's function.

The free-field propagator was discussed at length in Chapter 3, Section 2. It is given by

$$\begin{aligned} \Delta_+(x-x'\,;\,s) &= \langle 0|iT\phi(x)\phi(x')|0\rangle_s^{(0)} \\ &= i\theta(x^0-x'^0)\Delta^{(+)}(x-x'\,;\,s) + i\theta(x'^0-x^0)\Delta^{(+)}(x'-x\,;\,s) \\ &= \int \frac{(d^4p)}{(2\pi)^4}\frac{e^{ip(x-x')}}{p^2+s-i\epsilon}\,. \end{aligned} \tag{6.1.39}$$

Thus the Fourier transform of the interacting Green's function has the dispersion relation form[†]

$$G_+(p) = \int_{M^2}^\infty ds\,\rho(s)\frac{1}{p^2+s-i\epsilon}\,. \tag{6.1.40}$$

The Green's function in momentum space $G_+(p)$ appears as the boundary value of analytic function $G(z)$, $z \to -p^2 + i\epsilon$ where

$$G(z) = \int_{M^2}^\infty ds\,\rho(s)\frac{1}{s-z} \tag{6.1.41}$$

defines a function analytic in the whole z plane save for a cut along the positive real axis which starts at $z = M^2$, as shown in Fig. 6.2.

[†] Although the representation (6.1.17) for the unordered function always exists in the distribution theory sense, the dispersion relation (6.1.40) for the Green's function may not exist unless subtractions are made. The point is that the products of the two distributions, the θ step functions and the $G^{(+)}$, may not be a well-defined distribution. This is the case in the perturbative expansion of the $\lambda\phi^4$ theory where $\rho(s)$ does not decrease as $s \to \infty$, and the integral in Eq. (6.1.40) does not converge. In such cases a subtraction must be made, with the dispersion relation written as

$$G_+(p) = A - (p^2+s_0)\int ds\frac{\rho(s)}{s-s_0}\frac{1}{p^2+s}\,.$$

Since we are concerned only with the formal aspects of the theory, we shall neglect such complications.

2 Single-particle states

Let us now assume that the field ϕ creates a single-particle state of mass $m_{\rm phy}$ from the vacuum state as well as creating other multiparticle states. Then, separating out the single-particle state with the energy-momentum relation

$$p^0 = E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_{\rm phy}^2} \tag{6.1.42}$$

from the sum (6.1.15) that defines the spectral weight,

$$\rho(-p^2) = \int \frac{(d^3\mathbf{p}')}{(2\pi)^3}\frac{1}{2E(\mathbf{p}')}(2\pi)^3\delta^{(4)}(p-p')|\langle 0|\phi(0)|p'\rangle|^2 + \sum_{\text{multiparticles}}(2\pi)^3\delta^{(4)}(p-p_n)|\langle 0|\phi|n\rangle|^2 . \tag{6.1.43}$$

Here the same Lorentz invariant normalization for the single-particle state $|p'\rangle$ is used that has been previously employed in the free-field theory. Hence

$$|\langle 0|\phi(0)|p'\rangle|^2 = Z > 0 \tag{6.1.44}$$

defines a positive constant‡ Z. This is a *finite* wave-function renormalization constant which must appear since, although the field ϕ is the renormalized field, it is defined by a minimal subtraction scheme, and it does not couple the vacuum state to the single-particle state with unit strength. With $p^0 > 0$,

$$\int (d^3\mathbf{p}')\frac{1}{2E(\mathbf{p}')}\delta^{(4)}(p-p') = \frac{1}{2p^0}\delta(p^0 - E(\mathbf{p})) = \delta(p^2 + m_{\rm phy}^2)\,, \tag{6.1.45}$$

and so

$$\rho(-p^2) = Z\delta(-p^2 - m_{\rm phy}^2) + \bar{\rho}(-p^2)\,, \tag{6.1.46}$$

where now $\bar{\rho}(-p^2)$ stands for the contribution of the multiparticle states with

$$\bar{\rho}(s) = 0\,, \qquad s < M_{\rm th}^2\,. \tag{6.1.47}$$

The threshold mass $M_{\rm th}$ must be larger than the particle mass $m_{\rm phy}$, $M_{\rm th}^2 > m_{\rm phy}^2$, for otherwise the single particle created by ϕ would decay into the particles associated with $M_{\rm th}$; the particle would not be stable

‡ By Lorentz invariance, Z must be a function only of p'^2. But $-p'^2 = m_{\rm phy}^2$, and so Z must be a constant.

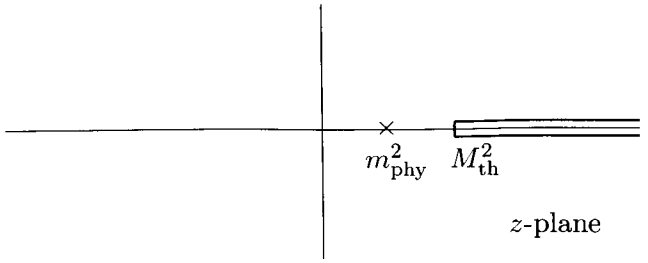


Fig. 6.3. Analytic structure of the Green's function with a single-particle pole.

and so it would not enter into the spectrum of physical states. Thus

$$G_+(p) = \frac{Z}{p^2 + m_{\rm phy}^2} + \int_{M_{\rm th}^2}^{\infty} ds \frac{\bar{\rho}(s)}{p^2 + s}\,. \tag{6.1.48}$$

The physical mass of the particle is the position of pole of the two-point Green's function. Moreover, the positive residue of this pole is a finite wave function renormalization in addition to the minimal but infinite renormalization that relates the renormalized field ϕ to the bare field ϕ_0. The analytic structure is now described by the picture in Fig. 6.3.

6.2 Reduction formula

The Green's functions of a quantum field theory yield all the (n-particle) scattering amplitudes of the theory. To see how this goes, we first examine the construction of single-particle states $|p\rangle$ in terms of the field operator $\phi(x)$ acting on the vacuum state $|0\rangle$.

If $\phi(x)$ is a free field, the single-particle state has the construction (Chapter 3, Section 2)

$$\langle p'| = \int (d^3\mathbf{x})e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\phi(x)\,, \tag{6.2.1}$$

with these states having the normalization

$$\langle p'|p\rangle = 2p^0(2\pi)^3\delta(\mathbf{p}' - \mathbf{p})\,. \tag{6.2.2}$$

For this free field, the wave function is given by

$$\langle p'|\phi(x)|0\rangle = e^{-ip'x}\,. \tag{6.2.3}$$

For an interacting field theory which has a single-particle state with the energy-momentum relation

$$p'^{\mu} = \left(\sqrt{\mathbf{p}'^2 + m_{\rm phy}^2}\,, \mathbf{p}'\right)\,, \tag{6.2.4}$$

we are motivated to try a construction akin to that given in Eq. (6.2.1) for a free particle. This construction is tested by examining the resulting "wave function" which is now given by an operation on the interacting two-point Green's function. Using the Lehmann representation,

$$\begin{aligned}
&\int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\phi(x)\phi(x')|0\rangle \\
&= \int (d^3\mathbf{x}) e^{-i\mathbf{p}'\cdot\mathbf{x}+ip'^0 t} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \int ds \left[Z\delta(s - m_{\text{phy}}^2) + \bar{\rho}(s)\right] \\
&\qquad \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \frac{1}{2E_s(\mathbf{p})} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}') - iE_s(p)(t-t')} \\
&= Ze^{-ip'x'} + \int_{M_{\text{th}}^2}^{\infty} ds\bar{\rho}(s) \frac{E_s(\mathbf{p}') + p'^0}{2E_s(\mathbf{p}')} e^{-iE_s(\mathbf{p}')(t-t')} e^{-i\mathbf{p}'\cdot\mathbf{x}'+ip'^0 t} .
\end{aligned} \tag{6.2.5}$$

The first term on the right-hand side of the last equality is just the free-particle wave function modified by the appearance of the finite wave-function renormalization constant Z. The second term represents the contribution of the continuum of states that is also produced when an interacting field acts on the vacuum state.

To understand the nature of the continuum contribution, it is worthwhile to first review the Riemann-Lebesgue Lemma. This lemma roughly states that if $f(\omega)$ is a "smooth" function which vanishes as $\omega \to \pm\infty$, then its Fourier transform vanishes in the infinite time limit,

$$t \to \infty : \qquad \int_{-\infty}^{\infty} d\omega f(\omega) e^{-i\omega t} \to 0 . \tag{6.2.6}$$

The point is that as $t \to \infty$, the exponential oscillates ever more rapidly and adjacent regions of the integration cancel. The proof follows by partial integration

$$\begin{aligned}
\int_{-\infty}^{\infty} d\omega f(\omega) e^{-i\omega t} &= \frac{i}{t} \int_{-\infty}^{\infty} d\omega f(\omega) \frac{\partial}{\partial\omega} e^{-i\omega t} \\
&= -\frac{i}{t} \int_{-\infty}^{\infty} d\omega \frac{df(\omega)}{d\omega} e^{-i\omega t} \\
&= \cdots\cdots \\
&= \left(\frac{-i}{t}\right)^N \int_{-\infty}^{\infty} d\omega \frac{d^N f(\omega)}{d\omega^N} e^{-i\omega t} .
\end{aligned} \tag{6.2.7}$$

This process can be repeated until a discontinuity or singularity is obtained in the N-th derivative, in which case one has proved that the Fourier transform vanishes at least as rapidly as $1/t^N$. If all the derivatives of $f(\omega)$ exist, then the Fourier transform vanishes faster than any

power of $1/t$ as $t \to \infty$. In view of this lemma, we expect that

$$\lim_{|t-t'|\to\infty} \int_{M_{\text{th}}^2}^{\infty} ds\, \bar{\rho}(s) \frac{E_s + p'^0}{2E_s} e^{-iE_s(t-t')} = 0 , \tag{6.2.8}$$

so that the continuum contribution vanishes in the infinite time limit. However the integrand is not smooth at the various n-particle thresholds – at the points $s = M_n^2$ where the states containing n particles start to make their contribution. Hence, the continuum contribution does not vanish faster than any power of $1/|t-t'|$.

To estimate the long-time limit of the continuum contribution, we note that the spectral weight $\bar{\rho}(s)$ has infinitely many thresholds corresponding to 2-particle, 3-particle, ... intermediate states. With s near the n-particle threshold, $s = M_n^2$, $\bar{\rho}(s)$ is proportional to the phase space for the production of n-particles (with additional factors of $\sqrt{s - M_n^2}$ appearing if the particles have spin). A straightforward calculation (see Problem 2) shows that near the n-particle threshold*

$$\bar{\rho}(s) \propto \left(\sqrt{s - M_n^2}\right)^{3n-5} . \tag{6.2.9}$$

The leading term in the long-time limit which arises from the threshold behavior of an n-particle intermediate state is obtained by the approximation

$$E_s(\mathbf{p}') = \sqrt{\mathbf{p}'^2 + s} \simeq E_{M_n}(\mathbf{p}') + \frac{s - M_n^2}{2E_{M_n}(\mathbf{p}')} , \tag{6.2.10}$$

with

$$E_{M_n}(\mathbf{p}') = \sqrt{\mathbf{p}'^2 + M_n^2} , \tag{6.2.11}$$

which is valid in the threshold region where s is near M_n. Thus

$$\begin{aligned}
&\int_{M_n^2}^{\infty} ds\, \bar{\rho}(s) \frac{E_s(\mathbf{p}') + p'^0}{2E_s(\mathbf{p}')} e^{-iE_s(\mathbf{p}')(t-t')} \\
&\sim \int_{M_n^2}^{\infty} ds (s - M_n^2)^{\frac{3n-5}{2}} \exp\left\{-\frac{i}{2} \frac{s - M_n^2}{E_{M_n}(\mathbf{p}')}(t-t')\right\} \exp\left\{-iE_{M_n}(\mathbf{p}')(t-t')\right\} \\
&\sim \left(\frac{1}{M|t-t'|}\right)^{\frac{3}{2}(n-1)} \exp\left\{-iE_{M_n}(\mathbf{p}')(t-t')\right\} ,
\end{aligned} \tag{6.2.12}$$

where M is some characteristic elementary particle mass scale.

Taking, say, $M \sim 1$ GeV corresponds to $M^{-1} \sim 10^{-23}$ sec. Therefore for $(t-t') \gg M^{-1} \sim 10^{-23}$ sec the continuum contribution is negligible.

* The simple square root behavior for a two particle threshold is evident from the calculation of the dispersion relation for the "bubble" graph found in Problem 2 of Chapter 4.

For example, with $t-t' \sim 10^{-13}$ sec, the contribution of an $n=2$-particle threshold is on the order of 10^{-15}. Although the continuum contribution vanishes only as an inverse power for large time difference, the scale is set by the very short elementary particle time M^{-1}. Hence

$$\lim_{|t-t'|\to\infty} \int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \langle 0|\phi(x)\phi(x')|0\rangle = Z e^{-ip'x'}, \quad (6.2.13)$$

where, in physical terms, the "infinite limit" means time differences $|t-t'|$ on the order of 10^{-13} sec or so.

We can now see how to construct a single-particle state for an interacting field system, namely

$$\langle p| = \lim_{t\to\pm\infty} \int (d^3\mathbf{x}) e^{-ipx} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \langle 0| \frac{1}{\sqrt{Z}} \phi(x), \quad (6.2.14)$$

for this procedure gives the properly normalized wave function

$$\langle p|\phi(x)|0\rangle = \sqrt{Z} e^{-ipx}. \quad (6.2.15)$$

The Hermitian adjoint yields

$$|p\rangle = \lim_{t\to\pm\infty} \int (d^3\mathbf{x}) e^{+ipx} (-i)(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \frac{1}{\sqrt{Z}} \phi(x)|0\rangle. \quad (6.2.16)$$

Using the wave function (6.2.15), the scalar product of the two states constructed by Eqs. (6.2.14) and (6.2.16) — but with opposite time limits [$t \to +\infty$ for the bra, $t \to -\infty$ for the ket (or the other way around)] so as to suppress the continuum contribution — is just that which has been used for free particles:

$$\langle p|p'\rangle = (2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p}-\mathbf{p}'). \quad (6.2.17)$$

It should be emphasized that the infinite time limits used to construct the single-particle states picks out only the pole term in the Green's function — the continuum contributions are "washed out" in accordance with the Riemann-Lebesgue Lemma.

We shall use this method of state construction not only to obtain single-particle states but also to construct multiparticle states as well. This does not follow strictly from what we have done. We will not give a rigorous proof, but the result is so plausible that it is easy to accept. Suppose that there is an initial ϕ-particle wave packet which later collides with the wave packet of another particle or system of particles which we denote by ζ. Initially the two wave packets are separated by a great spatial distance and there is no interaction between them. Therefore, the initial state $|\zeta-\rangle$ containing the other system is essentially the vacuum state as far as the well-separated ϕ-particle wave packet is concerned. Hence the initial state with the additional ϕ-particle wave packet is constructed by replacing the vacuum state in Eq. (6.2.16) by $|\zeta-\rangle$ with the momentum p which appears

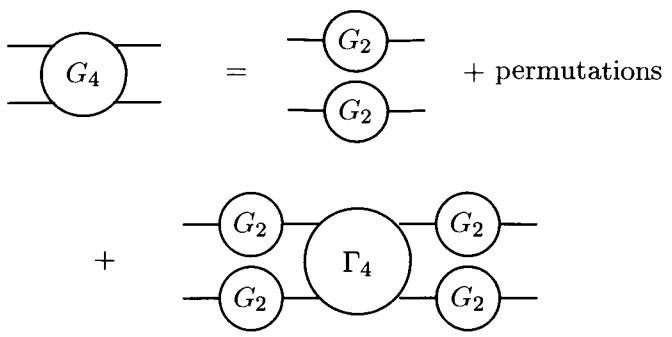


Fig. 6.4. Graphical description of the decomposition of the four-point Green's function into disconnected and single-particle irreducible parts.

in Eq. (6.2.16) integrated with the momentum-space wave function $f(p)$ which produces the well-separated ϕ-particle wave packet. In the limit in which the ϕ-particle wave packet becomes an incident plane-wave with momentum p, the initial state has the construction

$$|p,\zeta-\rangle = \lim_{t\to-\infty} \int (d^3\mathbf{x}) e^{ipx} (-i)(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \frac{1}{\sqrt{Z}} \phi(x)|\zeta-\rangle. \quad (6.2.18)$$

Similarly, the limit of the outgoing situation wave packet to a plane wave of momentum p' is given by

$$\langle p'\zeta'+| = \lim_{t\to+\infty} \int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \langle \zeta'+| \frac{1}{\sqrt{Z}} \phi(x). \quad (6.2.19)$$

Let us use this procedure to construct the transformation function $\langle p_1' p_2' + |p_1 p_2 -\rangle$ and thereby the two-particle scattering amplitude from the four-point Green's function

$$G(x_1,x_2,x_3,x_4) = \langle -T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle. \quad (6.2.20)$$

To do this, we first decompose the Green's function into disconnected and irreducible parts as shown by the graphical structure in Fig. 6.4. The first graphs represent the unscattered, "straight through" propagation of the particles, but with the "fully dressed" interacting propagators. The final graph represents the processes that give rise to the scattering, but again with the interacting propagator factors removed to define an amplitude Γ which is single-particle irreducible.† These graphs correspond to the

† It is clear that the sum of all the Feynman graphs fall into the categories illustrated in Fig. 6.4. It is not obvious, however, that they combine to form precisely the propagator factors G_2 that are shown. This is explicitly proven in Sections 4 and 5 below.

formula

$$G(x_1, x_2, x_3, x_4) = G(x_1 - x_2)G(x_3 - x_4) + G(x_1 - x_3)G(x_2 - x_4) + G(x_1 - x_4)G(x_2 - x_3) + \int (d\bar{x}_1)\cdots(d\bar{x}_4)G(x_1 - \bar{x}_1)\cdots G(x_4 - \bar{x}_4)\Gamma(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)\,, \tag{6.2.21}$$

or, in momentum space,

$$\begin{aligned}&(2\pi)^4\delta(\Sigma p)G(p_1\,, p_2\,, p_3\,, p_4)\\ &= (2\pi)^4\delta(p_1 + p_2)G(p_1)\,(2\pi)^4\delta(p_3 + p_4)G(p_3) + \text{perms}\\ &+(2\pi)^4\delta(\Sigma p)G(p_1)G(p_2)G(p_3)G(p_4)\Gamma(p_1\,, p_2\,, p_3\,, p_4)\,.\end{aligned} \tag{6.2.22}$$

Using Eqs. (6.2.19) and (6.2.18) to construct the states, the transformation function is given by

$$\begin{aligned}\langle p_1' p_2' + |p_1 p_2 -\rangle = &\int_{+\infty} (d^3\mathbf{x}_1)e^{-ip_1' x_1}(\overrightarrow{\partial}_{10} - \overleftarrow{\partial}_{10})\\ &\cdots \int_{-\infty} (d^3\mathbf{x}_4)e^{ip_2 x_4}(\overrightarrow{\partial}_{40} - \overleftarrow{\partial}_{40})\\ &\frac{1}{Z^2}\langle T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\,.\end{aligned} \tag{6.2.23}$$

Note that the time-ordering in the matrix element orders the fields appropriately, with the field operators creating the outgoing state appearing on the left and the field operators creating the incoming state appearing on the right. Making use of the decomposition (6.2.21) we encounter terms of the form

$$\begin{aligned}\int_{+\infty} (d^3\mathbf{x}_1)e^{-ip_1' x_1} i(\overrightarrow{\partial}_{10} - \overleftarrow{\partial}_{10})\frac{1}{\sqrt{Z}}G(x_1 - y)\\ = i\langle p_1'|\phi(y)|0\rangle = i\sqrt{Z}e^{-ip_1' y}\,,\end{aligned} \tag{6.2.24}$$

and

$$\begin{aligned}\int_{-\infty} (d^3\mathbf{x}_3)e^{ip_1 x_3}(-i)(\overrightarrow{\partial}_{30} - \overleftarrow{\partial}_{30})\frac{1}{\sqrt{Z}}G(x_3 - z)\\ = i\langle 0|\phi(z)|p_1\rangle = i\sqrt{Z}e^{ip_1 z}\,.\end{aligned} \tag{6.2.25}$$

Using Eqs. (6.2.24) and (6.2.25) it is easy to check that the "straight through" propagation terms in the decomposition (6.2.21) connecting x_1 with x_2 and x_3 with x_4 do not contribute while the other "straight through" terms give single-particle transformation functions. Also using Eqs. (6.2.24) and (6.2.25) it is easy to see that the remaining scattering term involves the Fourier transform of the Γ amplitude. Thus one finds

that

$$\begin{aligned}\langle p_1' p_2' + |p_1 p_2 -\rangle = &\langle p_1'|p_1\rangle\langle p_2'|p_2\rangle + \langle p_1'|p_2\rangle\langle p_2'|p_1\rangle\\ &-i(2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)T\,,\end{aligned} \tag{6.2.26}$$

in which

$$T = -iZ^2\Gamma(p_1'\,, p_2'\,, -p_1\,, -p_2)\,. \tag{6.2.27}$$

Note that as far as the scattering contribution described by T is concerned, the state construction just picks out the residue of the single-particle poles $(p^2 + m_{\text{phy}}^2)^{-1}$. A factor of $\sqrt{Z}$ is removed for each external particle. This leaves a finite wave function renormalization which must be taken into account, the factor Z^2 in relating the scattering amplitude to the irreducible vertex Γ. We have already learned in Chapter 3, Section 4 how to construct the scattering cross section from the structure (6.2.26).

We have gone through this "derivation" in some detail particularly to explain how the "straight-through-propagation" factors work out. Let us now give a simple but heuristic discussion which leads to the "reduction mnemonic". First we note that a particle of momentum p' is added to a final state by writing

$$\langle \zeta' p' + |\xi -\rangle = \int (d^3\mathbf{x})e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\frac{1}{\sqrt{Z}}\langle \zeta' + |\phi(x)|\xi -\rangle\,, \tag{6.2.28}$$

where the limit $t = x^0 \to +\infty$ is understood. A partial integration presents this as

$$\begin{aligned}\langle \zeta' p' + |\xi -\rangle = &\int_{-\infty}^{\infty} dt\,\partial_0 \int (d^3\mathbf{x})e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\frac{1}{\sqrt{Z}}\langle \zeta' + |\phi(x)|\xi -\rangle\\ &+ \int_{-\infty} (d^3\mathbf{x})e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\frac{1}{\sqrt{Z}}\langle \zeta' + |\phi(x)|\xi -\rangle.\end{aligned} \tag{6.2.29}$$

The second integration at $t = -\infty$ gives no "scattering" contribution – this term has a rapid phase variation that produces a vanishing result unless it corresponds to destroying a particle of momentum p' in the initial state, which gives a "straight-through-propagation" contribution. The overall time derivative on the right-hand side of the first line of the equation combines to give the form $i(\overrightarrow{\partial_0^2} - \overleftarrow{\partial_0^2})$. To this can be added the combination $\overrightarrow{\nabla^2} - \overleftarrow{\nabla^2}$ which gives no contribution as an integration by parts establishes. Hence

$$\begin{aligned}\langle \zeta' p' + |\xi -\rangle &= \int (d^4x)e^{-ip'x} i(-\overrightarrow{\partial^2} + \overleftarrow{\partial^2})\frac{1}{\sqrt{Z}}\langle \zeta' + |\phi(x)|\xi -\rangle\\ &= \int (d^4x)e^{-ip'x} i(-\overrightarrow{\partial^2} + m_{\text{phy}}^2)\frac{1}{\sqrt{Z}}\langle \zeta' + |\phi(x)|\xi -\rangle.\end{aligned} \tag{6.2.30}$$

One now takes p' slightly off mass shell, $-p'^2 \neq m^2_{\rm phy}$. Then there are rapid phase variations at the space-time boundaries, and integrations by parts can be freely performed to give

$$\langle \zeta' p' + | \xi - \rangle = \frac{i}{\sqrt{Z}} (p'^2 + m^2_{\rm phy}) \int (d^4x) e^{-ip'x} \langle \zeta' + | \phi(x) | \xi - \rangle \,. \quad (6.2.31)$$

The physical limit $p'^2 + m^2_{\rm phy} \to 0$ picks out the residue of the $(p'^2 + m^2_{\rm phy})^{-1}$ pole of the Fourier transform of the amplitude in Eq. (6.2.31) and produces the physical state matrix element. Similarly, an additional particle in an initial state is constructed as

$$\langle \zeta' + | \xi p - \rangle = \lim_{p^2 + m^2_{\rm phy} \to 0} \frac{i}{\sqrt{Z}} (p^2 + m^2_{\rm phy}) \int (d^4x) e^{ipx} \langle \zeta' + | \phi(x) | \xi - \rangle \,. \quad (6.2.32)$$

As an example of the use of this reduction mnemonic, let us consider the previous example of the two-particle scattering amplitude. Applying the mnemonic to Eq. (6.2.22) we see again that that the scattering amplitude is identified as the residue in the $(p^2 + m^2_{\rm phy})^{-1}$ poles,

$$\begin{aligned} \langle p'_1 p'_2 + | p_1 p_2 - \rangle = & \text{ straight through propagation terms} \\ & -(2\pi)^4 \delta(p'_1 + p'_2 - p_1 - p_2) Z^2 \Gamma(p'_1, p'_2, -p_1, -p_2) \,, \end{aligned} \quad (6.2.33)$$

in agreement with Eqs. (6.2.26) and (6.2.27).

6.3 Unstable particles

Let us first recall the analytic structure of the two-point Green's function or propagator when the theory contains a particle whose physical mass we will now denote simply by m. It has a pole at $-p^2 = m^2$ and a branch cut starting at $-p^2 = M^2_{\rm th}$ as shown in Fig. 6.3. The Lehmann representation (6.1.40) implies that $G(p)$ has no singularities except for $-p^2$ real and positive. In particular a pole in $G(p)$ must lie on the real $-p^2$ axis. Suppose now that the parameters of the theory are changed. Then the pole and branch point may move towards one another. If further changes in the parameters are made the pole will move into the branch point. Then what? Since the pole cannot move onto the complex plane it must move into new Riemann sheets. As will be described shortly, the Green's function originally defined above the cut can be analytically continued into a second Riemann sheet below the cut. The final position of the pole in this second sheet is sketched in Fig. 6.5. Parameters giving poles on unphysical sheets produce an unstable particle if these poles are close to the real axis.

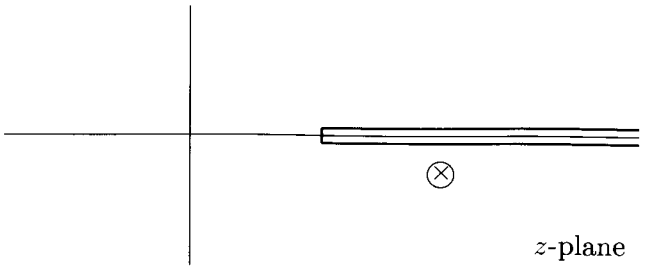


Fig. 6.5. When the parameters of the theory are varied, the pole at the mass of a stable particle can move through the branch point into other Riemann sheets. Shown here is the final position of the pole in the continuation into the second sheet which is described in the text.

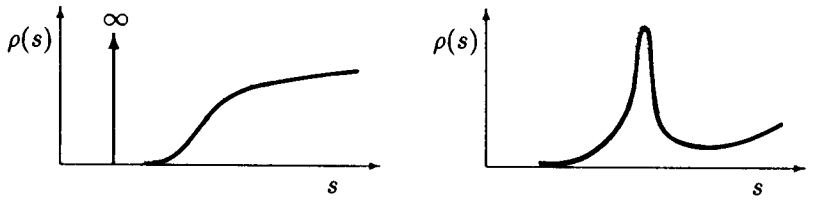


Fig. 6.6. Left: Spectral weight for a stable particle. Right: Spectral weight for an unstable particle.

The continuation of the mass parameter can also be cast in terms of the spectral weight of the Lehmann representation. Referring to Eq. (6.1.40) and using

$$\epsilon \to 0^+ : \qquad \mathrm{Im} \frac{1}{x \mp i\epsilon} = \pm \pi \delta(x) \,, \quad (6.3.1)$$

one sees that this weight is the imaginary part of the propagator,

$$\rho(-p^2) = \frac{1}{\pi} \mathrm{Im}\, G(p) \,. \quad (6.3.2)$$

Here $G(p)$ is the physical function obtained by the $\epsilon \to 0^+$ limit of Eq. (6.1.40). Note that the spectral weight also appears as the discontinuity across the cut in the analytic function $G(z)$ defined by Eq. (6.1.41),

$$G(x + i\epsilon) - G(x - i\epsilon) = 2\pi i \rho(x) \,. \quad (6.3.3)$$

In terms of the spectral weight, the stable and unstable particle cases are described by the sketches in Fig. 6.6. The presence of the unstable particle poles is reflected in a sharp peak in the spectral weight.

To describe the relevant "second sheet", we recall that the Green's func-

tion is defined along the real axis and just above the cut shown in Fig. 6.5. The effect of the unstable particle is revealed by analytically continuing the Green's function down though this cut. To do this continuation, it is convenient to consider the inverse function $G^{-1}(z)$ which exhibits the pole in $G(z)$ as a zero. This function has the same analytic structure as does $G(z)$, namely it is analytic in the entire $z = -p^2$ plane save for a cut running from $z = M_{\rm th}^2$ to plus infinity along the real axis. Just above this cut, the imaginary part

$$\sigma(s) = \frac{1}{\pi}\mathrm{Im}\, G^{-1}(s + i\epsilon) \tag{6.3.4}$$

can be expressed in terms of the propagator's spectral weight. Using $G^{-1} = G^*/|G|^2$ shows that this spectral weight is negative (or zero),

$$\sigma(s) = -\frac{\rho(s)}{|G(s+i\epsilon)|^2} \leq 0\,. \tag{6.3.5}$$

The imaginary part of $G^{-1}(z)$ is also related to the discontinuity of this function across the cut,

$$G^{-1}(x + i\epsilon) - G^{-1}(x - i\epsilon) = 2\pi i\sigma(x)\,. \tag{6.3.6}$$

The analytic continuation downwards into the second sheet is such that analytically continued function $G_{II}(z)$ just below the real axis is equal to the original function just above the real axis,

$$G_{II}^{-1}(x - i\epsilon) = G^{-1}(x + i\epsilon). \tag{6.3.7}$$

In view of Eq. (6.3.6), this may be written as

$$G_{II}^{-1}(x - i\epsilon) = G^{-1}(x - i\epsilon) + 2\pi i\sigma(x)\,. \tag{6.3.8}$$

The analytic extension of $\sigma(x)$ into the lower-half plane defines the propagator on the second sheet:

$$G_{II}^{-1}(z) = G^{-1}(z) + 2\pi i\sigma(z)\,. \tag{6.3.9}$$

To see that when parameters are varied the propagator pole — the zero of $G^{-1}(z)$ — does appear in this second sheet, we note that when the process starts, the zero lies close to the real axis so that we may approximate $G_{II}^{-1}(z)$, which is continuous across the real axis, by its value on the real axis

$$G_{II}^{-1}(z) \simeq Z^{-1}(m^2 - z) + \pi i\sigma(m^2)\,. \tag{6.3.10}$$

Since $\sigma(m^2)$ is negative, a positive quantity Γ is defined in writing

$$-\pi\sigma(m^2) = Z^{-1}m\Gamma\,. \tag{6.3.11}$$

Thus

$$G_{II}^{-1}(z) \simeq Z^{-1}[m^2 - im\Gamma - z] \tag{6.3.12}$$

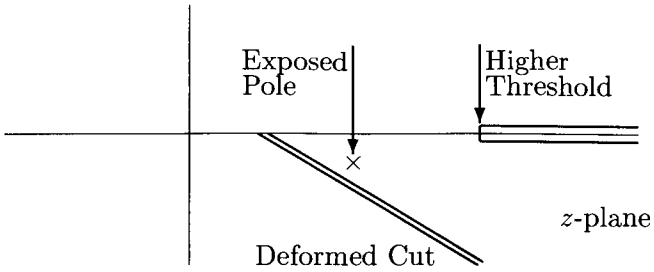


Fig. 6.7. The continuation into the second Riemann sheet is equivalent to deforming the branch cut as shown. This exposes the pole in the second sheet. The branch point positions cannot be altered.

does indeed have a zero which lies below the real axis.* Further variation of the parameters must keep this zero in the second sheet for, if it were to move into the upper-half plane, the original Lehmann representation would be violated. As will be discussed below, unstable particles can be observed only if $\Gamma/m \ll 1$. In this case of interest, Eq. (6.3.11) can be used to compute Γ. (In this regard see Problem 2 of Chapter 4.)

The analytic continuation below the cut given by the function $G_{II}(z)$ is equivalent to deforming the branch cut as shown in Fig. 6.7. It should be noted the positions of the branch points cannot be altered by such an analytic continuation. Moreover, we are interested in the analytic continuation which is equivalent to deforming the cut which originates from the lowest threshold branch point at $M_{\rm th}^2$ as shown in Fig. 6.7. This lowest branch point and its cut corresponds to the contribution of intermediate states which have a fixed number n of particles with the minimum invariant total mass that couple the field ϕ to the vacuum state. Other intermediate states with a different number of particles give contributions that have branch points at larger $z = -p^2$ values. The lowest of such higher thresholds is marked in Fig. 6.7.

1 Exponential decay

The fact that this altered analytic structure of the Green's function does indeed describe an unstable particle is shown in the behavior of the time-

* One could have started with $G(z)$ just below the cut and continued above through the cut to obtain an analytic continuation $G_{III}(z)$ defined in the upper- rather than the lower-half plane. The same considerations which have just been made show that $G_{III}(z)$ has a zero in the upper-half plane at $z = m^2 + im\Gamma$. The pole originally on the real axis moves into several Riemann sheets. However, the continuation $G_{II}(z)$ that we have made is the proper one for computing the physical Green's function.

dependence of the propagator,

$$G(\mathbf{p},\, t) = \int \frac{dp^0}{2\pi} e^{-ip^0 t} G(p)\,. \tag{6.3.13}$$

To expose the effect of an unphysical pole, we make use of the analytic continuation shown in Fig. 6.7 which is equivalent to writing

$$G(p) = \frac{Z}{p^2 + m^2 - im\Gamma} + G_{\text{"smooth"}}(p)\,, \tag{6.3.14}$$

where $G_{\text{"smooth"}}(p)$ is a smooth function except at the various thresholds. According to the Riemann-Lebesgue Lemma, for times that are long in comparison with the very short elementary particle time scale M^{-1},

$$\int \frac{dp^0}{2\pi} e^{-ip^0 t} G_{\text{"smooth"}}(p) \sim (1/Mt)^{\text{some power}}\,. \tag{6.3.15}$$

On the other hand, for $t > 0$, the contribution of the unphysical pole in the second sheet can be obtained by closing the p^0-integration contour in Eq. (6.3.13) the lower-half plane where the factor $\exp(-ip^0 t)$ becomes a damping exponential. This produces a simple contour integral which gives the evaluation

$$\begin{aligned} t > 0: \quad & \int \frac{dp^0}{2\pi} e^{-ip^0 t} \frac{Z}{p^2 + m^2 - im\Gamma} \\ & = i \frac{Z}{2\sqrt{\mathbf{p}^2 + m^2 - im\Gamma}} \exp\left\{-i\sqrt{\mathbf{p}^2 + m^2 - im\Gamma}\, t\right\}. \end{aligned} \tag{6.3.16}$$

This pole will dominate if $\Gamma \ll m$. If this is not the case, the enhancement in the spectral weight ρ will be so broad as to describe only a wide resonance but not an unstable particle whose propagation in time can be observed experimentally. If there is a resonance that is sharp enough to be called an unstable particle, one can always write

$$\sqrt{\mathbf{p}^2 + m^2 - im\Gamma} = E(p) - \frac{im\Gamma}{2E(p)}\,, \tag{6.3.17}$$

where

$$E(p) = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{6.3.18}$$

In this case, the long-time behavior of the propagator is given by

$$t \gg M^{-1}: \quad G(\mathbf{p},\, t) = \frac{iZ}{2E(p)} \exp\{-iE(p)t\} \exp\left\{-\frac{m\Gamma}{2E(p)}\, t\right\}. \tag{6.3.19}$$

There are power-law corrections to this result as indicated in Eq.(6.3.15). Although they dominate in a mathematical sense at very long times, they

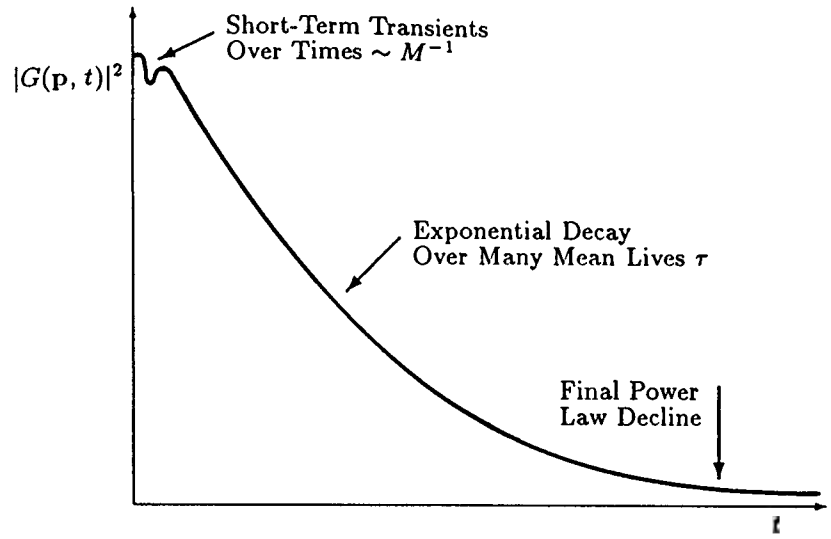


Fig. 6.8. Absolute square of the propagator for an unstable particle as a function of time. For very small times on the order of the time for light to cross the Compton wavelength of the particle, there are short-term transient effects. Over times that are very long on this scale, over many life times, the propagator decays exponentially. At times that are very long in comparison with the lifetime, there is a final power law decline, but at these times the probability is too small to detect.

are so small as to be never observable. For short times, times on the order of the basic elementary particle time scale M^{-1}, all domains of the p^0-integral contribute and there are various transient effects. The various behaviors are summarized in the plot displayed in Fig. 6.8. There is an exponential decay over a wide time span – essentially the only observable time span — with a mean life

$$\tau(p) = \frac{E(p)}{m}\Gamma^{-1}\,. \tag{6.3.20}$$

The factor $E(p)/m$ is just the proper relativistic time dilation factor,

$$\frac{E(p)}{m} = \frac{1}{\sqrt{1 - v^2}}\,. \tag{6.3.21}$$

Hence Γ^{-1} is the particle's proper mean life the mean lifetime in the particle's rest frame. Note, for example, that the π^+ meson with a mass of 140 MeV lives about 10^{-8} seconds, which gives $\Gamma/m \sim 10^{-15}$, illustrating the general smallness of this ratio.

To examine the power law correction that arises from the threshold contributions in more detail, we recall the discussion of Eq. (6.2.12) and

consider the case of a two-particle threshold. In this case, taking $\mathbf{p} = 0$, the threshold power decline dominates when $\exp(-\Gamma t/2)$ becomes small in comparison to some numerical constant (which arises from coupling constants and geometrical factors such as 2π) times $(1/Mt)^{3/2}$. Taking the logarithm of this condition which involves large numbers, the overall constant factors make a negligible contribution, and so the two-particle threshold contribution dominates when

$$\frac{1}{2}\Gamma t > \frac{3}{2}\ln Mt = \frac{3}{2}\ln\left[\left(\frac{M}{\Gamma}\right)(\Gamma t)\right] . \tag{6.3.22}$$

Since Γt is small in comparison with M/Γ, this may be written approximately as

$$\Gamma t > 3\ln\left(\frac{M}{\Gamma}\right) . \tag{6.3.23}$$

For $M/\Gamma \approx 10^{15}$ as for a π-meson, this gives $\Gamma t \approx 100$. Thus the exponential decay holds through very many mean lifetimes, so many, in fact, that when this decay law ceases to be valid there is essentially no chance of observing the particle. The power law correction will never be seen.

We have yet to explain how the propagator for an unstable particle can be observed. It enters into a production process as illustrated in Fig. 6.9. This is just the graphical decomposition into single-particle irreducible parts, with the fully interacting propagator $G(\mathbf{p}, t)$ displayed. The initial production mechanism can be localized in a time interval Δt around $t = 0$, where $M^{-1} \ll \Delta t \ll \Gamma^{-1}$, while the decay of the unstable particle can be detected at time t, again within a time uncertainty on the order of the previous Δt . Hence the probability for this process is proportional to $|G(\mathbf{p}\,,\, t)|^2$. In addition to the single-particle reducible graph shown in Fig. 6.9, there may be additional single-particle irreducible terms in which the particles that can compose the decay products of the unstable particle are produced without the intervention of the unstable particle's propagator. Such terms, however, do not contribute to the long-time behavior discussed here since they involve rapidly varying phases that wash out the amplitude in the manner of the Riemann-Lebesgue Lemma. Finally, we should note that there are threshold factors in the production and decay amplitudes which give very long time inverse power law corrections just as these factors do in the propagator, but they are generally negligible.

2 *Unstable particles as resonances*

If the proper lifetime Γ^{-1} is less than 10^{-12} sec or so, this type of experiment is very difficult, and instead of measuring a lifetime, it is much easier to measure the distribution in the overall center-of-mass energy s of

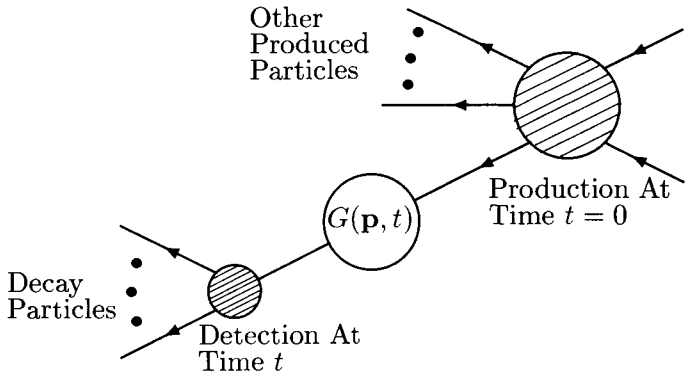


Fig. 6.9. Unstable particle production. The initial production and final detection of the decay products can be localized in time intervals that are very small in comparison with the life time. The probability of the overall process is the square of the amplitude show here and thus contains the decaying exponential in $|G(\mathbf{p}, t)|^2$.

the decay products which, with the neglect of non-resonant, "background terms" in the propagator, is proportional to

$$|G(p)|^2 = \frac{|Z|^2}{(m^2 - s)^2 + m^2\Gamma^2} . \tag{6.3.24}$$

To show how that goes, consider, for example, a 2-body decay into particles of mass m' as described by the sketch in Fig. 6.10. Since $-p^2 \simeq m^2$, there is little momentum variation in the decay vertex, and we can approximate it by a constant g to a very great accuracy. In view of the sketch, the partial cross section with an intermediate unstable particle is given by

$$d\sigma = \frac{1}{2}\int_\Delta \frac{(d^3\mathbf{k}_1)}{(2\pi)^3}\frac{1}{2k_1^0}\frac{(d^3\mathbf{k}_2)}{(2\pi)^3}\frac{1}{2k_2^0}\int\frac{(d^3\mathbf{q}_1)}{(2\pi)^3}\frac{1}{2q_1^0}\cdots$$
$$(2\pi)^4\delta^{(4)}(k_1 + k_2 + q_1 + \cdots - P)\,|\,g\,G(p)\,\mathcal{M}\,|^2 . \tag{6.3.25}$$

Here Δ indicates the range of the momenta over which the decay products are detected. The phase-space integrals over $(d^3\mathbf{q}_1)\cdots$ are those over the undetected particles in the experiment. Finally, $\mathcal{M}$ denotes the

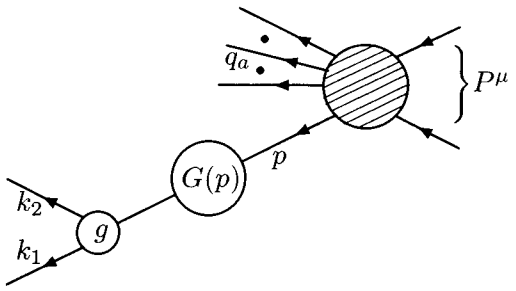


Fig. 6.10. Unstable particle production appearing as a resonant state of two specific final particles in an inelastic scattering process. The two detected particles have momenta k_1 and k_2, with $p = k_1 + k_2$. The momenta of the unobserved particles are labeled by q_a, $a = 1, \ldots$. The total initial four-momentum is denoted by P^μ.

production amplitude. We introduce

$$1 = \int_\Delta (d^4p)\delta^{(4)}(k_1 + k_2 - p)\,, \tag{6.3.26}$$

where now Δ restricts the range of final momenta, and also introduce

$$1 = \int ds \delta(p^2 + s) \tag{6.3.27}$$

and use the two-particle phase-space calculated in Problem 5 of Chapter 3,

$$\frac{1}{2}\int \frac{(d^3\mathbf{k}_1)}{(2\pi)^3}\frac{1}{2k_1^0}\int \frac{(d^3\mathbf{k}_2)}{(2\pi)^3}\frac{1}{2k_2^0}(2\pi)^4\delta(k_1 + k_2 - p) = \frac{1}{16\pi}\sqrt{1 - \frac{4m'^2}{s}}\,. \tag{6.3.28}$$

Since the distribution in s is sharply peaked at $s = m^2$ to within very small corrections (of relative order Γ/m),

$$d\sigma = \int ds \int_\Delta (d^4p)\delta(p^2 + s)\frac{g^2}{16\pi}\sqrt{1 - \frac{4m'^2}{m^2}}\,\frac{1}{(s - m^2)^2 + m^2\Gamma^2} \int \frac{(d^3\mathbf{q}_1)}{(2\pi)^3}\frac{1}{2q_1^0}\cdots\delta(p + q_1 + \cdots - P)|Z\mathcal{M}|^2\,. \tag{6.3.29}$$

The factor

$$\frac{1}{p^2 + m^2 - im\Gamma}$$

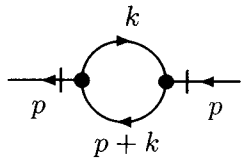


Fig. 6.11. Lowest order loop graph for the self-energy.

is obtained by approximating the self-energy function by its value at $-p^2 = m^2$. The lowest order self-energy function corresponds to the simple loop graph shown in Fig. 6.11. According to Problem 2 of Chapter 4, this graph gives

$$\Gamma = \frac{1}{2m}\frac{g^2}{16\pi}\sqrt{1 - \frac{4m'^2}{m^2}}\,. \tag{6.3.30}$$

With $\Gamma/m \ll 1$, higher order effects do not significantly modify this result. The point to be made is that the imaginary part of the self-energy is directly related to the square of the decay vertex amplitude and a phase-space factor.

We now have

$$\begin{aligned} d\sigma &= \int_\Delta ds \frac{1}{\pi}\frac{m\Gamma}{(s - m^2)^2 + m^2\Gamma^2}\int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0} \\ &\qquad \int \frac{(d^3\mathbf{q}_1)}{(2\pi)^3}\frac{1}{2q_1^0}\cdots(2\pi)^4\delta(p + q_1 + \cdots - P)\,|\sqrt{Z}\mathcal{M}|^2 \\ &= \int_{\Delta'} ds \frac{1}{\pi}\frac{m\Gamma}{(s - m^2)^2 + m^2\Gamma^2}\,\Delta\sigma_p\,. \end{aligned} \tag{6.3.31}$$

Here $\Delta\sigma_p$ is the cross section for the production with an effectively stable particle replacing the unstable particle. Note that this identification is consistent since in the stable particle limit

$$\lim_{\Gamma\to 0}\frac{1}{\pi}\frac{m\Gamma}{(s - m^2)^2 + m^2\Gamma^2} = \delta(s - m^2)\,, \tag{6.3.32}$$

and $d\sigma = \Delta\sigma$.

In general, with $\Gamma/m \ll 1$, the approximation

$$s - m^2 = (\sqrt{s} + m)(\sqrt{s} - m) \simeq 2\sqrt{s}(\sqrt{s} - m) \tag{6.3.33}$$

may be used to write

$$\frac{d\sigma}{d\sqrt{s}} = \frac{1}{\pi}\frac{\Gamma/2}{(\sqrt{s} - m)^2 + \Gamma^2/4}\,\Delta\sigma_p\,, \tag{6.3.34}$$

which is a Lorentzian or Breit-Wigner distribution as shown in the sketch in Fig. 6.12. The full width at half-maximum of this curve is Γ. We have just described the unstable particle, single-particle reducible contribution

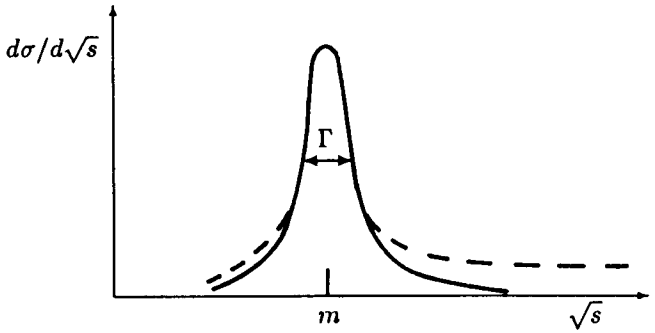


Fig. 6.12. Solid curve: Unstable particle contribution to production cross section as a function of the center-of-mass energy of the two decay particles. Dashed curve: Complete cross section including non-resonant, "background" contributions.

to the particle production process. There may be additional terms in the amplitude which do not involve this propagator. They produce a "background" contribution that varies slowly with s, and so the curve obtained in an actual experiment appears as the dashed curve shown in Fig. 6.12. Such resonances can be rather broad and the concept of an unstable particle merges into the notion of a resonant state. For example, the ρ meson has a mass m of 770 MeV and a width Γ of 150 MeV.

6.4 Graphical structure, effective action

Continuing the discussion of the general features of quantum field theory, we introduce and describe in a somewhat discursive fashion the effective action which is the generating function of connected Green's functions, Green's functions that contain only connected graphs. We first give heuristic arguments and relate the connectivity character to the cluster decomposition property before proving that the effective action does indeed contain only connected graphs. In the course of the proof, we shall show how powers of Planck's constant $\hbar$ count the number of loop graphs in an amplitude. Finally, we compute the effective action through one loop for the $\lambda\phi^4$ theory.

1 Connectivity

In ordinary units, Planck's constant $\hbar$ has the dimension of action, or energy times time. It will prove convenient to temporarily display this

Fig. 6.13. Graph representing the free-particle effective action of Eq. (4.4).

constant in writing the generating functional as

$$Z[\rho] = \int [d\phi] \exp\left\{\frac{i}{\hbar}\int (d^n x)[\mathcal{L} + \rho_0\phi]\right\} . \tag{6.4.1}$$

Note, however, that $1/\hbar$ appears only as an overall factor multiplying the action, but no factors of $\hbar$ occur inside the Lagrange function. Thus, for example,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \cdots , \tag{6.4.2}$$

where the mass parameter m_0 must be now considered to take on the units of inverse length rather than mass (it is now considered to be the inverse Compton wavelength of the particle). The effective action $W[\rho]$ is defined by

$$Z[\rho] = \exp\left\{\frac{i}{\hbar}W[\rho]\right\} . \tag{6.4.3}$$

It is the generating functional of *connected* graphs. We shall prove this property after we have discussed its consequences. The connected character is illustrated by the lowest order, free field limit. We found in Chapter 3. Section 2 that with no interactions

$$W[\rho] = \frac{1}{2}\int (d^n x)(d^n y)\, \rho(x)\Delta_+(x-y)\rho(y) , \tag{6.4.4}$$

corresponding to the graph in Fig. 6.13. Crosses denote the action of the source ρ. This graph is obviously connected. However, the expansion of the generating functional $Z[\rho]$,

$$Z[\rho] = \exp\left\{\frac{1}{\hbar}\ \text{X}\!\!-\!\!\text{X}\right\}$$
$$= 1 + \frac{1}{\hbar}\ \text{X}\!\!-\!\!\text{X} + \frac{1}{2!\,\hbar^2}\ \begin{matrix}\text{X}\!\!-\!\!\text{X}\\ \text{X}\!\!-\!\!\text{X}\end{matrix} + \cdots . \tag{6.4.5}$$

is obviously not connected.

To see why $W[\rho]$ should be connected even for an interacting theory, consider a source function $\rho(\mathbf{x}, t)$ which is non-vanishing only over some finite spatial extent and which slowly turns on to a time-independent value over the long interval $-T/2 < t < T/2$ and then slowly turns off. The

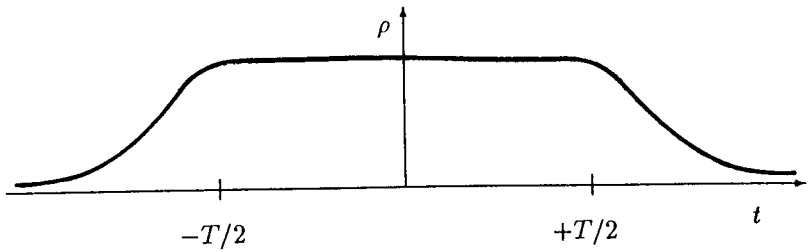


Fig. 6.14. Time behavior of the source ρ which gives an energy shift ΔE.

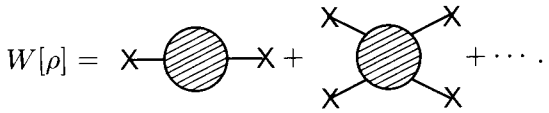


Fig. 6.15. Graphical expansion of the effective action. None of the amplitudes which appear here have disconnected pieces.

time dependence of the source is illustrated in Fig. 6.14. If we start with a system in its ground state and slowly turn on an external force which then becomes constant, the original ground state goes into the ground state of the forced system with unit probability. Similarly, slowly turning off the force puts the system back in the original ground state with unit probability. Hence for the source function $\rho(\mathbf{x}, t)$ which has been chosen,

$$Z[\rho] = \exp\left\{-\frac{i}{\hbar}\Delta E T\right\}, \tag{6.4.6}$$

where ΔE is the energy difference between the perturbed and unperturbed ground states. We have assumed that T is very large, and we have written the dominant contribution to the phase which is proportional to T . There are, of course, additional small phases associated with turning $\rho(\mathbf{x}, t)$ on and off, but with T very large these are relatively zero. Suppose in addition that $\rho(\mathbf{x}, t)$ is uniform in a large spatial volume V, but rapidly vanishes outside this region. Then the dominant phase is given by

$$Z[\rho] = \exp\left\{-\frac{i}{\hbar}\Delta\mathcal{E} V T\right\}, \tag{6.4.7}$$

where $\Delta\mathcal{E}$ is the energy density produced by the driving source. It is now easy to see that this behavior for a uniform source $\rho(\mathbf{x}, t)$ in very large space-time volume VT requires that $W[\rho]$ be connected. All we need do is to examine the graphical expansion shown in Fig. 6.15. With $\rho(\mathbf{x}, t)$ constant in VT, the overall translation invariance of each of the amplitudes which appear in this expansion in increasing powers of ρ implies that they each contain the overall factor of VT appearing in Eq. (6.4.7). On

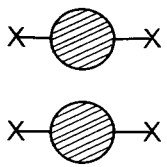


Fig. 6.16. Disconnected graphs.

the other hand, if $W[\rho]$ had disconnected pieces as shown in Fig. 6.16, we would get terms of order $(VT)^2$ or higher, which contradicts Eq. (6.4.7). Therefore $W[\rho]$ should be connected.

We have often made use of the fact that $Z[\rho]$ is the generating functional of all the Green's functions of the theory. For example,

$$\left(\frac{\hbar^2}{i}\right)^2 \frac{\delta}{\delta\rho(x_1)}\frac{\delta}{\delta\rho(x_2)}\frac{\delta}{\delta\rho(x_3)}\frac{\delta}{\delta\rho(x_4)} Z[\rho]\Big|_{\rho=0} = -\langle T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle . \tag{6.4.8}$$

The fact that $W[\rho]$ is the generating functional of connected Green's functions is illustrated by using Eq. (6.4.3) to write Z in terms of W in Eq. (6.4.8). Rearranging the terms, this substitution gives

$$\begin{aligned}\frac{\hbar^3}{i}\frac{\delta}{\delta\rho(x_1)}\frac{\delta}{\delta\rho(x_2)}\frac{\delta}{\delta\rho(x_3)}\frac{\delta}{\delta\rho(x_4)} W[\rho]\Big|_{\rho=0} &= -\langle T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\ &\quad - \{\langle iT\phi(x_1)\phi(x_2)\rangle\langle iT\phi(x_3)\phi(x_4)\rangle + \text{permutations}\} .\end{aligned} \tag{6.4.9}$$

The right-hand side of this equation defines the connected four-point Green's function, the Green's function with all possible lower-order correlations removed. A similar decomposition hold for the higher Green's functions where all lower-order correlations are removed to form the connected function.

2 Cluster decomposition

A scaling argument for the connected property of $W[\rho]$ has been made, a scaling that is akin to that in statistical mechanics where the partition function is the exponential of the volume times an intensive variable, the pressure over temperature, $\beta\mathcal{P}$. There is another very important reason why $W[\rho]$ must be connected, namely the *cluster decomposition property* implies that this must be so. To see how this goes, consider

$$\rho(x) = \rho_1(x) + \rho_2(x) , \tag{6.4.10}$$

where the two source functions have support in widely separated space-time regions; the two sources are non-vanishing only in small regions separated by a large space-like distance R^μ. A large separation means far on the scale of the range of the forces in the theory. To make our point, we can take $\rho_1(x)$ to be near the Earth and $\rho_2(x)$ on the other side of the Moon, although having $\rho_1(x)$ and $\rho_2(x)$ separated by 10^{-8} cm could suffice. With a great separation between sources, measurements in one region cannot effect the other region. Hence the probability amplitude must factor

$$Z[\rho_1 + \rho_2] = Z[\rho_1]Z[\rho_2]\,. \tag{6.4.11}$$

This is the cluster decomposition property. Equivalently, the effective action must add,

$$W[\rho_1 + \rho_2] = W[\rho_1] + W[\rho_2]\,. \tag{6.4.12}$$

Clearly, this requires that $W[\rho]$ be connected, for a disconnected set of graphs such as those shown in Fig. 6.16 do not have this additive property. It should be noted that the support of ρ_1 and ρ_2 must have a space-like separation, for otherwise the two regions could communicate.

It is worth explaining the cluster decomposition in terms of the behavior of the Green's functions for two sets of fields whose coordinates $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ are separated by a wide space-like interval R^μ, $R^2 > 0$. It is convenient to assume that all the x_a^μ coordinates are clustered near the origin and write

$$y_b^\mu = R^\mu + \bar{y}_b^\mu\,, \tag{6.4.13}$$

where the $\bar{y}_b^\mu$ are also clustered near the origin. All of the coordinate differences of the fields in the two regions are space-like, $(x_a - y_b)^2 > 0$. We shall assume that such fields in space-like separation commute. We have seen that this must be the case for the vacuum-value of two field operators. This "axiom" follows from the functional integral representation. Although a detailed proof is lengthy, the outline of the proof is clear. A Green's function, all of whose coordinate differences $x_a - x_b$ are space-like, is equal to its analytic continuation into Euclidean space. The Euclidean-space Green's function is a completely symmetrical function of its coordinates — it is invariant under any interchange $x_a \leftrightarrow x_b$. Hence the field operators in the original Green's function all commute with one another. This situation is changed when a coordinate difference is time-like, say $(x_c - x_d)^2 < 0$. In this case, the Euclidean-space Green's function must be analytically continued to go back to Minkowski space time, and the continuation with the two different orderings of the coordinates continues to different sides of a branch cut and so the fields do not commute. However, there is no difficulty in continuing the remaining

space-like coordinate intervals, and the corresponding field commute. For the case in hand, the $\phi(x_a)$ fields commute with the $\phi(y_b)$ fields and so the time ordering of the two sets of fields factors into the time orderings of the individual sets,

$$\begin{aligned}&\langle T\phi(x_1)\cdots\phi(x_m)\,\phi(y_1)\cdots\phi(y_n)\rangle\\ &\qquad = \langle [T\phi(x_1)\cdots\phi(x_m)][T\phi(y_1)\cdots\phi(y_n)]\rangle\,.\end{aligned} \tag{6.4.14}$$

Using

$$\phi(y_b) = e^{-iPR}\phi(\bar{y}_b)e^{iPR} \tag{6.4.15}$$

this factoring can further be expressed as

$$\begin{aligned}&\langle T\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)\rangle\\ &= \langle [T\phi(x_1)\cdots\phi(x_m)]e^{-iPR}[T\phi(\bar{y}_1)\cdots\phi(\bar{y}_n)]\rangle\\ &= \langle T\phi(x_1)\cdots\phi(x_m)\rangle\langle T\phi(\bar{y}_1)\cdots\phi(\bar{y}_n)\rangle\\ &+ \sum\nolimits_{N\neq\text{vac}} \langle 0|T\phi(x_1)\cdots\phi(x_m)|N\rangle e^{-ip_N R}\langle N|T\phi(\bar{y}_1)\cdots\phi(\bar{y}_n)|0\rangle\,.\end{aligned} \tag{6.4.16}$$

It follows from the Riemann-Lebesgue Lemma that the last sum over intermediate states which does not include the vacuum state vanishes when $R^2 \to \infty$. In fact, it can be proven that when $R^2 \to \infty$, this sum vanishes as $\exp\{-m\sqrt{R^2}\}$, where m is the lowest mass in the theory. Thus, if R^μ is a large space-like vector, then

$$\begin{aligned}&\langle T\phi(x_1)\cdots\phi(x_m)\,\phi(y_1)\cdots\phi(y_n)\rangle\\ &\qquad = \langle T\phi(x_1)\cdots\phi(x_m)\rangle\langle T\phi(y_1)\cdots\phi(y_n)\rangle\,.\end{aligned} \tag{6.4.17}$$

Here we have replaced $\bar{y}_a$ by y_a since the vacuum expectation value is translation invariant. Equation (6.4.17) is precisely the cluster decomposition property* for the Green's functions that is implied by the generating functional cluster relation (6.4.11).

3 Graphical proof

We turn now to prove directly that $W[\rho]$ does indeed contain only connected graphs. To do this, we work with bare field quantities that are generated by the bare source $\rho_0(x)$. The theory is regulated (by continuation to n dimensions) so that we always work with well-defined quantities. First we note that with

$$\exp\left\{\frac{i}{\hbar}W[\rho]\right\} = Z[\rho] = \int [d\phi]\exp\left\{\frac{i}{\hbar}\int[\mathcal{L} + \rho_0\phi]\right\}\,, \tag{6.4.18}$$

* Spontaneously broken theories may not have a unique vacuum state, and they may not obey the cluster property. See Problem 4.

the source derivative appears as

$$\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(x)}Z[\rho]=\int[d\phi]\phi(x)\exp\left\{\frac{i}{\hbar}\int[\mathcal{L}+\rho_0\phi]\right\}$$
$$=\frac{\delta W[\rho]}{\delta\rho_0(x)}Z[\rho]\,. \tag{6.4.19}$$

We use the notation

$$\langle X\rangle\equiv\frac{\int[d\phi]X\exp\left\{\frac{i}{\hbar}\int[\mathcal{L}+\rho_0\phi]\right\}}{\int[d\phi]\exp\left\{\frac{i}{\hbar}\int[\mathcal{L}+\rho_0\phi]\right\}} \tag{6.4.20}$$

to define the normalized expectation value of any operator X in the presence of the source ρ. In terms of this notation

$$\frac{\delta W[\rho]}{\delta\rho_0(x)}=\langle\phi_0(x)\rangle\,. \tag{6.4.21}$$

Since the functional derivative now also acts on the denominator, the second variational derivative entails

$$\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(y)}\frac{\delta W[\rho]}{\delta\rho_0(x)}=\langle T\phi_0(x)\phi_0(y)\rangle-\langle\phi_0(x)\rangle\langle\phi_0(y)\rangle\,. \tag{6.4.22}$$

Next we recall how the field equation appears in our functional language. We define

$$-\mathcal{F}(\phi(x))=\frac{\delta}{\delta\phi(x)}\int\mathcal{L}\,, \tag{6.4.23}$$

which is the Euler-Lagrange equation operating upon ϕ. Since the functional integral of a total variational derivative vanishes,

$$0=\int[d\phi]\frac{\hbar}{i}\frac{\delta}{\delta\phi(x)}e^{\frac{i}{\hbar}\int[\mathcal{L}+\rho_0\phi]}$$
$$=\int[d\phi][-\mathcal{F}(\phi(x))+\rho_0(x)]e^{\frac{i}{\hbar}\int[\mathcal{L}+\rho_0\phi]}\,. \tag{6.4.24}$$

Thus the field equation is transcribed into the functional-differential equation

$$\mathcal{F}\left(\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(x)}\right)Z[\rho]=\rho_0(x)Z[\rho]\,. \tag{6.4.25}$$

Writing Eq. (6.4.25) in terms of the effective action [Eq. (6.4.3)] gives

$$e^{-\frac{i}{\hbar}W[\rho]}\mathcal{F}\left(\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(x)}\right)e^{\frac{i}{\hbar}W[\rho]}=\mathcal{F}\left(\frac{\delta W}{\delta\rho_0(x)}+\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(x)}\right)=\rho_0(x)\,, \tag{6.4.26}$$

or, in view of Eq. (6.4.21),

$$\mathcal{F}\left(\langle\phi_0(x)\rangle+\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(x)}\right)=\rho_0(x)\,. \tag{6.4.27}$$

For the $\lambda\phi^4$ theory where

$$\mathcal{F}(\phi_0(x))=(-\partial^2+m_0^2)\phi_0(x)+\frac{\lambda_0}{3!}\phi_0(x)^3\,, \tag{6.4.28}$$

this reads

$$(-\partial^2+m_0^2)\langle\phi_0(x)\rangle+\frac{\lambda_0}{3!}\left(\langle\phi_0(x)\rangle+\frac{\hbar}{i}\frac{\delta}{\delta\rho_0(x)}\right)^3=\rho_0(x)\,, \tag{6.4.29}$$

or

$$(-\partial^2+m_0^2)\langle\phi_0(x)\rangle+\frac{\lambda_0}{3!}\langle\phi_0(x)\rangle^3$$
$$=\rho_0(x)-\frac{\lambda_0}{2}\langle\phi_0(x)\rangle\frac{\hbar}{i}\frac{\delta\langle\phi_0(x)\rangle}{\delta\rho_0(x)}+\hbar^2\frac{\lambda_0}{3!}\frac{\delta^2\langle\phi_0(x)\rangle}{\delta\rho_0(x)^2}\,. \tag{6.4.30}$$

4 *Tree graphs and the classical limit*

To discuss this functional-differential equation, we first take the classical limit $\hbar\to 0$. In this limit, the field expectation value $\langle\phi_0(x)\rangle$ obeys

$$(-\partial^2+m_0^2)\langle\phi_0(x)\rangle+\frac{\lambda_0}{3!}\langle\phi_0(x)\rangle^3=\rho_0(x)\,, \tag{6.4.31}$$

which is, of course, just the classical field equation. To prove that $W[\rho]$ is connected in this $\hbar=0$ limit, we first take the source function to be a constant, $\rho_0(x)=\rho_0$. Then the solution of

$$m_0^2\langle\phi_0\rangle+\frac{\lambda_0}{3!}\langle\phi_0\rangle^3=\rho_0 \tag{6.4.32}$$

is also a constant, $\langle\phi_0(x)\rangle=\langle\phi_0\rangle$. In this case Eq. (6.4.21) can be integrated in terms of an ordinary integral to obtain

$$W[\rho]=VT\int\langle\phi_0\rangle d\rho_0=-VT\mathcal{E}\,. \tag{6.4.33}$$

Thus $W[\rho]$ is connected in the classical limit because it is of first order in VT when $\rho_0(x)$ becomes constant. For non-constant $\rho_0(x)$, the classical field equation can be solved in perturbation theory. Starting with $\lambda_0=0$,

$$(-\partial^2+m_0^2)\langle\phi_0(x)\rangle=\rho_0(x)\,, \tag{6.4.34}$$

which gives the non-interacting solution

$$\langle\phi_0(x)\rangle=\int(d^ny)\Delta_+(x-y)\rho_0(y)\,. \tag{6.4.35}$$

This solution appears graphically as

$\langle\phi_0(x)\rangle=$. (6.4.36)

Functionally integrating Eq. (6.4.21) produces

$$W[\rho] = \frac{1}{2}\int (d^n x)(d^n y)\rho_0(x)\Delta_+(x-y)\rho_0(y)\,, \tag{6.4.37}$$

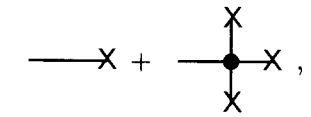
which is the familiar non-interacting result. Having the zeroth order solution in hand, we may insert it in Eq. (6.4.31) to get the solution to first order in λ_0,

$$(-\partial^2 + m_0^2)\langle\phi_0(x)\rangle = \rho_0(x) - \frac{\lambda_0}{3!}\left[\int (d^n y)\Delta_+(x-y)\rho_0(y)\right]^3\,, \tag{6.4.38}$$

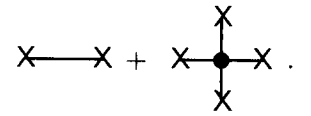
or

$$\langle\phi_0(x)\rangle = \int (d^n y)\Delta_+(x-y)\rho_0(y) - \frac{\lambda_0}{3!}\int (d^n y)\Delta_+(x-y)\left[\int (d^n z)\Delta_+(y-z)\rho_0(z)\right]^3 \tag{6.4.39}$$

This appears graphically as

$\langle\phi(x)\rangle =$  , (6.4.40)

and a functional integration presents the effective action to this order as the graphical sum

$W[\rho] =$  . (6.4.41)

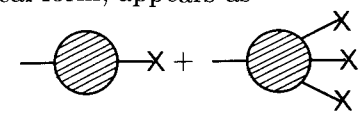
Going on to higher orders it should be clear that one gets a graphical series of higher order tree graphs of an obvious structure. The *classical limit corresponds to tree graphs.* The classical limit for the field expectation value $\langle\phi_0(x)\rangle$ or the corresponding action functional $W[\rho]$ corresponds to the set of all possible *connected* tree graphs. This iterative solution of the classical field equation must give connected graphs for otherwise it would not do so in the limit of a constant source where we have proved that it does.

5 *Proof completed; $\hbar$ counts loops*

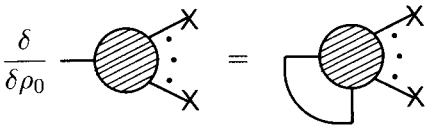
We now return to the general functional field equation

$$(-\partial^2 + m_0^2)\langle\phi_0(x)\rangle + \frac{\lambda_0}{3!}\langle\phi_0(x)\rangle^3 = \rho_0(x) - \frac{\lambda_0}{2}\langle\phi_0(x)\rangle\frac{\hbar}{i}\frac{\delta\langle\phi_0(x)\rangle}{\delta\rho_0(x)} + \frac{\lambda_0}{3!}\hbar^2\frac{\delta^2\langle\phi_0(x)\rangle}{\delta\rho_0(x)^2}\,, \tag{6.4.42}$$

whose solution has an expansion in the number of source terms which, expressed in graphical form, appears as

$\langle\phi_0(x)\rangle =$  + $\cdots$. (6.4.43)

Consider the term $(\hbar/i)(\delta\langle\phi_0(x)\rangle/\delta\rho_0(x))$. The functional derivative removes a source $\rho_0(\bar{x})$ and identifies the coordinate of the end of the line which was connected to this source with the coordinate of the field expectation value. In terms of a graph,

$\frac{\hbar}{i}\frac{\delta}{\delta\rho_0}$  . (6.4.44)

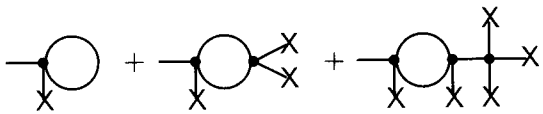
The $(\hbar/i)(\delta/\delta\rho_0)$ operation *adds a loop.* The general functional field equation may be solved by expanding in powers of $\hbar$:

$$\langle\phi_0(x)\rangle = \langle\phi_0(x)\rangle^{(0)} + \hbar\langle\phi_0(x)\rangle^{(1)} + \frac{1}{2}\hbar^2\langle\phi_0(x)\rangle^{(2)} + \ldots\,. \tag{6.4.45}$$

Here $\langle\phi_0(x)\rangle^{(0)}$ is the classical solution that we discussed in the previous paragraph. It corresponds to the set of all connected tree graphs, and it drives the order $\hbar$ term:

$$\left[-\partial^2 + m_0^2 + \frac{\lambda_0}{2}\left(\langle\phi_0(x)\rangle^{(0)}\right)^2\right]\langle\phi_0(x)\rangle^{(1)} = -\frac{\lambda_0}{2}\langle\phi_0(x)\rangle^{(0)}\frac{1}{i}\frac{\delta\langle\phi_0(x)\rangle^{(0)}}{\delta\rho_0(x)}\,. \tag{6.4.46}$$

Solving this equation we see that $\langle\phi_0(x)\rangle^{(1)}$ consists of all connected graphs which contain a *single* closed loop. This is pictured in the graphical sum

$\langle\phi_0(x)\rangle^{(1)} =$  + $\cdots$. (6.4.47)

Only connected graphs appear because $\langle\phi_0\rangle^{(0)}$ is connected. Only single loop graphs appear because the driving term on the right-hand side of Eq. (6.4.46) has only a single loop. Iterating again using $\langle\phi_0(x)\rangle^{(0)}$ and $\langle\phi_0(x)\rangle^{(1)}$ to compute $\langle\phi_0(x)\rangle^{(2)}$, we see that $\langle\phi_0(x)\rangle^{(2)}$ consists of connected graphs with *two* closed loops. In general $\langle\phi_0(x)\rangle^{(\ell)}$ contains connected graphs with ℓ loops. Functionally integrating

$$\frac{\delta W[\rho]}{\delta\rho_0(x)} = \langle\phi_0(x)\rangle \tag{6.4.48}$$

gives

$$W[\rho] = \sum_{\ell=0}^{\infty}\frac{\hbar^\ell}{\ell!}W^{(\ell)}[\rho]\,, \tag{6.4.49}$$

in which $W^{(\ell)}[\rho]$ contains all connected graphs with ℓ loops. We have now proven that $W[\rho]$ is indeed the generating functional of *connected* graphs. Moreover, we have also shown that counting the power of $\hbar$ counts the number of loops.†

The fact that the order of $\hbar$ is the loop order can also be easily proven using the "power counting" method of Chapter 5, Section 2. The propagator corresponding to a line of a graph is the inverse of the differential operator occurring in the quadratic terms in $(1/\hbar)\mathcal{L}$ and so it contains a factor of $\hbar$. Each vertex carries a factor of $1/\hbar$. Hence a graph with I internal lines and V vertices carries the power

$$P = I - V \tag{6.4.50}$$

of $\hbar$. The number of internal momentum integrations is the number of loops L. This number [c.f. Eq. (5.2.5)] is the number of internal lines I minus the number of vertices V since each vertex contains a momentum conserving δ-function – except that one combination of these δ-functions gives the overall momentum conservation of the graph,

$$L = I - V + 1 . \tag{6.4.51}$$

Combining this with Eq. (6.4.50) shows that

$$L = P + 1 . \tag{6.4.52}$$

Since the effective action was defined with an overall factor of $1/\hbar$, the power P which appears here is given by $P = \ell - 1$, where ℓ is the index in Eq. (6.4.49). The number of loops is given by $L = \ell$.

6 $W[\rho]$ to one loop

The loop expansion can be computed from the functional integral representation

$$\exp\left\{\frac{i}{\hbar}W[\rho]\right\} = \int [d\phi] \exp\left\{\frac{i}{\hbar}\int [\mathcal{L} + \rho_0\phi]\right\} . \tag{6.4.53}$$

In the limit‡ $\hbar \to 0$ only the stationary phase path will contribute, the path for which

$$\delta \int [\mathcal{L} + \rho_0\phi] = 0 . \tag{6.4.54}$$

† This feature is of importance for theories that contain various kinds of symmetries. Although individual graphs of such theories do not necessarily obey the symmetries they may be transformed into one another by a symmetry operation the sum of all the graphs in a given loop order does represent the symmetry. The reason for this is that the symmetry must hold order-by-order in the expansion in powers in $\hbar$ since $1/\hbar$ multiplies the entire Lagrange function $\mathcal{L}$ which has the symmetry.

‡ This classical limit was previously discussed in Chapter 1, Section 4.

This defines the classical solution which gives the lowest order, tree graph approximation to the vacuum value of the field operator in the presence of the external source,

$$\phi_{\rm cl}(x)[\rho] = \langle \phi_0(x) \rangle_{\rm tree} . \tag{6.4.55}$$

The functional integral may be calculated by expanding about this stationary phase path. We write the field integration variable as

$$\phi(x) = \phi_{\rm cl}(x) + \hbar^{1/2}\chi . \tag{6.4.56}$$

Since the action $\int [\mathcal{L} + \rho_0\phi]$ is stationary for infinitesimal variations of ϕ about $\phi_{\rm cl}$, there are no terms linear in χ. Moreover, $\phi(x)$ and $\chi(x)$ differ only by a constant translation (in functional space) and so§ $[d\phi] = [d\chi]$. Hence

$$\exp\left\{\frac{i}{\hbar}W[\rho]\right\} = \exp\left\{\frac{i}{\hbar}W^{(0)}[\rho]\right\} \int [d\chi] \exp\left\{-i\int (d^n x) \left[\frac{1}{2}\chi\left(-\partial^2 + m_0^2 + \frac{\lambda_0}{2}\phi_{\rm cl}(x)^2\right)\chi + \frac{\lambda_0}{3!}\hbar^{1/2}\phi_{\rm cl}\chi^3 + \frac{\lambda_0}{4!}\hbar\chi^4\right]\right\} , \tag{6.4.57}$$

where

$$W^{(0)}[\rho] = \int (d^n x)[\mathcal{L}_{\rm cl} + \rho_0\phi_{\rm cl}] \tag{6.4.58}$$

is the total classical action which is a function of the classical field $\phi_{\rm cl}$ that is the solution of the classical field equation driven by the source ρ_0. The terms involving χ^3 and χ^4 in the exponential can be expanded using the Taylor's series for that part of the exponential which contains them. Since the remainder of the integral is even in χ, only even terms in χ will contribute. Hence there are no terms which are odd in $\hbar^{1/2}$ and we do get a power series in terms of integer powers of $\hbar$.

We shall be content to compute the first two terms

$$W[\rho] \simeq W^{(0)}[\rho] + \hbar W^{(1)}[\rho] . \tag{6.4.59}$$

The classical action (6.4.58) involves the the Lagrange function $\mathcal{L}_{\rm cl}$ computed from the classical solution $\phi_{\rm cl}$. To verify that this explicit construction of $W^{(0)}[\rho]$ is identical with our previous implicit definition we compute $\delta W^{(0)}[\rho]/\delta\rho(x)$. Varying ρ makes $\phi_{\rm cl}$ vary. But $W^{(0)}[\rho]$ is unchanged by variations in $\phi_{\rm cl}$, since $\phi_{\rm cl}$ is the solution to the classical field equation. Hence

$$\frac{\delta}{\delta\rho_0(x)} W^{(0)}[\rho] = \phi_{\rm cl}(x) , \tag{6.4.60}$$

§ The factors of $\hbar^{1/2}$ are absorbed in the normalization to make $Z[0] = 1$.

which is just our previous implicit definition of $W^{(0)}[\rho]$. Passing now to the first order term, we introduce

$$\mathcal{G}^{-1}(\phi_{\rm cl}) = -\partial^2 + m_0^2 + \frac{\lambda_0}{2}\phi_{\rm cl}(x)^2 , \tag{6.4.61}$$

and use an operator notation to write

$$\exp\{iW^{(1)}[\rho]\} = \int [d\chi] \exp\Big\{-\frac{i}{2}\chi\mathcal{G}^{-1}\chi\Big\} . \tag{6.4.62}$$

This is the familiar Gaussian integral which produces a Fredholm determinant. The integration measure must be such as to give $W^{(1)}[\rho = 0] = 0$. Hence the proper normalization is achieved by writing

$$\exp\{iW^{(1)}[\rho]\} = \mathrm{Det}^{-1/2}[\mathcal{G}^{-1}(\phi_{\rm cl})\mathcal{G}(0)] , \tag{6.4.63}$$

since in the limit $\rho \to 0$, $\mathcal{G}^{-1}(\phi_{\rm cl})\mathcal{G}(0) \to 1$ and $\mathrm{Det}[1] = 1$. Here $\mathcal{G}(0) = \Delta_+$ is the familiar free field propagator and

$$\mathcal{G}^{-1}(\phi_{\rm cl})\Delta_+ = 1 + \frac{\lambda_0}{2}\phi_{\rm cl}^2\Delta_+ . \tag{6.4.64}$$

Employing

$$\mathrm{Det}[1 + X] = \exp\{\mathrm{Tr}\ln[1 + X]\} , \tag{6.4.65}$$

the explicit form

$$W^{(1)}[\rho] = \frac{i}{2}\,\mathrm{Tr}\ln\Big[1 + \frac{\lambda_0}{2}\phi_{\rm cl}^2\Delta_+\Big] , \tag{6.4.66}$$

is obtained. This is a well-defined result with the regularization to n dimensions. The renormalization needed will be discussed towards the end of the next section.

6.5 Effective potential

We turn now to introduce $\Gamma[\langle\phi\rangle]$, the generating functional of connected, single-particle irreducible Green's functions and its restriction to uniform background fields, the effective potential $\mathcal{V}(\bar{\phi})$. These quantities are particularly useful for the discussion of spontaneously broken theories — theories which are relativistic analogs of the spontaneously symmetry breakdown in non-relativistic many-body systems discussed in Chapter 2, Section 7 with the illustration of superfluid helium. But before introducing these new quantities, other aspects of the effective action $W[\rho]$ need to be described.

1 Convexity

We return to natural units, $\hbar = 1$, and also return to Euclidean space with the effective action presented as

$$\exp\{W[\rho]\} = \int [d\phi] \exp\Big\{-\int (d_E^n x)[\mathcal{L}_E - \rho_0\phi]\Big\} . \tag{6.5.1}$$

It is sometimes convenient to write this in terms of the functional measure

$$[d\mu] = [d\phi] \exp\Big\{-\int (d_E^n x)\mathcal{L}_E\Big\} \tag{6.5.2}$$

which is normalized,

$$\int [d\mu] = 1 . \tag{6.5.3}$$

Thus the exponential of the effective action appears as an average value of the source term,

$$\exp\{W[\rho]\} = \int [d\mu] \exp\Big\{\int (d_E^n x)\rho_0\phi\Big\} . \tag{6.5.4}$$

In particular, this form may be used to prove that the effective action is a *convex* functional.

A proof of this assertion involves the use of the Hölder inequality

$$\int [d\mu] F^\alpha G^\beta \le \Big[\int [d\mu] F\Big]^\alpha \Big[\int [d\mu] G\Big]^\beta , \tag{6.5.5}$$

where F and G are positive but otherwise arbitrary functionals and where α, β are positive with

$$\alpha + \beta = 1 . \tag{6.5.6}$$

Writing

$$F = \exp\Big\{\int (d_E^n x)\rho_0^{(1)}\phi\Big\} , \tag{6.5.7}$$

and

$$G = \exp\Big\{\int (d_E^n x)\rho_0^{(2)}\phi\Big\} , \tag{6.5.8}$$

the Hölder inequality (6.5.5) implies that

$$\exp\Big\{W[\alpha\rho^{(1)} + \beta\rho^{(2)}]\Big\} \le \exp\Big\{\alpha W[\rho^{(1)}] + \beta W[\rho^{(2)}]\Big\} , \tag{6.5.9}$$

or that

$$W[\alpha\rho^{(1)} + \beta\rho^{(2)}] \le \alpha W[\rho^{(1)}] + \beta W[\rho^{(2)}] , \tag{6.5.10}$$

which is just the statement that $W[\rho]$ is a convex functional. In a picture in which the source function is replaced by a single variable, this corresponds to the sketch in Fig. 6.17.

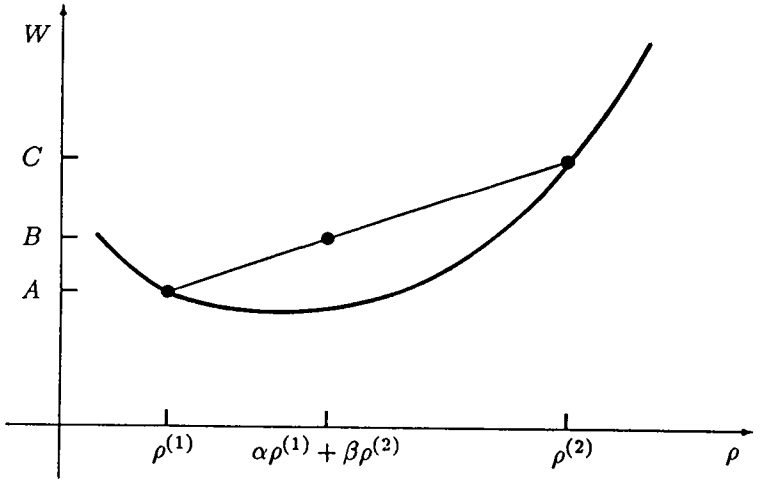


Fig. 6.17. A convex function lies beneath any chord: Values are labeled by $A = W[\rho^{(1)}]$, $C = W[\rho^{(2)}]$, and B the straight-line interpolated value, $B = \alpha W[\rho^{(1)}] + \beta W[\rho^{(2)}]$. According to the convexity, B lies above $W[\alpha\rho^{(1)} + \beta\rho^{(2)}]$.

Setting $\rho^{(1)} = \rho$ and $\rho^{(2)} = \rho + f$ where $f(x)$ is a small but arbitrary function, and then expanding the convexity statement (6.5.10) in powers of $f(x)$, the zeroth and first order terms cancel while the quadratic terms require that

$$\int (d_E^n x)(d_E^n y) f(x) f(y) \frac{\delta^2 W[\rho]}{\delta\rho(x)\delta\rho(y)} \geq 0 . \tag{6.5.11}$$

This is the statement that the second variational derivatives of the effective action produce a positive (semi-definite) operator.

The proof of the Hölder inequality follows from an elementary inequality which we now derive. Consider

$$M(\theta) = \alpha a e^{\theta/\alpha} + \beta b e^{-\theta/\beta} , \tag{6.5.12}$$

where a, b are positive and α, β non-negative, with $\alpha + \beta = 1$. Clearly $d^2M(\theta)/d\theta^2 > 0$. The unique minimum of $M(\theta)$ is determined by setting $dM(\theta)/d\theta = 0$, which yields $e^{\theta} = (b/a)^{\alpha\beta}$. The value at the minimum gives the inequality

$$M(\theta) \geq a^{\alpha} b^{\beta} , \tag{6.5.13}$$

which holds for all θ. In particular, taking $\theta = 0$ yields

$$a^{\alpha} b^{\beta} \leq \alpha a + \beta b . \tag{6.5.14}$$

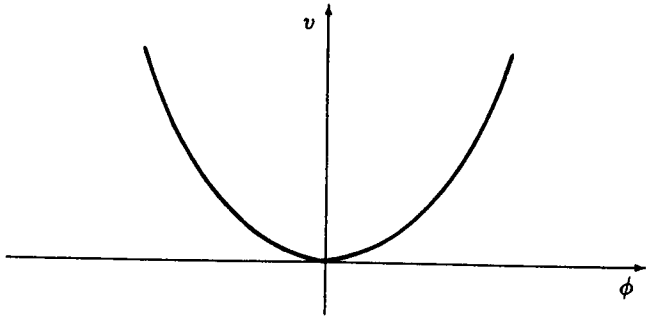


Fig. 6.18. The "potential function" of Eq. (5.17).

This result implies that

$$\int [d\mu] \left[\frac{F}{\int [d\mu] F}\right]^{\alpha} \left[\frac{G}{\int [d\mu] G}\right]^{\beta} \leq \int [d\mu] \left[\frac{\alpha F}{\int [d\mu] F} + \frac{\beta G}{\int [d\mu] G}\right]$$
$$= \alpha + \beta = 1 , \tag{6.5.15}$$

which is just the Hölder inequality (6.5.5).

It is worth illustrating the character of the convexity of the effective action with the simplest model where it is replaced by the zeroth order classical action of Eq. (6.4.58) with the source and field taken to be uniform in space-time. In this model, the effective action is equal to the space-time volume VT times an ordinary function $w(\rho)$ with

$$w(\rho) = \rho\phi - v(\phi) , \tag{6.5.16}$$

in which

$$v(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 . \tag{6.5.17}$$

A plot of the "potential function" $v(\phi)$ appears as in Fig. 6.18.

The field ϕ is a function of the source ρ determined by the "classical field equation"

$$\frac{dv(\phi)}{d\phi} = m^2\phi + \frac{\lambda}{6}\phi^3 = \rho . \tag{6.5.18}$$

The solution $\phi(\rho)$ may be determined graphically by plotting $v' = dv/d\phi$ and finding the value of ϕ when the curve $v'(\phi)$ intersects the value $v' = \rho$ as shown in Fig. 6.19.

A given value of ρ determines a unique value of $\phi(\rho)$. Since the action is stationary under variations of ϕ with ρ fixed,

$$\frac{dw(\rho)}{d\rho} = \phi(\rho) . \tag{6.5.19}$$

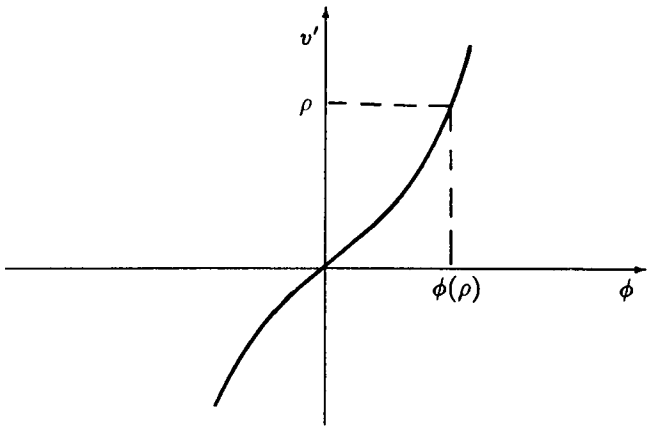


Fig. 6.19. Graphical determination of $\phi(\rho)$.

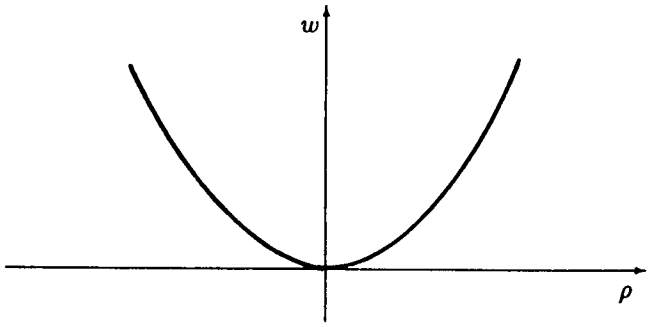


Fig. 6.20. The "effective action function".

Hence the second derivative is given by

$$\frac{d^2w(\rho)}{d\rho^2} = \frac{d\phi(\rho)}{d\rho}\,. \tag{6.5.20}$$

From the sketch in Fig. 6.19 it is clear that as ρ increases so does $\phi(\rho)$. Accordingly,

$$\frac{d^2w(\rho)}{d\rho^2} > 0\,, \tag{6.5.21}$$

which is equivalent to the statement that $w(\rho)$ is a convex function, in agreement with the general theorem proven above. The function $w(\rho)$ has the form illustrated in Fig. 6.20.

This case is to be contrasted to the situation of spontaneous symmetry

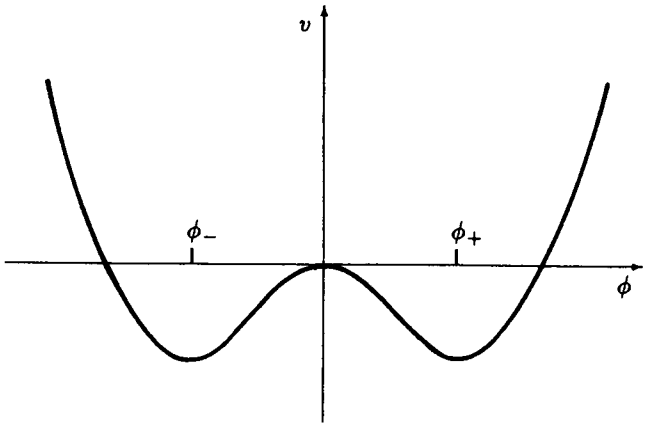


Fig. 6.21. Double-well "potential".

breakdown which occurs if m^2 is replaced by $-m^2$, with

$$v(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\,. \tag{6.5.22}$$

This potential is of the same form as that appearing in Eq. (2.7.48) which was used in the model of superfluid helium. The potential now has a double-well shape; it has two minima, $v'(\phi) = 0$, at

$$\phi = \phi_\pm = \pm\sqrt{\frac{6m^2}{\lambda}}\,, \tag{6.5.23}$$

as shown in Fig. 6.21.

As we shall see, there will be two possible ground states of the system corresponding to $\phi = \phi_+$ or to $\phi = \phi_-$. Thus the symmetry $\phi \to -\phi$ is spontaneously broken. Just as before, the field $\phi(\rho)$ is determined by setting $v'(\phi) = \rho$ where now

$$v'(\phi) = -m^2\phi + \frac{1}{6}\lambda\phi^3\,. \tag{6.5.24}$$

The graphical solution is illustrated in the sketch of Fig. 6.22.

The stationary points of this curve, $v''(\phi) = 0$, appear at $\phi = \pm\phi_S = \pm\sqrt{2m^2/\lambda}$ and define $\rho_\pm = v'(\mp\phi_S)$. For $\rho > \rho_+$ or $\rho < \rho_-$, the solution $\phi(\rho)$ is unique. However, for $\rho_- < \rho < \rho_+$ there are three solutions; the cubic equation has three real roots in this region. In the previous section it was shown that

$$\mathcal{E} = -w(\rho) = v(\phi(\rho)) - \rho\phi(\rho) \tag{6.5.25}$$

gives the energy density of the vacuum state adiabatically perturbed by the source ρ. The root must be chosen so as to make this energy min-

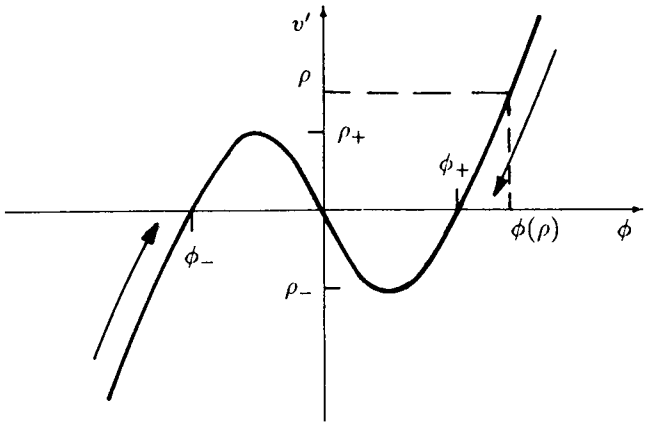


Fig. 6.22. Graphical determination of $\phi(\rho)$ for the double-well "potential".

imum. (Another root of higher energy would give an amplitude (in Euclidean space) suppressed by an exponential involving VT which vanishes in the $VT \to \infty$ limit.) As ρ approaches $\rho_+ > 0$ from above, a double root develops at $\phi = -\phi_S$. However, it is easy to check that the energy $\mathcal{E}$ associated with this double root is positive in contrast to that of the original root which is negative. Thus the original root must be chosen when $\rho = \rho_+$ no discontinuity appears here. With ρ decreasing further, continuity implies that the proper root $\phi(\rho)$ is given by the path through positive ϕ-values of the continuation of the original root. This persists to the limit $\rho \to 0^+$ with $\phi = \phi_+ > 0$ in the limit. This behavior is indicated by the line with the arrow on the right-hand side of the sketch. Similarly, for negative values of ρ, the proper root $\phi(\rho)$ continuously runs through negative values and approaches $\phi = \phi_- < 0$ in the limit $\rho \to 0^-$. There is a discontinuity in $\phi(\rho)$ at $\rho = 0$. There is no discontinuity in $w(\rho)$ at $\rho = 0$, with

$$w(0) = -v(\phi_\pm) = \frac{3}{2}\frac{m^4}{\lambda}\,. \tag{6.5.26}$$

However, $w(\rho)$ does have a discontinuity in slope at $\rho = 0$,

$$\rho \to 0^\pm \;:\quad \frac{dw(\rho)}{d\rho} = \phi(\rho) \to \phi_\pm\,. \tag{6.5.27}$$

This function has a cusp at $\rho = 0$. Nonetheless, the graphical construction of the root $\phi(\rho)$ makes it clear that we still have $d^2w(\rho)/d\rho^2 = d\phi(\rho)/d\rho > 0$ and $w(\rho)$ remains a convex function — as it must — even in the case of spontaneously broken symmetry. The results that we have obtained in this case are illustrated in Fig. 6.23. The cusp is the hallmark of spontaneous symmetry breakdown.

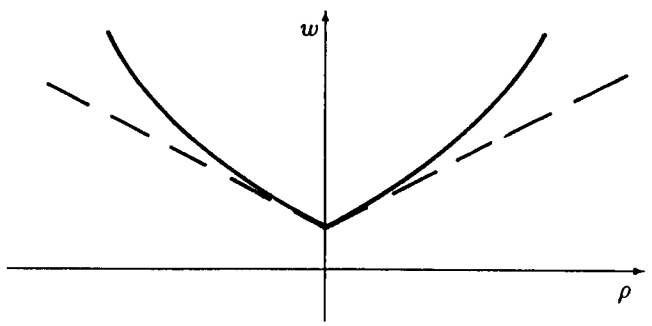


Fig. 6.23. The "effective action function" for the case of spontaneous symmetry breakdown. Note that this function has a cusp.

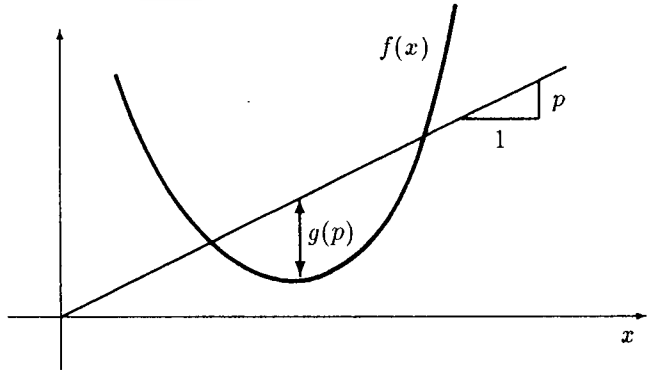


Fig. 6.24. The Legendre transform is the maximum separation between the function $f(x)$ and the line with slope p passing through the origin.

2 Legendre transform

The generating functional of the connected, single-particle irreducible Green's functions is called Γ. It is the Legendre transform of the effective action W, and so we first review the nature of this transform for simple functions of a single variable. The Legendre transform is a rule which given one function $f(x)$ produces another function $g(p)$. This rule can be expressed as a graphical construction involving the two curves defined by

$$y = px \tag{6.5.28}$$

and

$$y = f(x)\,. \tag{6.5.29}$$

Namely, $g(p)$ is defined to be the maximum separation between the two curves as shown in Fig. 6.24. To obtain the explicit formula for $g(p)$, one

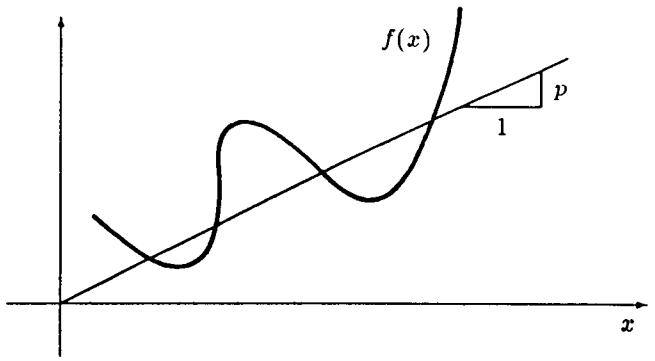


Fig. 6.25. The Legendre transform for an arbitrary function may be non-differentiable at points where the maximum separation jumps from one region to another.

defines

$$F(p,x) = px - f(x)\,, \tag{6.5.30}$$

which is the separation in y between the two curves. Then

$$g(p) = \max_x F(p,x)\,. \tag{6.5.31}$$

If $f(x)$ is convex and differentiable as in Fig. 6.24, there is a single maximum which occurs when $\partial F(x)/\partial x = 0$, or when

$$p = \frac{df(x)}{dx}\,, \tag{6.5.32}$$

which defines $x = x(p)$. In this case

$$g(p) = px(p) - f(x(p))\,, \tag{6.5.33}$$

and the Legendre transform is a continuous and differentiable function. The Legendre transform (6.5.31) exists for a general function as shown Fig. 6.25. As in the case shown in this sketch, the Legendre transform is not always differentiable — the maximum separation between the curve and the straight line may jump from one region to another as the slope p of the straight line is altered.

Legendre transforms of convex functions are *involutive*. That is, if $g(p)$ is the Legendre transform of $f(x)$, then the Legendre transform of $g(p)$ gives back $f(x)$. The proof is straightforward. The transform $h(q)$ of $g(p)$ is obtained from

$$G(q,p) = qp - g(p)\,; \tag{6.5.34}$$

namely

$$h(q) = G(q,p(q))\,, \tag{6.5.35}$$

where $p(q)$ is determined by

$$q = \frac{dg(p)}{dp}\,. \tag{6.5.36}$$

Using Eq. (6.5.33) and the chain rule of differentiation,

$$q = x + \left[p - \frac{df(x(p))}{dx(p)}\right]\frac{dx(p)}{dp} = x\,, \tag{6.5.37}$$

and so

$$h(x) = xp - [xp - f(x)] = f(x)\,. \tag{6.5.38}$$

We shall need one more property. The Legendre transform is *convex* (in any region where it is well-defined). This may be shown as follows. With α, β positive and $\alpha + \beta = 1$,

$$\begin{aligned} g(\alpha p_1 + \beta p_2) &= \max_x\{\alpha[p_1 x - f(x)] + \beta[p_2 x - f(x)]\} \\ &= \max_x\{\alpha F(p_1,x) + \beta F(p_2,x)\}\,. \end{aligned} \tag{6.5.39}$$

The following artifice is useful: The maximum may be considered to be the maximum over two separate variables x_1 and x_2 but with the constraint that $x_1 = x_2 = x$,

$$g(\alpha p_1 + \beta p_2) = \text{constrained } \max_{x_1,x_2}\{\alpha F(p_1,x_1) + \beta F(p_2,x_2)\}\,. \tag{6.5.40}$$

The constrained maximum is less than the unconstrained maximum. Thus

$$\begin{aligned} g(\alpha p_1 + \beta p_2) &\le \alpha \max_{x_1} F(p_1,x_1) + \beta \max_{x_2} F(p_2,x_2) \\ &= \alpha g(p_1) + \beta g(p_2)\,, \end{aligned} \tag{6.5.41}$$

which is the statement of convexity.

3 $\Gamma[\langle\phi\rangle]$

With this background material in mind, we now define $\Gamma[\langle\phi\rangle]$, the generating functional of connected, single-particle irreducible Green's functions. It is the functional Legendre transform of the effective action $W[\rho]$. The extension of the previous description of the Legendre transform of a simple function of a single variable to a functional should be clear. In particular, $\Gamma[\langle\phi\rangle]$ is well defined because $W[\rho]$ is convex, with*

$$\Gamma[\langle\phi\rangle] = \int (d_E^n x)\,\rho(x)\langle\phi(x)\rangle - W[\rho]\,. \tag{6.5.42}$$

* Our sign convention here differs from that in most of the literature.

This is a functional Legendre transform which interchanges the roles of $\rho(x)$ and $\langle\phi(x)\rangle$. It is the functional generalization of the transform discussed above since

$$\delta W[\rho] = \int (d_E^n x)\, \langle\phi(x)\rangle \delta\rho(x)\,, \tag{6.5.43}$$

gives a stationary value to this expression if ρ is treated as an independent variable. Moreover, with ρ now considered to be a functional of $\langle\phi\rangle$, the variations that appear when ρ is varied cancel and so

$$\delta\Gamma[\langle\phi\rangle] = \int (d_E^n x)\rho(x)\delta\langle\phi(x)\rangle\,, \tag{6.5.44}$$

or

$$\frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi(x)\rangle} = \rho(x)\,. \tag{6.5.45}$$

To see how $\Gamma[\langle\phi\rangle]$ works, consider the second variation

$$\frac{\delta^2\Gamma[\langle\phi\rangle]}{\delta\langle\phi(y)\rangle\delta\langle\phi(x)\rangle} = \frac{\delta\rho(x)}{\delta\langle\phi(y)\rangle}\,. \tag{6.5.46}$$

Now

$$\frac{\delta\langle\phi(y)\rangle}{\delta\rho(x)} = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle = G(x,y) \tag{6.5.47}$$

defines the connected two-point Green's function or propagator in the presence of the source ρ. Hence Eq. (6.5.46) tells us that

$$\frac{\delta^2\Gamma[\langle\phi\rangle]}{\delta\langle\phi(y)\rangle\delta\langle\phi(x)\rangle} = G^{-1}(y,x) \tag{6.5.48}$$

defines the inverse propagator in an operator sense. That is,

$$\int (d_E^n z) G^{-1}(x,z)G(z,y) = \delta(x-y)\,. \tag{6.5.49}$$

If the source is turned off, $\rho(x) \to 0$, then $G(x,y)$ becomes the ordinary propagator and the Fourier transforms $G(p)$ and $G^{-1}(p)$ are simply the ordinary numerical inverses of one another.

Since $\Gamma[\langle\phi\rangle]$ is a well-defined Legendre transform, $\Gamma[\langle\phi\rangle]$ is *convex*. This convexity implies that $\delta^2\Gamma[\langle\phi\rangle]/\delta\langle\phi(x)\rangle\delta\langle\phi(y)\rangle$ is a positive kernal, which is to say that, in view of Eq. (6.5.48),

$$\int (d_E^n x)(d_E^n y) f(x) G^{-1}(x,y) f(y) \geq 0 \tag{6.5.50}$$

for any $f(x)$.

To take further functional derivatives we use the operator result

$$\delta G^{-1} = -G^{-1}\delta G G^{-1} \tag{6.5.51}$$

to compute

$$\frac{\delta^3\Gamma[\langle\phi\rangle]}{\delta\langle\phi(z)\rangle\delta\langle\phi(y)\rangle\delta\langle\phi(x)\rangle} = -\int (d_E^n w)(d_E^n w')\, G^{-1}(y,w)\frac{\delta G(w,w')}{\delta\langle\phi(z)\rangle}G^{-1}(w',x)\,. \tag{6.5.52}$$

Here

$$\begin{aligned}\frac{\delta G(w,w')}{\delta\langle\phi(z)\rangle} &= \int (d_E^n\bar{z})\frac{\delta\rho(\bar{z})}{\delta\langle\phi(z)\rangle}\frac{\delta G(w,w')}{\delta\rho(\bar{z})} \\ &= \int (d_E^n\bar{z}) G^{-1}(z,\bar{z})\{\langle\phi(x)\phi(y)\phi(\bar{z})\rangle - \langle\phi(\bar{z})\rangle G(x,y) \\ &\quad -\langle\phi(x)\rangle G(\bar{z},y) - \langle\phi(y)\rangle G(\bar{z},x) - \langle\phi(x)\rangle\langle\phi(y)\rangle\langle\phi(\bar{z})\rangle\}\,.\end{aligned} \tag{6.5.53}$$

Taking one more functional derivative with respect to $\langle\phi\rangle$ and then setting $\rho = 0$ gives the result for

$$\Gamma(x,y,z,w) = \left.\frac{\delta^4\Gamma[\langle\phi\rangle]}{\delta\langle\phi(w)\rangle\delta\langle\phi(z)\rangle\delta\langle\phi(y)\rangle\delta\langle\phi(x)\rangle}\right|_{\rho=0}, \tag{6.5.54}$$

namely

$$\begin{aligned}\Gamma(x,y,z,w) = -\int &(d_E^n\bar{x})(d_E^n\bar{y})(d_E^n\bar{z})(d_E^n\bar{w})G^{-1}(x,\bar{x})G^{-1}(y,\bar{y}) \\ &G^{-1}(z,\bar{z})G^{-1}(w,\bar{w})\,\{G(\bar{x},\bar{y},\bar{z},\bar{w}) - G(\bar{x},\bar{y})G(\bar{z},\bar{w}) \\ &-G(\bar{x},\bar{z})G(\bar{y},\bar{w}) - G(\bar{x},\bar{w})G(\bar{y},\bar{z})\}\,.\end{aligned} \tag{6.5.55}$$

The quantity within the curly brackets is just the connected four-point Green's function which is generated by $W[\rho]$. The integrations with the G^{-1} factors remove the external propagator from this connected four-point Green's function. Such amplitudes with the external propagator factors removed are called "*truncated*".

We shall leave the exercise of taking six functional derivatives of $\Gamma[\langle\phi\rangle]$ and then setting $\rho = 0$ as a problem — the result of this exercise shows that $\Gamma(x_1,\ldots,x_6)$ consists of the connected and truncated 6-point function with all of its single particle reducible parts removed. That is, $\Gamma(x_1,\ldots,x_6)$ is *completely single-particle irreducible*. In general, $\Gamma[\langle\phi\rangle]$ generates connected, truncated, and single-particle irreducible Green's functions.

Functional methods may be used to give the outline of a general proof that $\Gamma[\langle\phi\rangle]$ is the generating functional of single-particle irreducible amplitudes. Suppose that the parameters in the Lagrange function are varied,

changing $\mathcal{L}(\phi)$ to $\mathcal{L}(\phi)+\delta\mathcal{L}(\phi)$. Using the functional integral representation

$$e^{W[\rho]} = \int [d\phi] e^{\int [\mathcal{L}+\rho_0\phi]}, \tag{6.5.56}$$

this variation appears as

$$\begin{aligned}\delta e^{W[\rho]} &= \int [d\phi] \int \delta\mathcal{L}(\phi) e^{\int [\mathcal{L}+\rho_0\phi]} \\ &= \int \delta\mathcal{L}\left(\frac{\delta}{\delta\rho_0}\right) e^{W[\rho]},\end{aligned} \tag{6.5.57}$$

giving

$$\begin{aligned}\delta W[\rho] =& e^{-W[\rho]} \int \delta\mathcal{L}\left(\frac{\delta}{\delta\rho_0}\right) e^{W[\rho]} \\ =& \int \delta\mathcal{L}\left(\langle\phi_0\rangle + \frac{\delta}{\delta\rho_0}\right) \\ =& \int \delta\mathcal{L}\left(\langle\phi_0\rangle + G\frac{\delta}{\delta\langle\phi_0\rangle}\right).\end{aligned} \tag{6.5.58}$$

This is the resulting variation in $W[\rho]$ with ρ_0 fixed. Since $\rho\langle\phi\rangle = \rho_0\langle\phi_0\rangle$, the Legendre transform may equally well be written in terms of the unrenormalized fields,

$$\Gamma[\langle\phi_0\rangle] = \int (d_E^n x)\rho_0(x)\langle\phi_0(x)\rangle - W[\rho_0]. \tag{6.5.59}$$

We now add to the previous variation of the parameters a variation δ_ρ of ρ_0 such that the total variation of $\langle\phi_0(x)\rangle$ vanishes. Since

$$\delta_\rho W[\rho_0] = \int (d_E^n x)\langle\phi_0(x)\rangle\delta\rho_0(x), \tag{6.5.60}$$

the additional variation makes no alteration in $\Gamma[\langle\phi_0\rangle]$. Hence, the induced parameter variation in $\Gamma[\langle\phi_0\rangle]$ with $\langle\phi_0\rangle$ fixed is given by

$$\delta\Gamma[\langle\phi_0\rangle] = -\int \delta\mathcal{L}\left(\langle\phi_0\rangle + G\frac{\delta}{\delta\langle\phi_0\rangle}\right). \tag{6.5.61}$$

Graphically, the operation of $G\delta/\delta\langle\phi_0\rangle$ entails the addition of a loop. Hence, if $\Gamma[\langle\phi\rangle]$ is single-particle irreducible for some values of the parameters, it remains single-particle irreducible when these parameters are varied. In particular, we may vary the coupling constant. Starting from the free theory where $\Gamma[\langle\phi\rangle]$ is explicitly single-particle irreducible, we see that this property holds in general.

Let us briefly discuss the renormalization group behavior of the generating functionals that have been introduced. The original functional $Z[\rho]$ knows nothing of the arbitrary mass unit μ and so this property carries

over to the effective action

$$\mu\frac{d}{d\mu}W[\rho] = 0. \tag{6.5.62}$$

Following the discussion of Chapter 5, particularly that which led to Eq. (5.4.3), this independence of μ is expressed by

$$\begin{aligned}\Big\{\mu\frac{\partial}{\partial\mu} + \beta(\lambda;n)\frac{\partial}{\partial\lambda} + \delta(\lambda)m^2\frac{\partial}{\partial m^2} \\ + \gamma(\lambda)\int (d_E^n x)\rho(x)\frac{\delta}{\delta\rho(x)}\Big\}W[\rho;\lambda,m^2,\mu^2] = 0.\end{aligned} \tag{6.5.63}$$

Taking the functional derivative of this equation with respect to $\rho(x)$ gives

$$\mu\frac{d}{d\mu}\langle\phi(x)\rangle = -\gamma(\lambda)\langle\phi(x)\rangle, \tag{6.5.64}$$

where the total derivative $\mu d/d\mu$ is again the same as the operation in the curly bracket in Eq. (6.5.63). This result is, in fact, obvious from the remark that the product $\rho\phi = \rho_0\phi_0$ is independent of μ while

$$\mu\frac{d}{d\mu}\rho(x) = \gamma(\lambda)\rho(x). \tag{6.5.65}$$

The field expectation value $\langle\phi\rangle$ must obey Eq. (6.5.64) to keep this product invariant. Hence $\Gamma[\langle\phi\rangle]$ defined by Eq. (6.5.42) must also be independent of the arbitrary mass unit μ,

$$\mu\frac{d}{d\mu}\Gamma[\langle\phi\rangle] = 0. \tag{6.5.66}$$

Expressing this total derivative in terms of all of its separate variations gives the renormalization group equation

$$\begin{aligned}\Big\{\mu\frac{\partial}{\partial\mu} + \beta(\lambda;n)\frac{\partial}{\partial\lambda} + \delta(\lambda)m^2\frac{\partial}{\partial m^2} \\ - \gamma(\lambda)\int (d_E^n x)\langle\phi(x)\rangle\frac{\delta}{\delta\langle\phi(x)\rangle}\Big\}\Gamma[\langle\phi\rangle;\lambda,m^2,\mu^2] = 0.\end{aligned} \tag{6.5.67}$$

This is of the same form as that discussed in Chapter 5, Section 4, and it has the solution

$$\begin{aligned}&\Gamma[\langle\phi\rangle;\lambda(\mu_2),m^2(\mu_2),\mu_2^2] = \\ &\qquad \Gamma\left[\exp\left\{\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda'\frac{\gamma(\lambda')}{\beta(\lambda';n)}\right\}\langle\phi\rangle;\lambda(\mu_1),m^2(\mu_1),\mu_1^2\right].\end{aligned} \tag{6.5.68}$$

Previously we computed the effective action $W[\rho]$ through one loop order. Let us transcribe this result in terms of $\Gamma[\langle\phi\rangle]$. Transforming

Eq. (6.4.66) into Euclidean space gives

$$\Gamma[\langle\phi\rangle] = \int (d_E^n x)\rho(x)\langle\phi(x)\rangle - \int (d_E^n x)\left\{\mathcal{L}(\phi_{\rm cl}) + \rho\phi_{\rm cl}\right\} + \frac{1}{2}{\rm Tr}\ln\left[1 + \frac{\lambda_0}{2}\phi_{\rm cl}^2\Delta_E\right]. \tag{6.5.69}$$

To this one-loop order

$$\langle\phi(x)\rangle = \phi_{\rm cl}(x) + \langle\phi(x)\rangle^{(1)}, \tag{6.5.70}$$

where $\langle\phi(x)\rangle^{(1)}$ is the one-loop correction to the field expectation value. Writing $\phi_{\rm cl} = \langle\phi\rangle - \langle\phi\rangle^{(1)}$ in the classical action piece of Eq. (6.5.69) gives no term which is linear in $\langle\phi\rangle^{(1)}$ since the classical action is stationary under variations of $\phi_{\rm cl}$. The remaining, non-linear terms in $\langle\phi\rangle^{(1)}$ are of higher order than one loop, and they may be omitted. Similarly, to one-loop order, we can substitute $\langle\phi\rangle$ for $\phi_{\rm cl}$ in the trace-log contribution which is already of first order. Hence through the one-loop order:

$$\Gamma[\langle\phi\rangle] = \frac{1}{2}{\rm Tr}\ln\left[1 + \frac{\lambda_0}{2}\langle\phi\rangle^2\Delta_E\right] - \int (d_E^n x)\mathcal{L}(\langle\phi\rangle). \tag{6.5.71}$$

4 *Effective potential*

Finally, let us consider the nature of $\Gamma[\langle\phi\rangle]$ when the source $\rho(x)$ is a constant $\bar\rho$. In this case the field expectation value will also be constant,

$$\langle\phi\rangle = \bar\phi, \tag{6.5.72}$$

and $\Gamma[\bar\phi]$ will contain a factor of the volume of space-time and so we write

$$\Gamma[\bar\phi] = \mathcal{V}(\bar\phi)\int (d_E^n x). \tag{6.5.73}$$

The function $\mathcal{V}(\bar\phi)$ is called the effective potential. According to the discussion in the previous section, in the presence of a constant source

$$W[\rho_0] = -\Delta\mathcal{E}\int (d_E^n x), \tag{6.5.74}$$

where $\Delta\mathcal{E}$ is the energy shift of the perturbed ground state relative to the original vacuum. So

$$\mathcal{V}(\bar\phi) = \Delta\mathcal{E} + \bar\rho\bar\phi. \tag{6.5.75}$$

In the presence of the source $\bar\rho$ the energy density is described by the Hamiltonian density

$$\mathcal{H}(\bar\rho) = \mathcal{H} - \bar\rho\phi, \tag{6.5.76}$$

with

$$\Delta\mathcal{E} = \langle\mathcal{H}(\bar\rho)\rangle = \langle\mathcal{H}\rangle - \bar\rho\bar\phi. \tag{6.5.77}$$

Hence

$$\mathcal{V}(\bar\phi) = \langle\mathcal{H}\rangle, \tag{6.5.78}$$

and we find that the effective potential is simply the expectation value of the original Hamiltonian in the new ground state which gives $\langle\phi\rangle = \bar\phi$.

Remembering Eq. (6.5.45), we now have

$$\frac{d\mathcal{V}(\bar\phi)}{d\bar\phi} = \bar\rho. \tag{6.5.79}$$

Thus the ground state of the system without a source is fixed by the condition that

$$\frac{d\mathcal{V}(\bar\phi)}{d\bar\phi} = 0. \tag{6.5.80}$$

The effective potential $\mathcal{V}(\bar\phi)$ is convex since it is a restriction of the convex functional $\Gamma[\langle\phi\rangle]$. Thus the ground state condition (6.5.80) requires that the effective potential be at its minimum. This point should be emphasized strongly. The effective potential provides a clear determination of the ground state of a system. In particular, the nature of the ground state is not obvious when a system undergoes spontaneous symmetry breaking, but it is clearly defined by the effective potential.

It is interesting to illustrate the nature of the effective potential with the simple model discussed above where the effective action was replaced by $VTw(\rho)$, with $w(\rho) = \rho\phi - v(\phi)$. For this model the naive application of the Legendre transform [Eq. (6.5.42)] yields

$$\mathcal{V}(\bar\phi) = v(\bar\phi). \tag{6.5.81}$$

Moreover, the condition (6.5.80) for the ground state determines the obvious value $\bar\phi = 0$. This is correct for the case of unbroken symmetry. But it is wrong for the case of spontaneous symmetry breakdown, for in this case the potential $v(\bar\phi)$ has a double-well structure which is not convex. In this case one must be careful and return to the precise definition of the Legendre transform as the maximum distance between a curve and a straight line. No paradox can arise if this is done since the convexity follows from this precise definition. So we return to the previous plot of $w(\rho)$ in Fig. 6.23 and add a straight line of slope $\bar\phi$ as shown in Fig. 6.26. The Legendre transform involves the maximum of the distance *from* the curve *to* the straight line. In the present case, the curve of $w(\rho)$ sometimes lies above the straight line and this distance is negative. When this happens it is easier to think of the Legendre transform as being defined as the negative of the *minimum* of the distance measured the other way around which is positive. This happens in the region $-\phi_0 < \bar\phi < \phi_0$, where ϕ_0 is defined by $v(\phi_0) = 0$. We recall that $w(\rho)$ has a cusp at $\rho = 0$, with slopes

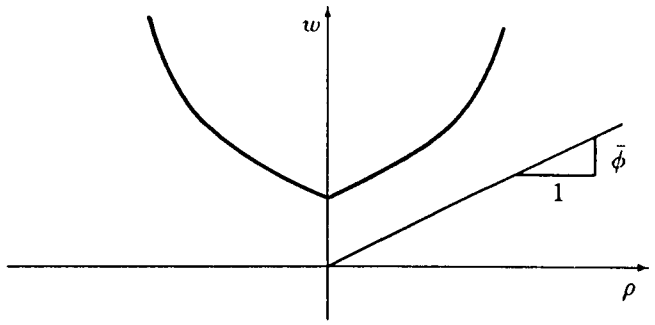


Fig. 6.26. Legendre transform construction for the case of spontaneous symmetry breakdown where the original function has a cusp.

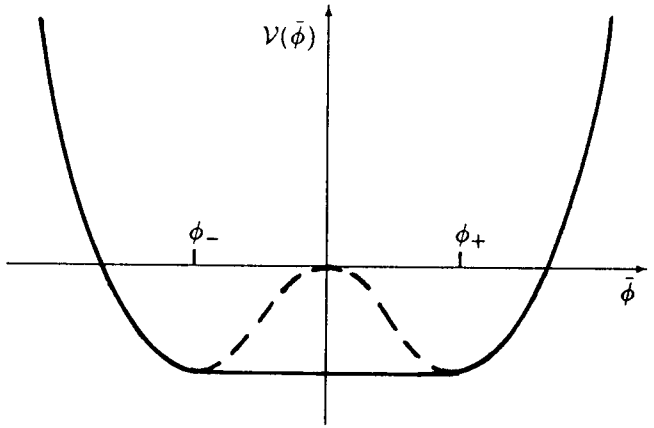


Fig. 6.27. Effective potential for a theory which has a double well "potential" in the Lagrange function.

ϕ_+ on the right and ϕ_- on the left. With $\bar{\phi} > \phi_+$, there is a maximum separation for $\bar{\phi} > \phi_0$ and a minimum separation for $\bar{\phi} < \phi_0$ at finite, positive values of ρ, and the Legendre transform produces $\mathcal{V}(\bar{\phi}) = v(\bar{\phi})$. However, when $\bar{\phi}$ decreases below $\bar{\phi} = \phi_+$, the minimum separation stays fixed at $\rho = 0$ until $\bar{\phi} = \phi_- = -\phi_+$, and in this region $\mathcal{V}(\bar{\phi}) = v(\phi_+)$. For $\bar{\phi} < \phi_-$, the Legendre transform again produces $\mathcal{V}(\bar{\phi}) = v(\bar{\phi})$.

The resulting effective potential shown in Fig. 6.27 is clearly continuous and convex. The dashed line represents the analytic continuation of $\mathcal{V}(\bar{\phi})$ into the region $\phi_- < \bar{\phi} < \phi_+$ which is just the function $v(\bar{\phi})$. As was discussed in the superfluid helium example of Chapter 2, Section 7, an isolated system with spontaneous symmetry breakdown is not well-defined. Such systems must be defined with an extra external perturbation such as the source $\bar{\rho}$ and then the limit taken where the external perturbation

vanishes. In our case, the system appears in the state with $\bar{\phi} = \phi_+$ if it is initially perturbed with a positive source $\bar{\rho}$ and then the limit $\bar{\rho} \to 0^+$ taken, while the system appears with $\bar{\phi} = \phi_-$ with an initial negative source and the limit $\bar{\rho} \to 0^-$. Although the Legendre transform $\mathcal{V}(\bar{\phi})$ is flat in the region $\phi_- < \bar{\phi} < \phi_+$, this is just a formal construction. It should be borne in mind that the effective potential is a physical quantity — the energy density in the ground state constrained to give $\langle\phi\rangle = \bar{\phi}$. The analytic continuation of the effective potential to $\mathcal{V}(\bar{\phi}) = v(\bar{\phi})$ in this region may be of physical significance - the situation here is akin to a supercooled phase of a thermodynamic system.

To compute the effective potential through one loop, we first note that to this order the bare parameters may be replaced by their renormalized values in the trace-log term in Eq. (6.5.71) since it is already of first order. We replace m_0^2 by m^2, write $\lambda_0 = \lambda\mu^{4-n}$, and use the usual quantum mechanical notation to compute

$$\begin{aligned}\mathrm{Tr}\ln\left[1+\frac{\lambda\mu^{4-n}}{2}\bar{\phi}^2\Delta_E\right] &= \int (d_E^n x)\langle x|\ln[\cdots]|x\rangle \\ &= \int (d_E^n x)(d_E^n p)\langle x|p\rangle\langle p|\ln[\cdots]|x\rangle \\ &= \int (d_E^n x)(d_E^n p)|\langle x|p\rangle|^2 \ln\left[1+\frac{\lambda\mu^{4-n}}{2}\bar{\phi}^2\frac{1}{p^2+m^2}\right]. \quad (6.5.82)\end{aligned}$$

In this quantum mechanical notation $|\langle x|p\rangle|^2 = 1/(2\pi)^n$, and so including the tree level contribution, Eq. (6.5.71) produces

$$\mathcal{V}(\bar{\phi}) = \frac{1}{2}\int \frac{(d_E^n p)}{(2\pi)^n}\ln\left[1+\frac{\lambda\mu^{4-n}}{2}\bar{\phi}^2\frac{1}{p^2+m^2}\right] + \frac{1}{2}m_0^2\bar{\phi}^2 + \frac{\lambda_0}{4!}\bar{\phi}^4 . \quad (6.5.83)$$

The logarithm in the momentum integral may be expressed as

$$\ln\left[\frac{p^2+v''(\bar{\phi})}{p^2+m^2}\right] = \int_0^\infty \frac{ds}{s}\, e^{-sp^2}\left[\exp\{-m^2 s\} - \exp\{-v''(\bar{\phi})s\}\right], \quad (6.5.84)$$

where

$$v''(\bar{\phi}) = m^2 + \frac{\lambda\mu^{4-n}}{2}\bar{\phi}^2 . \quad (6.5.85)$$

Using this integral representation, the momentum integral may be calculated in the standard manner. There is no need to do this, however, because the result is just the difference of vacuum energies for mass squared $v''(\bar{\phi})$ and m^2. The vacuum energy, given in Eq. (4.2.10), provides the

evaluation

$$\mathcal{V}(\bar{\phi}) = -\frac{1}{2}\frac{\Gamma(-n/2)}{(4\pi)^2}\mu^{n-4}\left\{[v''(\bar{\phi})]^2\left[\frac{v''(\bar{\phi})}{4\pi\mu^2}\right]^{n/2-2} - m^4\left[\frac{m^2}{4\pi\mu^2}\right]^{n/2-2}\right\} + \frac{1}{2}m_0^2\bar{\phi}^2 + \frac{1}{4!}\lambda_0\bar{\phi}^4 . \tag{6.5.86}$$

Using the $n \to 4$ limit

$$\Gamma(-n/2) = -\frac{1}{n-4} + \frac{3}{4} - \frac{\gamma}{2}, \tag{6.5.87}$$

a finite result is obtained if

$$m_0^2 = m^2\left[1 - \frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right], \tag{6.5.88}$$

and

$$\lambda_0 = \lambda\mu^{4-n}\left[1 - \frac{3\lambda}{(4\pi)^2}\frac{1}{n-4}\right]. \tag{6.5.89}$$

These are just the renormalizations that have been obtained previously, but the calculation here is much simpler.

The limit $n \to 4$ produces

$$\mathcal{V}(\bar{\phi}) = \frac{1}{2}m^2\left[1 - \frac{\lambda}{(4\pi)^2}\left(\frac{3}{4} - \frac{\gamma}{2}\right)\right]\bar{\phi}^2 + \frac{1}{4!}\lambda\left[1 - \frac{3\lambda}{(4\pi)^2}\left(\frac{3}{4} - \frac{\gamma}{2}\right)\right]\bar{\phi}^4 + \frac{1}{4}\frac{1}{(4\pi)^2}\left\{[v''(\bar{\phi})]^2\ln\left[\frac{v''(\bar{\phi})}{4\pi\mu^2}\right] - m^4\ln\left[\frac{m^2}{4\pi\mu^2}\right]\right\}. \tag{6.5.90}$$

Note that the expression in the curly brackets vanishes when $\bar{\phi}$ vanishes. Moreover, this expression is non-singular in the massless limit, $m^2 \to 0$, as must be the case with the minimal subtraction method.

Problems

> ... and it was not clear to me as a student that the approach to a more profound knowledge of the basic principles of physics is tied up with the most intricate mathematical methods. This dawned upon me only gradually after years of independent scientific work.
>
> Albert Einstein

1. The matrix element $\langle 0 - |\phi(x)|0-\rangle^\rho$ gives the average value of the field observed at the space-time event x^μ if the system, perturbed by the external source $\rho(x)$, is initially in the vacuum state. This matrix element may be written as

$$\langle 0 - |\phi(x)|0-\rangle^\rho = \sum\nolimits_n \langle 0 - |n+\rangle^\rho \langle n + |\phi(x)|0-\rangle^\rho = \sum\nolimits_n [\langle n + |0-\rangle^\rho]^* \langle n + |\phi(x)|0-\rangle^\rho .$$

The expansion of the amplitudes $\langle n+|0-\rangle^\rho$ and $\langle n+|\phi(x)|0-\rangle^\rho$ in powers of ρ entail the familiar time-ordered products produced by the functional integral. Use these expressions to first order in ρ to prove that

$$\langle 0 - |\phi(x)|0-\rangle^\rho = \int (d^4x')\,\theta(x^0 - x'^0)\langle 0|i[\phi(x),\,\phi(x')]|0\rangle\rho(x') + \ldots .$$

2. The contribution of an N-particle intermediate state to the propagator's spectral weight is given by

$$\rho_N(-p^2) = \text{const.}\int \frac{(d^3\mathbf{q}_1)}{2q_1^0}\cdots\frac{(d^3\mathbf{q}_N)}{2q_N^0}\delta^{(4)}(\Sigma q_a - p)|\langle 0|\phi(0)|q_1\cdots q_N\rangle|^2 .$$

Let $m_a^2 = -q_a^2$ be the mass squared of the a-th particle. The N-particle threshold is at

$$-p^2 = s = M_N^2 ,$$

where

$$M_N = \sum_{a=1}^{N} m_a .$$

Compute the behavior of $\rho_N(s)$ for s near M_N^2. Hints: Use the frame in which $\mathbf{p} = 0$. In this frame near threshold all of the particles $a =$

$1, \cdots, N$ are non-relativistic. The matrix element $|\langle 0|\phi(0)|q_1 \cdots q_N\rangle|^2$ may be taken to be a constant.

3. The amplitude $\Gamma(x_1, \cdots, x_6)$ is obtained by taking six functional derivatives of $\Gamma[\langle\phi\rangle]$ with respect to $\langle\phi(x)\rangle$ and then setting $\langle\phi(x)\rangle = 0$. Compute this amplitude for the interacting scalar field theory in terms of the connected 6, 4, and 2-point Green's functions. Express the result in terms of a graphical decomposition. Explain how $\Gamma(x_1, \cdots, x_6)$ is a connected, single-particle irreducible amplitude.

4. Spontaneous Symmetry Breakdown

The spontaneously broken theory discussed in the text has two degenerate vacuum states $|0_\pm\rangle$ which give

$$\langle 0_\pm|\phi(x)|0_\pm\rangle = \phi_\pm\,,$$

where $\phi_- = -\phi_+$. The purpose of this problem is to give rough but heuristic proofs of the following facts. In the infinite volume limit, the two vacua are orthogonal, $\langle 0_-|0_+\rangle = 0$. Moreover, in this limit, all the states formed from these two vacua are orthogonal. That is, if $|S_+\rangle$ is a state constructed by a linear combination of field operators $\phi(x_1)\cdots\phi(x_M)$ acting upon $|0_+\rangle$ and $|S'_-\rangle$ a state similarly constructed by $\phi(x_1)\cdots\phi(x_N)$ acting upon $|0_-\rangle$, then $\langle S'_-|S_+\rangle = 0$. Therefore, coherent superpositions of the states built upon the two vacua behave as an incoherent mixture, a density matrix, since

$$[\,a^*\langle S_+| + b^*\langle S'_-|\,]\,[\,c|S''_+\rangle + d|S'''_-\rangle\,] = a^*c\,\langle S_+|S''_+\rangle + b^*d\,\langle S'_-|S'''_-\rangle\,.$$

In view of this orthogonality, the states

$$|\theta\rangle = |0_+\rangle\cos\theta + |0_-\rangle\sin\theta$$

yield

$$\langle\theta|\phi|\theta\rangle = \phi_+[\cos^2\theta - \sin^2\theta]\,,$$

and thus an expectation value $\langle\phi\rangle$ that ranges between ϕ_+ and ϕ_- as θ moves from 0 to $\pi/2$. The effective potential for these states is fixed at $\mathcal{V}(\phi_+)$ and so they realize the "flat bottom" which the Legendre transform produces. It should be noted, however, that these states violate the cluster decomposition property for their matrix elements

$$\langle\theta|T\phi(x_1)\cdots\phi(x_M)\phi(y_1)\cdots\phi(y_N)|\theta\rangle\,.$$

The rough "proof" follows by making the lowest-order but non-trivial approximation which we now describe. We neglect all renormalizations and write the Hamiltonian for the spontaneously broken theory as

$$H = \int(d^3\mathbf{r})\left\{\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + v(\phi)\right\}\,,$$

with

$$v(\phi) = \frac{1}{4!}\lambda(\phi^2 - \phi_+^2)^2\,.$$

a) Prove that

$$\exp\left\{i\int(d^3\mathbf{r})f(\mathbf{r})\Pi(\mathbf{r})\right\}\phi(\mathbf{r})\exp\left\{-i\int(d^3\mathbf{r})f(\mathbf{r})\Pi(\mathbf{r})\right\} = \phi(\mathbf{r}) + f(\mathbf{r})\,.$$

To make the following manipulations well defined, the system must now be restricted to a box of finite volume V. Then

$$\exp\left\{i\phi_\pm\int_V(d^3\mathbf{r})\Pi(\mathbf{r})\right\}H\exp\left\{-i\phi_\pm\int_V(d^3\mathbf{r})\Pi(\mathbf{r})\right\} = H_0\,,$$

where, with the neglect of cubic and quartic interactions,

$$H_0 = \int_V(d^3\mathbf{r})\left\{\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\,,$$

in which

$$m^2 = \frac{1}{3}\lambda\phi_+^2\,.$$

In this approximation, and in the finite box, the two vacua $|0_\pm\rangle$ can be obtained from the vacuum state $|0\rangle$ of H_0 — the vacuum state for a free particle of mass m. Namely,

$$|0_\pm\rangle = \exp\left\{-i\phi_\pm\int_V(d^3\mathbf{r})\Pi(\mathbf{r})\right\}|0\rangle\,.$$

Moreover,

$$\langle 0_\pm|\phi|0_\pm\rangle = \langle 0|(\phi + \phi_\pm)|0\rangle = \phi_\pm\,.$$

b) We now have

$$\langle 0_-|0_+\rangle = \langle 0|\exp\left\{-2i\phi_+\int_V(d^3\mathbf{r})\Pi(\mathbf{r})\right\}|0\rangle\,.$$

Compute the matrix element on the right-hand side of this equation in the large volume limit and show that it vanishes exponentially when $V \to \infty$.

c) Explain why the states $|S_+\rangle$ and $|S'_-\rangle$ are orthogonal.

d) Explain why $\langle\theta|T\phi(x_1)\cdots\phi(x_M)\phi(y_1)\cdots\phi(y_N)|\theta\rangle$ does not obey the cluster decomposition property.

5. With a uniform source $\bar{\rho}$, the effective action is an ordinary function times the space-time volume,

$$W[\bar{\rho}] = VTw(\bar{\rho})\,.$$

Using the measure introduced in Eq. (6.5.2), this function is given by

$$\exp\{VTw(\bar{\rho})\} = \int [d\mu] \exp\left\{\bar{\rho}\int (d_E^n x)\phi(x)\right\} .$$

We define a function $\tilde{\mathcal{V}}(\bar{\phi})$ by the large VT limit

$$\exp\left\{-VT\tilde{\mathcal{V}}(\bar{\phi}) + 0(\ln VT)\right\} = \int [d\mu]\delta\left(\bar{\phi} - \frac{1}{VT}\int (d_E^n x)\phi(x)\right) .$$

Here the $0(\ln VT)$ in the exponential is the same as a prefactor of the exponential that depends in an algebraic manner on VT. The expression has been written in this way to emphasize that $\tilde{\mathcal{V}}(\bar{\phi})$ is determined by the exponential damping in VT. Prove that this defines the usual effective potential, $\tilde{\mathcal{V}}(\bar{\phi}) = \mathcal{V}(\bar{\phi})$. Hints that give the problem away: Write

$$1 = \int d\bar{\phi}\,\delta\left(\bar{\phi} - \frac{1}{VT}\int (d_E^n x)\phi(x)\right)$$

inside the functional integral above that defines $w(\bar{\rho})$, interchange integrals and express the resulting functional integral in terms of $\tilde{\mathcal{V}}(\bar{\phi})$. Now do the final $d\bar{\phi}$ integral by the method of steepest descent which is appropriate to the $VT \to \infty$ limit that is always implied. Finally remember that the Legendre transform is involutive.

6. Pions, Chiral Symmetry

Mesons are composite bound states of quark-antiquark pairs with a cloud of gluons. Nonetheless, if the energy is low so that the internal states of these composite systems are rarely excited, the mesons can be described by effective elementary fields.

There are three pions which we describe by $\phi_a(x)$, $a = 1, 2, 3$. Isospin is found to be a good symmetry. Hence one might try to describe the low-energy interactions of pions by the Lagrange function

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_a)^2 - \frac{1}{2}m^2\phi_a^2 - \frac{1}{4!}\lambda(\phi_a^2)^2 .$$

(Notice, incidentally, that this is renormalizable. If non-renormalizable couplings had been included, since they have dimensions involving inverse powers of a mass, their effects involve energy/mass to some power and they should be negligible at low energy.) The Lagrange function is invariant under the infinitesimal isospin rotation

$$\delta\phi_a(x) = \delta\omega_b\,\epsilon_{abc}\,\phi_c(x) .$$

Now let the rotation parameter vary from point to point in space-time $\delta\omega_a \to \delta\omega_a(x)$ so that there is a varying local isospin rotation. Since $\mathcal{L}$ is invariant for constant $\delta\omega_a$, the response must now have the form

$$\delta\mathcal{L} = V_a^\mu(x)\,\partial_\mu\delta\omega_a(x) ,$$

which identifies a current operator $V_a^\mu(x)$. Since the action is stationary for all variations about the solution to the field equations, we conclude that

$$\int (d^4x)\delta\mathcal{L}\Big|_{\phi=\text{sol.}} = 0 ,$$

and hence find that this current is conserved,

$$\partial_\mu V_a^\mu(x) = 0 ,$$

assuming that the field equations are obeyed.

a) Compute $V_a^\mu(x)$ in terms of the fields ϕ_a and show directly from the equations of motion that $V_a^\mu(x)$ is conserved. It follows from the current conservation that the total isospin

$$T_a = \int (d^3\mathbf{r})V_a^0(\mathbf{r}, t)$$

is a constant of the motion.

b) Recall that the conjugate field momentum $\pi_a(x)$ is defined by

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\partial_0\phi_a} ,$$

and that at equal times

$$[\phi_a(x),\, \pi_b(x')] = i\delta_{ab}\delta(\mathbf{r} - \mathbf{r}') .$$

Hence the formula

$$T_a\,\delta\omega_a = \int (d^3\mathbf{r})\,\pi_a\delta\phi_a$$

follows directly from the definition of V_a^μ . *Explain* in more detail what has just been stated, and *explain* why

$$[\phi_a(x),\, T_b] = i\,\epsilon_{abc}\,\phi_c(x) .$$

The isospin generators T_a must commute with the equal-time commutator $[\phi_b,\, \pi_c]$ since the latter is a pure number. Use the Jacobi identity $[A,[B,C]] = [[A,B],C] + [B,[A,C]]$ to show that

$$[\pi_a(x),\, T_b] = i\epsilon_{abc}\,\pi_c(x) .$$

[This proof involves the completeness of the operators ϕ_a , π_b. Any operator which commutes with ϕ_a (at equal times) must be a function of ϕ_a . So there could be some additional function of ϕ on the right-hand side of the last equation. However, it is very easy to see from an explicit calculation that there is no such additional term.] Now prove that

$$[T_a,\, T_b] = i\epsilon_{abc}T_c .$$

Thus the T_a provide an explicit operator realization of the Lie algebra of SU(2) = S0(3).

c) We shall assume that there is a unique minimum energy state, a unique vacuum state $|0\rangle$. Hence the vacuum state must be left invariant by an infinitesimal isospin rotation,

$$(1 + iT_a\delta\omega_a)|0\rangle = |0\rangle\,,$$

or

$$T_a\,|0\rangle = 0\,.$$

As discussed in the text, single particle states can be constructed by the action of a field operator on the vacuum. These states may be chosen to be eigenstates $|t\rangle$ of T_3,

$$T_3\,|t\rangle = |t\rangle\,t\,,$$

with $t = +1, 0, -1$. The states $|t\rangle$ are constructed in terms of fields $\phi_t(x)$ acting on the vacuum state, where ϕ_t are simple combinations of the ϕ_a. What are these combinations? Why? What field operator combinations are to be used to compute $\langle t|$?

d) The $\lambda\phi^4$ effective theory which has just been described does not agree with nature. The physics of low energy mesons is approximately chiral symmetric, and the previous Lagrangian did not have this symmetry. (This chiral symmetry follows from the structure of the underlying QCD theory which, except for small corrections, is symmetric under transformations that perform separate isospin rotations on the two different helicity components of the quark fields.) To take account of this symmetry, a new field $\sigma(x)$ is adjoined to the theory. This is a scalar field of zero isospin in contrast to ϕ_a which is a pseudoscalar field of unit isospin. Under an infinitesimal isospin rotation

$$\delta_I\phi_a = \delta\omega_b\epsilon_{abc}\,\phi_c\,, \qquad \delta_I\sigma = 0\,.$$

The chiral symmetry is obtained by demanding invariance under the transformation

$$\delta_C\phi_a = \delta\nu_a\sigma\,, \qquad \delta_C\sigma = -\delta\nu_a\phi_a\,.$$

This is a *chiral* transformation because fields of different parity are interchanged. Think of $(\phi_a\,,\,\sigma)$ as the components of a vector in a four-dimensional (abstract) Euclidean space, and show that the combined transformation δ_C and δ_I correspond to the six possible and different rotations in this space. Hence the symmetry we are discussing corresponds to the group SO(4).

e) A Lagrangian which is invariant under the SO(4) chiral group is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_a)^2 - \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m^2(\phi_a^2 + \sigma^2) - \frac{1}{4!}\lambda(\phi_a^2 + \sigma^2)^2\,,$$

since it involves only the square of the "four-vector" $(\phi_a\,,\,\sigma)$. Compute (in terms of ϕ_a and σ) the currents V_a^μ and A_a^μ that are defined by the responses to the δ_I and δ_C variations. These currents are conserved. Why? The total charges are defined by

$$T_a = \int (d^3\mathbf{r})V_a^0\,, \qquad X_a = \int (d^3\mathbf{r})A_a^0\,.$$

Show that the commutators of T_a and X_a with ϕ_a and σ produce the δ_I and δ_C variations of ϕ_a and σ. Compute the commutators $[T_a, T_b]$, $[T_a, X_b]$, and $[X_a, X_b]$. We define

$$X_a^{(\pm)} = \frac{1}{2}(T_a \pm X_a)\,.$$

Prove that $X_a^{(+)}$ and $X_b^{(-)}$ commute while the commutator of different components of $X_a^{(+)}$ or $X_a^{(-)}$ give the SU(2) algebra. Hence you have proven that SO(4) = SU(2) × SU(2).

The theory just presented has a scalar σ particle which is degenerate in mass with the three π particles. No such σ particle is observed, and so this is not the right theory. The right low-energy theory (except for small corrections) has the structure

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_a)^2 - \frac{1}{2}(\partial_\mu\sigma)^2 - V\,,$$

in which

$$V = \frac{1}{4!}\lambda(\phi_a^2 + \sigma^2 - f^2)^2\,.$$

This is just the previous theory with $m^2 = -\lambda f^2/6$. All that has been done is to make m^2 negative in the previous theory and write it in a convenient form. To discover the physical content of the new theory, we note that in general the ground state is determined by setting the slope of the effective potential to zero, $d\mathcal{V}(\langle\phi\rangle)/d\langle\phi\rangle = 0$. The expansion in powers of $\hbar$ corresponds to an expansion in the number of loops and the no loop or tree graph terms correspond to the classical solution. We start with the classical solution and perturb about it. The sum of all the terms in a given order in $\hbar$ maintains the full symmetry of the theory. Throughout the remainder of this problem we shall always work at the lowest order, tree level. In this approximation, $\langle\phi\rangle$ coincides with the classical field ϕ and the effective potential equals the classical potential V. The minimum of this potential appears at $\sigma = f$. Hence we write

$\sigma = f + \sigma'$, and use σ' as a new field variable. Note that in lowest order, the tree level,

$$\langle\sigma\rangle = \langle 0|\sigma|0\rangle = f\,.$$

f) Write V in terms of ϕ_a and σ' and identify the pion mass m_π, the σ mass m_σ, and all the various ϕ_a and σ' couplings. Note that $m_\pi = 0$. This is obvious from the picture of V because the field mass corresponds to the radius of curvature of V at its minimum in the direction of the field and the potential is flat in the ϕ_a directions at the minimum.

We should note that the isospin symmetry is an ordinary symmetry with $T_a|0\rangle = 0$. However, the chiral symmetry is not an ordinary symmetry but rather a spontaneously broken symmetry. The classical potential attains a minimum value for any fields which obey $\phi_a^2 + \sigma^2 = f^2$. This corresponds to the fact that there is not a unique vacuum state in the theory, but rather a continuous infinity of vacuum states. The choice of the ground state field configuration $\phi_a = 0$, $\sigma = f$ selects a particular vacuum state. Although the chiral symmetry is a good symmetry of the Lagrangian and hence of the field equations, it is not a good symmetry of the states. This lack of symmetry of the states is obvious from the fact that the pions and the σ are not degenerate in mass. There would be such a symmetry in the states if $X_a|0\rangle = 0$, but this is not true here since a chiral transformation moves the vacuum state which has been selected into another equivalent vacuum state.

g) Although the physical pion mass is very small, it is not zero. The pion mass can be accounted for by adding a small symmetry breaking term to the Lagrangian

$$\mathcal{L} \to \mathcal{L} - \epsilon\sigma\,.$$

Compute the new value of $\langle\sigma\rangle$ and the new values of m_π, m_σ and the ϕ_a and σ' couplings in terms of λ, f, and ϵ. Compute $\partial_\mu A_a^\mu$ by the simple and elegant procedure of computing $\delta_c\mathcal{L}$ for a local transformation $\delta\nu_a(x)$ and then making the action stationary. The pion decays via $\pi \to \mu\nu$. The decay rate is controlled by the matrix element $\langle 0|A_a^\mu|bp\rangle$ where p^μ is the four-momentum of the pion and b its isospin label. (Here it is convenient to use $b = 1,2,3$ rather than the physical t labels.) Lorentz and isospin invariance tell us that

$$\langle 0|A_a^\mu(x)|bp\rangle = \delta_{ab} i p^\mu F_\pi e^{ipx}\,.$$

Here F_π is the "pion decay constant" which has the units of mass and which is directly measured by the pion decay rate. Express $\partial_\mu A_a^\mu$ in terms of m_π, F_π, and ϕ_a, and write $\mathcal{L}$ in terms of the measurable constants m_π, m_σ, and F_π.

h) Compute the pion-pion scattering amplitude T in the tree approximation. Denote the initial isospin and momenta of the particles by $(a\,p_1)$ $(b\,p_2)$, the final isospin and momenta by $(a'\,p_1')$ $(b'\,p_2')$. The σ particle is somewhat murky, and it was introduced in our theory only so that we could have a simple, linear realization of chiral symmetry. Let us now dispense with it by taking the limit $m_\sigma \to \infty$ with m_π and F_π fixed. Write down T in this limit. Calculate the $\pi^+\pi^-$ differential scattering cross section $d\sigma/d\Omega$ in the center-of-mass frame. How is this cross section measured experimentally?

7. Spontaneous Symmetry Breakdown, Functional Methods

The previous problem described a specific example of spontaneous symmetry breaking, that provided by the chiral symmetry of pions. With that background in mind, this problem provides a general treatment of spontaneous symmetry breaking using the functional methods developed in this chapter. Here we consider a multicomponent, real scalar field ϕ_α, $\alpha = 1,2,\cdots,N$, with its dynamics described by the Lagrange function

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha - V(\phi_\alpha)\,.$$

The potential term $V(\phi_\alpha)$ is assumed to be invariant under some group of transformations whose infinitesimal elements may be parameterized by $\delta\omega_b$, $b = 1,2,\cdots,g$, and give the field variation

$$\delta\phi_\alpha = i t_{\alpha b\gamma}\,\delta\omega_b\,\phi_\gamma\,.$$

The $t_{\alpha b\gamma}$ must be purely imaginary to keep the field real. These numbers are the matrix elements of matrices which represent the Lie algebra of the group, $t_{\alpha b\gamma} = (t_b)_{\alpha\gamma}$. The group property is conveyed by the commutation relation

$$[t_a,\, t_b] = i f_{abc} t_c\,,$$

where f_{abc} are the structure constants of the group. To keep the kinetic energy term $\frac{1}{2}\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha$ invariant, the t-matrices must be anti-symmetric, $t_a^T = -t_a$. Hence these imaginary matrices are Hermitian and therefore the structure constants f_{abc} are real.

a) Since the t_a are Hermitian,

$$g_{ab} = \mathrm{tr}\, t_a t_b$$

are the elements of a real, symmetric, and positive-definite matrix. Explain why a basis may be chosen – a new set of t_a' that are linear combinations of the old t_a - such that

$$\mathrm{tr}\, t_a' t_b' = \delta_{ab}\,.$$

We shall now drop the prime and always work in such a basis except that the normalization may be chosen so that the δ_{ab} above may be multiplied by some convenient constant. The closure of the t_a under commutation defined by the f_{abc} above requires that $f_{bac} = -f_{abc}$. Prove, by contemplating $\mathrm{tr}[t_a, t_b]t_c$, that $f_{abc} = f_{cab}$, and thus that the structure constants are completely anti-symmetric.

The invariance of the Lagrange function under the transformations with constant $\delta\omega_a$, a "rigid", global invariance, implies the existence of a set of conserved currents $j_a^\mu(x)$. These currents are best obtained by promoting the global invariance to a *local* symmetry where the infinitesimal group parameters are arbitrary functions on space-time, $\delta\omega_a(x)$. This is done by introducing a set of external vector potential functions $A_{\mu a}(x)$ and defining the covariant derivative

$$D_\mu\phi_\alpha(x) = \partial_\mu\phi_\alpha(x) - it_{\alpha b\gamma}A_{\mu b}(x)\phi_\gamma(x)\,.$$

b) Prove that if $A_{\mu a}(x)$ undergoes the non-Abelian gauge transformation

$$\delta A_{\mu a}(x) = \partial_\mu\delta\omega_a(x) + f_{abc}\delta\omega_b(x)A_{\mu c}(x)\,,$$

with $\phi_\alpha(x)$ also being transformed as above, then

$$\delta(D_\mu\phi_\alpha(x)) = it_{\alpha b\gamma}\delta\omega_b(x)D_\mu\phi_\gamma(x)\,.$$

Hence the Lagrange function

$$\mathcal{L} = -\frac{1}{2}D_\mu\phi_\alpha D^\mu\phi_\alpha - V(\phi_\alpha)$$

is invariant under the local group operation. Note that the gauge transformation can also be expressed as a covariant derivative in the adjoint representation,

$$\delta A_{\mu\alpha} = D_\mu\delta\omega_a\,,$$

where

$$D_\mu\delta\omega_a = \partial_\mu\delta\omega_a - f_{abc}A_{\mu b}\delta\omega_c\,.$$

Note that the same symbol D_μ is used for the covariant derivative in both the original and in the adjoint representations. The actual form of the covariant derivative is clear from the context in which it is used.

c) The currents j_a^μ are defined by

$$\frac{\partial}{\partial A_{\mu a}(x)}\mathcal{L} = j_a^\mu(x)\,.$$

It has just been shown that the Lagrange function is invariant when the fields ϕ_α and external vector potentials $A_{\mu a}$ are simultaneously transformed by an infinitesimal local group operation. The action is stationary

for field variations $\delta\phi_\alpha$ about the classical solution. Use these two facts to prove that

$$D_\mu j_a^\mu(x) = 0\,,$$

if ϕ_α obeys the classical field equation.

d) We turn now to the quantum field theory with the generating functional

$$Z[A\,,\rho] = \int[d\phi]\exp\left\{i\int(d^nx)[\mathcal{L} + \rho_\alpha\phi_\alpha]\right\}\,.$$

The action in this expression is invariant under the infinitesimal local group transformations $\phi_\alpha \to \bar{\phi}_\alpha = \phi_\alpha + \delta\phi_\alpha$, $A_{\mu a} \to \bar{A}_{\mu a} = \bar{A}_{\mu a} + D_\mu\delta\omega_a$ if the source function is also altered, $\rho_\alpha \to \bar{\rho}_\alpha = \rho_\alpha + \delta\rho_\alpha$, with $\delta\rho_\alpha = it_{\alpha b\gamma}\delta\omega_b\rho_\gamma$. The change of the field $\phi_\alpha \to \bar{\phi}_\alpha$ can be made since this is a change of the (dummy) functional integration variable. It entails the Jacobian factor

$$\mathrm{Det}\frac{\delta\bar{\phi}}{\delta\phi} = 1 + i\int(d^nx)\delta^{(n)}(0)\delta\omega_a(x)\mathrm{tr}\,t_a\,.$$

Prove this equation. In dimensional regularization, $\delta^{(n)}(0) = 0$ since no quantity is available to make up the dimensions of $\delta^{(n)}(0)$. Moreover, $\mathrm{tr}\,t_a = 0$ since t_a is an antisymmetrical matrix. Hence the Jacobian is unity. Give a little more detail to show that

$$Z[\bar{A}\,,\bar{\rho}] = Z[A\,,\rho]\,.$$

Show that this implies that

$$\left\{D_\mu\frac{\delta}{\delta A_{\mu a}(x)} + i\rho_\alpha(x)t_{\alpha a\beta}\frac{\delta}{\delta\rho_\beta(x)}\right\}Z[A\,,\rho] = 0\,.$$

We shall refer to this equation as the divergence condition.

e) The divergence condition implies that the operators

$$T_a = \int(d\mathbf{r})j_a^0\,,$$

with $A_{\mu a} = 0$, represent the group's Lie algebra. To prove this, consider

$$\langle Tj_a^\mu(x)\phi_{\alpha_1}(x_1)\cdots\phi_{\alpha_n}(x_n)\rangle\,,$$

which is obtained by the appropriate functional derivatives of $Z[A\,,\rho]$ with $A\,,\rho$ then taken to vanish. The time-ordered product is discontinuous, with

$$\int_{t'-\epsilon}^{t'+\epsilon}dt\frac{\partial}{\partial t}\langle TX(t)Y(t')\cdots\rangle = \langle T[X(t')\,,Y(t')]\cdots\rangle$$

implying that for t near t' one has the singular behavior

$$\frac{\partial}{\partial t}\langle TX(t)Y(t')\cdots\rangle = \delta(t-t')\langle T[X(t'),\,Y(t')]\cdots\rangle + \cdots .$$

Use the divergence condition to calculate

$$\partial_\mu\langle Tj_a^\mu(x)\phi_{\alpha_1}(x_1)\cdots\phi_{\alpha_n}(x_n)\rangle = ?\,,$$

and compare this result with the singular behavior of the time-derivative of the time-ordered product illustrated above to prove that

$$[\phi_\alpha(x),\,T_b] = t_{\alpha b\gamma}\phi_\gamma(x)\,.$$

By considering

$$\langle Tj_a^\mu(x)j_b^\nu(y)\varphi_{\alpha_1}(x_1)\cdots\phi_{\alpha_n}(x_n)\rangle\,,$$

prove in a similar way that

$$[T_a,\,T_b] = if_{abc}T_c\,.$$

[Hint: Take $\delta/\delta A_{\nu b}(y)\,\delta/\delta\rho_{\alpha_1}(x_1)\cdots\delta/\delta\rho_{\alpha_n}(x_n)$ of the divergence condition and then set $A=0=\rho$.]

f) We shall work in Euclidean space-time for the remainder of this problem. The effective action $W[A\,,\rho]$ and then $\Gamma[A\,,\langle\phi\rangle]$ are defined in the usual way. Prove that

$$\left\{D_\mu\frac{\delta}{\delta A_{\mu a}(x)} + i\langle\phi_\alpha(x)\rangle t_{\alpha a\beta}\frac{\delta}{\delta\langle\phi_\beta(x)\rangle}\right\}\Gamma[A\,,\langle\phi\rangle] = 0\,.$$

Take $\delta/\delta\langle\phi_\gamma(y)\rangle$ of this divergence condition and then set $A=0=\rho$ and take the Fourier transform to prove that

$$i\langle\phi_\alpha\rangle t_{\alpha a\beta}G^{-1}(p)_{\beta\gamma} = p^2F_{a\gamma}(p)\,,$$

where

$$\left.\frac{\delta^2\Gamma[A,\langle\phi\rangle]}{\delta A_{\mu a}(x)\delta\langle\phi_\gamma(y)\rangle}\right|_{A=\rho=0} = \int\frac{(d_E^np)}{(2\pi)^n}e^{ip(x-y)}ip^\mu F_{a\gamma}(p)\,.$$

In the case of spontaneously broken symmetry, some components of the field vacuum expectation value $\langle\phi_\alpha\rangle$ are non-vanishing. In this case, the equation above for $G^{-1}(p)_{\beta\gamma}$ implies that some of these components vanish as p^2 when $p^2\to 0$. [$F_{a\gamma}(p)$ cannot have a pole at $p^2=0$ since $\Gamma[A,\langle\phi\rangle]$ is single particle irreducible.] Thus these components describe massless particles. This is the Goldstone Theorem: Spontaneous symmetry breakdown implies the existence of massless excitations. To see how this works out in detail, consider the example of pions with 0(4) symmetry examined in the previous problem. Here $\langle\phi_4\rangle = \langle\sigma\rangle \neq 0$. With $a=1,2,3$ labeling isospin rotations, $t_{4a\beta}=0$ and so $\langle\phi_\alpha\rangle t_{\alpha a\beta}=0$. This is to say

that the spontaneously broken vacuum — and the theory — is invariant under isospin rotations; the isospin subgroup is an isometry. On the other hand, for the chiral rotations, $a=4,5,6$, $\langle\phi_\alpha\rangle t_{\alpha a\beta}\neq 0$ when $\beta=1,2,3$, and we see that for these pion components $G^{-1}(p)_{\beta\gamma} = p^2f(p^2)\delta_{\beta\gamma}$, implying that the pions are massless particles.

g) Functional derivatives of $W[A,\rho]$ with respect to A with ρ held fixed produce connected current-correlation functions. Thus

$$\frac{\delta W[A,\rho]}{\delta A_{\mu a}(x)} = \langle j_a^\mu(x)\rangle\,,$$

and

$$\frac{\delta^2W[A,\rho]}{\delta A_{\mu a}(x)\delta A_{\nu b}(y)} = \langle Tj_a^\mu(x)j_b^\nu(y)\rangle - \langle j_a^\mu(x)\rangle\langle j_b^\mu(y)\rangle = \langle Tj_a^\mu(x)j_b^\nu(y)\rangle_c\,.$$

These results are to be compared to functional derivatives of $\Gamma[A,\langle\phi\rangle]$ with respect to A but now with $\langle\phi\rangle$ held fixed. Use the Legendre transform

$$\Gamma[A,\langle\phi\rangle] = \int(d_E^nx)\rho_\alpha\langle\phi_\alpha\rangle - W[A\,,\rho]\,,$$

and the chain rule for functional differentiation to prove that

$$\frac{\delta\Gamma[A,\langle\phi\rangle]}{\delta A_{\mu a}(x)} = -\langle j_a^\mu(x)\rangle\,,$$

and

$$\frac{\delta^2\Gamma[A,\langle\phi\rangle]}{\delta A_{\mu a}(x)\delta A_{\nu b}(y)} = -\langle Tj_a^\mu(x)j_b^\nu(y)\rangle_c$$
$$+\int(d_E^mz)(d_E^mw)\,\frac{\delta^2\Gamma[A,\langle\phi\rangle]}{\delta A_{\mu a}(x)\delta\langle\phi_\alpha(z)\rangle}G_{\alpha\beta}(z,w)\,\frac{\delta^2\Gamma[A,\langle\phi\rangle]}{\delta A_{\nu b}(y)\delta\langle\phi_\beta(w)\rangle}\,.$$

Let us examine this last equation in the unperturbed limit $A=0=\rho$. Using the Fourier transform definitions

$$\frac{\delta^2\Gamma[A,\langle\phi\rangle]}{\delta A_{\mu a}(x)\delta A_{\nu b}(y)} = \int\frac{(d_E^np)}{(2\pi)^n}\,e^{ip(x\,-y)}\Gamma_{ab}^{\mu\nu}(p)\,,$$

$$\langle Tj_a^\mu(x)j_b^\nu(y)\rangle = \int\frac{(d_E^np)}{(2\pi)^n}\,e^{ip(x-y)}K_{ab}^{\mu\nu}(p)\,,$$

and the previously defined Fourier transform for $\delta^2\Gamma/\delta A\delta\langle\phi\rangle$, the equation becomes

$$\Gamma_{ab}^{\mu\nu}(p) = -K_{ab}^{\mu\nu}(p) + p^\mu p^\nu F_{a\alpha}(p)G_{\alpha\beta}(p)F_{b\beta}(p)\,.$$

Using the equation involving $F_{a\alpha}(p)$ derived in part f) above, this may be written as

$$\Gamma_{ab}^{\mu\nu}(p) = -K_{ab}^{\mu\nu}(p) + \frac{p^\mu p^\nu}{p^2}\langle\phi_\alpha\rangle t_{\alpha a\beta}\frac{1}{p^2}G^{-1}(p)_{\beta\gamma}t_{\gamma b\delta}\langle\phi_\delta\rangle\,.$$

The point to be made is that $\Gamma^{\mu\nu}_{ab}(p)$ is single-particle irreducible and hence has no pole at $p^2 = 0$. Thus $K^{\mu\nu}_{ab}(p)$ has a pole at $p^2 = 0$ which is determined by the last set of terms on the right-hand side of the equation above. This pole in $K^{\mu\nu}_{ab}(p)$ arises from the single-particle intermediate states contributed by the massless, Goldstone particles. Work this out for the pion case and show that

$$F_\pi^2 = \langle\sigma\rangle^2 Z^{-1}(0)\,,$$

where, for $\alpha\,,\beta = 1,2,3$, $\delta_{\alpha\beta}Z^{-1}(0) = \lim_{p^2\to 0}\frac{1}{p^2}G^{-1}(p)_{\alpha\beta}$. Note that $\langle\sigma\rangle^2$ and $G(p)$ have the same renormalization factor so that this relation for the observable quantity F_π is independent of the unit of mass μ as it must be.

We first encountered spontaneous symmetry breakdown in Chapter 2, Section 7 in the discussion of superfluid helium. It is instructive to treat this case using functional methods. For this purpose it is convenient to write the non-relativistic destruction operator in terms of real fields, $\psi = \phi_1 + i\phi_2$ and assume that the phase is chosen such that the condensate is real, $\langle\psi\rangle = \langle\phi_1\rangle$. The broken symmetry here is the phase symmetry $\psi \to e^{i\theta}\psi$. Using the thermal ensemble functionals in imaginary time, it is not difficult to show that the broken symmetry relation of part f) above becomes

$$\langle\phi_1\rangle G_{22}^{-1}(\omega_\nu\,,\mathbf{p}) = \omega_\nu F_1(\omega_\nu,\mathbf{p}) + \mathbf{p}\cdot\mathbf{F}_2(\omega_\nu,\mathbf{p})\,.$$

Here $G_{22}(\omega_\nu\,,\mathbf{p})$ is the Fourier transform of the time-ordered product of two ϕ_2 fields, with the frequency taking on the discrete values $\omega_\nu = 2\pi\nu/\beta$. The description of time-reversal invariance in Problem 5 of Chapter 2 implies that $G_{22}(\omega_\nu\,,\mathbf{p})$ is an even function of ω_ν. It is convenient to exhibit a factor of $\langle\phi_1\rangle$ and also to display the rotational invariance in writing

$$F_1(\omega_\nu\,,p) = \omega_\nu\langle\phi_1\rangle A(\omega_\nu^2,\mathbf{p}^2)\,.$$

The thermal ensemble defines a rest frame and so there is no relativity principle to relate $\mathbf{F}_2(\omega_\nu\,,\mathbf{p})$ to $F_1(\omega_\nu\,,\mathbf{p})$. The system is, however, rotationally invariant and so $\mathbf{F}_2$ must be $\mathbf{p}$ times a function of $\mathbf{p}^2$. Taking account of the time-reversal invariance, we may write $\mathbf{F}_2$ in the convenient form

$$\mathbf{F}_2(\omega_\nu\,,\mathbf{p}) = \mathbf{p}\langle\phi_1\rangle A(\omega_\nu^2\,,\mathbf{p}^2)C^2(\omega_\nu^2\,,\mathbf{p}^2)\,.$$

With this notation,

$$G_{22}^{-1}(\omega_\nu\,,\mathbf{p}) = A(\omega_\nu^2\,,\mathbf{p}^2)\left\{\omega_\nu^2 + \mathbf{p}^2C^2(\omega_\nu^2\,,\mathbf{p}^2)\right\}\,.$$

We have seen that spontaneously broken chiral symmetry requires that pions (in the absence of explicit breaking) be massless. The analogous

situation in the spontaneously broken phase symmetry of superfluid helium produces low frequency sound waves. To see how this goes, we note that the functions $A(\omega_\nu^2\,,\mathbf{p}^2)$ and $C^2(\omega_\nu^2\,,\mathbf{p}^2)$ correspond to single particle-irreducible functions and thus are well-behaved in the $\omega_\nu^2 \to 0\,,\ \mathbf{p}^2 \to 0$ limit. Calling these limits A and C^2, we have, for small ω_ν^2 and $\mathbf{p}^2$,

$$G_{22}^{-1}(\omega_\nu\,,\mathbf{p}^2) \simeq A\{\omega_\nu^2 + \mathbf{p}^2C^2\}\,.$$

The spectral representation derived in Problem 4c) of Chapter 2 shows that $G_{22}(\omega_\nu\,,\mathbf{p})$ can have no singularity for real ω_ν. Hence the constant C^2 must be positive (as written). The function $G_{22}(\omega_\nu\,,p^2)$ describes time-ordered correlations in imaginary time rather than real-time, physical propagation. However, as explained in Problem 4 of Chapter 2, the analytic continuation $\omega_\nu \to i(\omega + i\epsilon)$ from the discrete ω_ν to continuous, real ω values produces the Fourier transform of the real-time retarded commutator. This function describes the response of the system to time-dependent, external perturbations. Since

$$G_{22}(i\omega\,,\mathbf{p}^2) \simeq \frac{A^{-1}}{C^2\mathbf{p}^2 - (\omega + i\epsilon)^2}\,,$$

this function describes waves with the frequency-wave number relation $\omega = C|\mathbf{p}|$, which is characteristic of a sound wave of speed C.

One must still ask: How is the function $G_{22}(i\omega\,,\mathbf{p})$ observed? The answer is that the thermal average of the retarded density-density commutator directly describes the propagation of waves in the liquid. It has the structure of the time-time component of the current-current correlation function discussed in part g) above. In our non-relativistic context, this function $K(\omega\,,\mathbf{p})$ has the structure

$$K(\omega\,,\mathbf{p}) = F_1(i\omega\,,\mathbf{p})G_{22}(i\omega\,,\mathbf{p})F_1(i\omega\,,\mathbf{p}) - \Gamma(\omega\,,\mathbf{p})\,,$$

where $\Gamma(\omega\,,\mathbf{p})$ is single-particle irreducible and therefore regular for small ω and $\mathbf{p}$. Hence, for small ω and $\mathbf{p}$, using the results above gives

$$K(\omega\,,\mathbf{p}) \simeq \frac{\omega^2|\langle\psi\rangle|^2 A}{C^2\mathbf{p}^2 - (\omega + i\epsilon)^2}\,.$$

This formula demonstrates that the sound-wave excitations do indeed travel in superfluid liquid helium.

7
Fields with spin

Non-relativistic quantum states of any given angular momentum S can be built out of the direct product of spin $\frac{1}{2}$ states. Suitable linear combinations of these product states can be formed to produce the irreducible representation of the rotation group which describes angular momentum S. In a similar fashion, relativistic quantum fields describing particles with spin S can be built out of spin $\frac{1}{2}$ fields. We shall, therefore, dwell on the nature of spin $\frac{1}{2}$ fields at some length. Moreover, it is the basic spin $\frac{1}{2}$ field, the Weyl spinor, which is used in the modern electro-weak theory of weak interactions, and so these spinors will be discussed extensively with the more commonly known Dirac field related to them. Since the behavior of integer-spin fields, such as the spin-one photon field, is described by tensor fields that should be familiar, not much attention will be paid to them. Throughout this chapter, we shall work in our observable world of four dimensions. Higher dimensional spaces, however, are needed in the regularization procedure produced by dimensional continuation which we employ throughout this book. Moreover, there has been considerable theoretical speculation which involves higher-dimension space times. Much of the machinery for the extension to higher dimensions appears in the problems at the end of this chapter. The problems also deal with some of the peculiarities of a world with two space-time dimensions.

7.1 Irreducible representations of the Lorentz group

Let us first consider the response of some general field $\chi(x)$ to a simple rotation of the spatial coordinate frame. Such a rotation produces a coordinate transformation $\mathbf{r} \to \mathbf{r}'$. These are different labels for the same spatial point. The symbol $\chi(x)$ denotes a collection of field variables which could be indexed by some letter, but it is often simpler to consider

$\chi(x)$ to be a column vector of these fields. The rotation of the coordinate frame produces the transformation $\chi(x) \to \chi'(x')$, where $\chi'(x')$ is the field as seen by observers attached to the new frame who use the coordinates $x' = (t, \mathbf{r}')$ to describe the same space-time event originally denoted by $x = (t, \mathbf{r})$. Since the field $\chi(x)$ provides a complete description of the system of interest, the new field variables must be a linear combination of the old field variables or, in our matrix notation,

$$\chi'(x') = R\chi(x)\,. \tag{7.1.1}$$

Performing a sequence of such changes, we see that the matrix R must form a representation of the rotation group. Making a rotation labeled by 1 followed by a rotation labeled by 2 is equivalent to a rotation labeled by 3 and $R_1 R_2 = R_3$, which is just the group representation property. Because of the group property, it suffices to consider only infinitesimal rotations which have the form

$$R = 1 + i\delta\vec{\omega} \cdot \mathbf{S}\,. \tag{7.1.2}$$

Finite rotations may be obtained by the repeated application of such infinitesimal transformations. As discussed in Problem 1 of Chapter 2, the group property of the rotation is conveyed in the commutation relation of the spin matrices S_k,

$$[S_k\,, S_l] = i\epsilon_{klm}\, S_m\,. \tag{7.1.3}$$

The simplest, non-trivial representation is the spin $\frac{1}{2}$ representation in which

$$S_k = \frac{1}{2}\sigma_k\,, \tag{7.1.4}$$

where σ_k are the 2×2 Pauli spin matrices. We shall always use these matrices in the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.1.5}$$

This is an *irreducible* representation of the rotation group since no subspace of the spinor space is left invariant under the action of the three σ_k. As was remarked before, any higher spin irreducible representation may be obtained by taking direct products of spin $\frac{1}{2}$ representations and then projecting out the irreducible part. Hence we concentrate on the spin $\frac{1}{2}$ representation which has the general rotation matrix

$$R = \exp\left\{\frac{i}{2}\vec{\omega} \cdot \vec{\sigma}\right\}. \tag{7.1.6}$$

The familiar properties of the Pauli matrices follow from the explicit forms written above. For example, all the σ_k are Hermitian. Their algebraic

properties — they all square to give the identity matrix, they all anti-commute with one another, and they obey (except for a factor of two) the angular momentum commutation relations - are contained in the formula

$$\sigma_k \sigma_l = \delta_{kl} + i\epsilon_{klm}\,\sigma_m\,. \tag{7.1.7}$$

The fact that σ_2 is imaginary while σ_1 and σ_3 are real is conveniently expressed as

$$\sigma_k^* = -\sigma_2\sigma_k\sigma_2 = -\sigma_2\sigma_k\sigma_2^{-1}\,. \tag{7.1.8}$$

Therefore

$$R^* = \sigma_2 R \sigma_2^{-1}\,, \tag{7.1.9}$$

which states that R^* is, in the group theory sense, *equivalent* to R. Such representations where the complex conjugation is the same as an equivalence are called *real* representations. Since the Pauli matrices are Hermitian, the spin-$\frac{1}{2}$ representation (7.1.6) is unitary,

$$R^\dagger = R^{-1}\,. \tag{7.1.10}$$

Hence the Hermitian adjoint of the complex conjugate equivalence relation (7.1.9) expresses the transposed rotation matrix as

$$R^T = \sigma_2 R^{-1} \sigma_2\,. \tag{7.1.11}$$

Multiplying this on the right by $\sigma_2 R$ produces

$$R^T \sigma_2 R = \sigma_2\,. \tag{7.1.12}$$

To explain the significance of this formula, we note that

$$(i\sigma_2)_{\alpha\beta} = \epsilon_{\alpha\beta}\,, \tag{7.1.13}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$. Hence Eq. (7.1.12) may be written as

$$\epsilon_{\alpha'\beta'} R_{\alpha'\alpha} R_{\beta'\beta} = \epsilon_{\alpha\beta}\,. \tag{7.1.14}$$

Since for a general 2×2 matrix

$$\epsilon_{\alpha'\beta'} M_{\alpha'\alpha} M_{\beta'\beta} = \epsilon_{\alpha\beta} \det M\,, \tag{7.1.15}$$

we conclude that the rotation matrix R must be unimodular,

$$\det R = 1\,. \tag{7.1.16}$$

This property follows, of course, from the vanishing trace of the Pauli matrices,

$$\det R = \exp\{\mathrm{tr} \ln R\} = \exp\left\{\frac{i}{2}\vec{\omega}\cdot \mathrm{tr}\vec{\sigma}\right\} = 1\,. \tag{7.1.17}$$

One essential property of the spin-$\frac{1}{2}$ rotation representation remains to be listed. It is

$$R^{-1}\sigma_k R = r_{kl}\,\sigma_l\,, \tag{7.1.18}$$

where r_{kl} comprise the elements of the 3×3 matrix which describes the spatial rotation of a 3-vector. This relation is easily established by going to the limit of an infinitesimal rotation, $\vec{\omega} \to \delta\vec{\omega}$. Since $R^{-1} = \sigma_2^{-1} R^T \sigma_2$, we may write the relation (7.1.18) as

$$R^T \sigma_2 \sigma_k R = r_{kl}\sigma_2\sigma_l\,, \tag{7.1.19}$$

or as

$$(\sigma_2\sigma_k)_{\alpha'\beta'} R_{\alpha'\alpha} R_{\beta'\beta} = r_{kl}(\sigma_2\sigma_l)_{\alpha\beta}\,. \tag{7.1.20}$$

This is the statement that $(\sigma_2\sigma_k)_{\alpha\beta}$ is a linear combination of outer products of spinors which transforms as the irreducible $S = 1$ representation. On the other hand, Eq. (7.1.14) asserts that $\epsilon_{\alpha\beta}$ transforms as the irreducible $S = 0$ representation. These remarks make the decomposition $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ explicit.

With this preparation in mind, we can turn now to discuss the action of Lorentz transformations. (The Lorentz group was reviewed in the problems of Chapter 3.) Observers in a new coordinate frame, produced by a Lorentz transformation from the old inertial frame, use the coordinates

$$x'^\mu = l^\mu{}_\nu x^\nu\,. \tag{7.1.21}$$

They measure the field $\chi'(x')$ that is related to the field $\chi(x)$ measured in the old frame by the linear combination

$$\chi'(x') = L\chi(x)\,. \tag{7.1.22}$$

For an infinitesimal transformation

$$L = 1 + \frac{i}{2}\delta\omega^{\mu\nu} S_{\mu\nu}\,. \tag{7.1.23}$$

The structure of the Lorentz group requires that the generators $S_{\mu\nu}$ obey the commutation relations of the $S(3,1)$ Lie algebra,

$$-i\,[S_{\mu\nu}\,, S_{\rho\sigma}] = g_{\mu\rho}S_{\nu\sigma} - g_{\nu\rho}S_{\mu\sigma} - g_{\mu\sigma}S_{\nu\rho} + g_{\nu\sigma}S_{\mu\rho}\,. \tag{7.1.24}$$

As was shown in the problems in Chapter 3, the nature of the Lorentz group is brought out by writing

$$S_k = \frac{1}{2}\epsilon_{klm}\,S_{lm}\,, \qquad K_l = S_{0l}\,, \tag{7.1.25}$$

so that

$$L = 1 + i\delta\vec{\omega}\cdot\mathbf{S} + i\delta\vec{\nu}\cdot\mathbf{K}\,. \tag{7.1.26}$$

Here $\delta\omega_{lm} = \epsilon_{lmk}\,\delta\omega_k$ parameterizes infinitesimal spatial rotations while $\delta\omega^{0k} = \delta\nu_k$ are the parameters for infinitesimal boosts to moving coordinate frames. To completely exhibit the structure of the Lorentz group one defines

$$S_k^{(\pm)} = \frac{1}{2}[S_k \mp iK_k]\,, \tag{7.1.27}$$

for then the commutation relation (7.1.24) reduces to the set

$$\left[S_k^{(+)}, S_l^{(-)}\right] = 0\,, \tag{7.1.28}$$

$$\left[S_k^{(+)}, S_l^{(+)}\right] = i\epsilon_{klm}\,S_m^{(+)}\,, \tag{7.1.29}$$

and

$$\left[S_k^{(-)}, S_l^{(-)}\right] = i\epsilon_{klm}\,S_m^{(-)}\,. \tag{7.1.30}$$

Except for the factor of i, this set of commutation relations asserts the Lie algebra equivalence SO(3,1)=SO(4)=SU(2) $\otimes$ SU(2).

In this new form, a finite Lorentz transformation appears as

$$L = \exp\{i(\vec{\omega} + i\vec{\nu})\cdot\mathbf{S}^{(+)}\}\exp\{i(\vec{\omega} - i\vec{\nu})\cdot\mathbf{S}^{(-)}\}\,. \tag{7.1.31}$$

This implies that the irreducible representations are given by the fields $\chi^{(j^{(+)},j^{(-)})}(x)$ where $j^{(+)}$ and $j^{(-)}$ label ordinary angular momentum representations. That is

$$(\mathbf{S}^{(\pm)})^2\chi^{(j^{(+)},j^{(-)})}(x) = j^{(\pm)}(j^{(\pm)}+1)\chi^{(j^{(+)},j^{(-)})}(x)\,, \tag{7.1.32}$$

where $j^{(+)}$ and $j^{(-)}$ can take on any integer or half-integer value. Since $S_{\mu\nu}$ behaves as a tensor, we see that under a spatial reflection (parity) transformation $\mathbf{r} \to -\mathbf{r}$, $\mathbf{S} \to \mathbf{S}$ while $\mathbf{K} \to -\mathbf{K}$ and so

$$\text{spatial reflection}: \quad \mathbf{S}^{(\pm)} \to \mathbf{S}^{(\mp)}\,. \tag{7.1.33}$$

This implies that under a spatial reflection $\chi^{(j^{(+)},j^{(-)})} \to \chi^{(j^{(-)},j^{(+)})}$. Only irreducible representations with $j^{(+)} = j^{(-)} = j$ can be parity eigenstates.

Let us give some examples of these representations. Clearly $\chi^{(0,0)}$ is the trivial, spin zero scalar field. The field $\chi^{(\frac{1}{2},\frac{1}{2})}$ has $2\times 2 = 4$ components. The two spatial spin $\frac{1}{2}$ parts that it contains may be combined to form spin 1 and spin 0. This field corresponds to a 4-vector field A^μ; under spatial rotations A^0 transforms as spin 0 and A^k as spin 1. Under spatial rotations, the field $\chi^{(1,0)}$ also transforms as spin 1, but it is not a parity eigenstate. In terms of tensor notation, it corresponds to an antisymmetrical tensor $F_{\mu\nu}^{(+)} = -F_{\nu\mu}^{(+)}$ which is self-dual, $F_{\mu\nu}^{(+)} = \frac{1}{2}\epsilon_{\mu\nu\lambda k}F^{(+)\lambda k}$. In four dimensions, an anti-symmetrical tensor has $4\cdot 3/2 = 6$ independent components. The self-duality restriction halves this number to 3, which is just the number of independent components in $\chi^{(1,0)}$. An antisymmetrical 4-tensor may be expressed in terms of two 3-vectors, $F_{kl} = \epsilon_{klm}B_m$ and $F^{0k} = E_k$. The duality transformation sends $\mathbf{B} \to \mathbf{E}$ and $\mathbf{E} \to \mathbf{B}$. Thus a self-dual tensor has $\mathbf{B} = \mathbf{E}$. The parity operation takes $\mathbf{B} \to \mathbf{B}$ and $\mathbf{E} \to -\mathbf{E}$ and hence converts a self-dual tensor $F_{\mu\nu}^{(+)}$ into an anti-self-dual tensor $F_{\mu\nu}^{(-)} = -\frac{1}{2}\epsilon_{\mu\nu\lambda k}F^{(-)\lambda k}$. This anti-self-dual tensor $F_{\mu\nu}^{(-)}$ is just the $\chi^{(0,1)}$ representation which is the parity transform of $\chi^{(1,0)}$. Finally, we note that there are two spin $\frac{1}{2}$ representations $\chi^{(1/2,0)}$ and $\chi^{(0,1/2)}$. These are the Weyl spinors. The Dirac field, which is a parity eigenstate, corresponds to the linear combination $(1/2,0)\oplus(0,1/2)$. We turn now to discuss these fields in detail.

7.2 Weyl, Majorana, Dirac spinors

1 Weyl spinors

So as to simplify the notation, the Weyl spinor fields are now written as

$$\chi^{(\frac{1}{2},0)}(x) = \chi_+(x)\,, \qquad \chi^{(0,\frac{1}{2})}(x) = \chi_-(x)\,. \tag{7.2.1}$$

The spinors have two components, $\chi_{\pm\alpha}$, $\alpha = 1,2$. The spinor χ_+ is called left-handed, χ_- right-handed. The reason for these names will be explained later. The action of a Lorentz transformation appears as

$$\chi'_\pm(x') = L_\pm\chi_\pm(x)\,, \tag{7.2.2}$$

in which

$$L_\pm = \exp\left\{\frac{i}{2}(\vec{\omega}\pm i\vec{\nu})\cdot\vec{\sigma}\right\}\,. \tag{7.2.3}$$

Note that $L_\pm^\dagger \neq L_\pm^{-1}$, the representations are not unitary. This is a consequence of the fact that the Lorentz group is not compact. (Only compact groups have finite-dimensional, unitary representations.) We have, in fact,

$$L_\pm^\dagger = L_\mp^{-1}\,. \tag{7.2.4}$$

On the other hand, just as in the case of the rotation group,

$$L_\pm^T = \sigma_2 L_\pm^{-1}\sigma_2\,, \tag{7.2.5}$$

or

$$L_\pm^T\sigma_2 L_\pm = \sigma_2\,. \tag{7.2.6}$$

In component form this gives

$$\epsilon_{\alpha'\beta'}L_{\pm\alpha'\alpha}L_{\pm\beta'\beta} = \epsilon_{\alpha\beta}\,, \tag{7.2.7}$$

which implies that the $L_\pm$ are proper transformations,

$$\det L_\pm = +1\,, \tag{7.2.8}$$

and moreover that $\epsilon_{\alpha\beta}$ also transforms as a scalar under Lorentz transformations. Combining the equations for the transpose and adjoint gives

$$L_\pm^* = (L_\mp^T)^{-1} = \sigma_2 L_\mp \sigma_2\,. \tag{7.2.9}$$

Hence the $L_\pm$ are also not real representations. In view of Eq. (7.2.9), the Lorentz transform of $\sigma_2\chi_\pm^*$ is given by

$$\begin{aligned}(\sigma_2\chi_\pm^*)' &= \sigma_2 L_\pm^* \chi_\pm^* = \sigma_2\sigma_2 L_\mp \sigma_2 \chi_\pm^* \\ &= L_\mp(\sigma_2\chi_\pm^*)\,.\end{aligned} \tag{7.2.10}$$

This is to say that $\sigma_2\chi_+^*$ transforms as* χ_-, $\sigma_2\chi_-^*$ as χ_+.

Lorentz scalars are needed for the construction of Lagrange functions which describe the dynamics of the theory. The quantities

$$S_\pm = -i\epsilon_{\alpha\beta}\chi_{\pm\alpha}\chi_{\pm\beta} = \chi_\pm\sigma_2\chi_\pm \tag{7.2.11}$$

transform as Lorentz scalars since, in view of Eq. (7.2.6),

$$S_\pm{}' = \chi_\pm L_\pm^T \sigma_2 L_\pm \chi_\pm = S_\pm\,. \tag{7.2.12}$$

If the $\chi_{\pm\alpha}$ were ordinary numbers, $\chi_{\pm\alpha}\chi_{\pm\beta} = \chi_{\pm\beta}\chi_{\pm\alpha}$ would be symmetrical in α, β. In this case $S_\pm$ would vanish identically since it would involve the contractions of the antisymmetrical $\epsilon_{\alpha\beta}$ with a symmetrical spinor combination. This, in fact, does not happen because the spinors represent Fermions and are thus anticommuting, Grassmann variables, not ordinary numbers. The $S_\pm$ are not the only scalars which can be constructed from bilinear spinor combinations since, as we have just remarked, $\sigma_2\chi_\mp^*$ transforms as $\chi_\pm$. The most general form for the "mass terms" in a Lagrange function containing both left- and right-handed Weyl spinors – the complete set of bilinear terms containing no derivatives - is given by

$$\begin{aligned}\mathcal{L}_{\text{mass}} =& m_+\chi_+\sigma_2\chi_+ + m_+^*\chi_+^*\sigma_2\chi_+^* \\ &+ m_-\chi_-\sigma_2\chi_- + m_-^*\chi_-^*\sigma_2\chi_-^* \\ &+ m_{+-}\chi_+^*\chi_- + m_{+-}^*\chi_-^*\chi_+\,.\end{aligned} \tag{7.2.13}$$

It involves three complex mass parameters m_+, m_-, and m_{+-}. This is the structure that appears in a functional integral over the Grassmann variables. Complex conjugation applied to an expression containing Grassmann variables reverses their order (just as does Hermitian conjugation

* The relation of the right-hand spinor to the complex conjugate of the left-hand spinor is the basis of the "dotted index" notation of spinor calculus. This is reviewed briefly in Problem 1.

of quantum operators). Since σ_2 is an Hermitian matrix, it follows that the mass parameters and their complex conjugates must enter as given in Eq. (7.2.13) so as to make it real, $\mathcal{L}_{\text{mass}}^* = \mathcal{L}_{\text{mass}}$.

The $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations can be combined to form quantities that transform as a four vector. This can be done in two ways corresponding to $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ or $(0, \frac{1}{2}) \otimes (\frac{1}{2}, 0)$, and the construction is made explicit by defining

$$\sigma_\pm^\mu = (1, \mp\sigma^k)\,. \tag{7.2.14}$$

Just as in the previous spatial rotation case, by considering infinitesimal transformations (together with the group multiplication property so as to go to a finite transformation), it is not difficult to prove that

$$L_\pm^\dagger \sigma_\pm^\mu L_\pm = l^\mu{}_\nu \sigma_\pm^\nu\,, \tag{7.2.15}$$

where $l^\mu{}_\nu$ is the usual vector representation of the Lorentz group, $x^\mu \to x'^\mu = l^\mu{}_\nu x^\nu$. The transformation law (7.2.15) for the $\sigma_\pm^\mu$ is clarified if Eqs. (7.2.4) and (7.2.5) are used to write it as

$$(\sigma_2\sigma_\pm^\mu)_{\alpha'\beta'} L_{\mp\alpha'\alpha} L_{\pm\beta'\beta} = l^\mu{}_\nu (\sigma_2\sigma_\pm^\nu)_{\alpha\beta}\,. \tag{7.2.16}$$

Thus $\sigma_2\sigma_+^\mu$ corresponds to $(0, \frac{1}{2}) \otimes (\frac{1}{2}, 0)$ while $\sigma_2\sigma_-^\mu$ corresponds to $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$. These matrix transformation properties mesh with the field transformation properties to produce four vectors. For example,

$$V_\pm^\mu = (\sigma_2\sigma_\pm^\mu)_{\alpha\beta}(\sigma_2\chi_\pm^*)_\alpha \chi_{\pm\beta} = -\chi_\pm^*\sigma_\pm^\mu\chi_\pm \tag{7.2.17}$$

transform as four vectors,

$$V_\pm'^\mu = l^\mu{}_\nu V_\pm^\nu\,. \tag{7.2.18}$$

Replacing $\chi_\pm$ by $\sigma_2\chi_\mp^*$ in various ways gives many other four-vectors. From any of these four-vectors, a scalar may be formed by taking the gradient of one of the spinor factors and contracting the two vector indices. In this way "kinetic energy" terms of a Lagrange function may be constructed. Noting that

$$\sigma_2\sigma_\pm^{\mu T}\sigma_2 = \sigma_\mp^\mu\,, \tag{7.2.19}$$

it is not difficult to prove that the most general form of the kinetic energy part of the Lagrange function is given by

$$\begin{aligned}\mathcal{L}_{\text{kin}} =& a_1\chi_+^*\sigma_+^\mu i\overleftrightarrow{\partial_\mu}\chi_+ + a_2\chi_-^*\sigma_-^\mu i\overleftrightarrow{\partial_\mu}\chi_- \\ &+ c\chi_+^*\sigma_+^\mu i\overleftrightarrow{\partial_\mu}\sigma_2\chi_-^* + c^*\chi_-\sigma_2\sigma_+^\mu i\overleftrightarrow{\partial_\mu}\chi_+\,,\end{aligned} \tag{7.2.20}$$

where $\overleftrightarrow{\partial_\mu} = \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$ is the difference of the derivative acting to the right and to the left, and a_1, a_2 are real, and c complex constants.

2 Majorana fields

Although the Weyl spinor form displays the Lorentz group representation content explicitly, the cumbersome forms displayed above make it clear that a more compact notation is needed. This is provided by the introduction of a four-component field ψ. As a prelude to the general case — which will turn out to be a rather lengthy prelude — we shall first work out the example of a single Weyl spinor, the theory described by the Lagrange function

$$\mathcal{L} = \frac{1}{2}\chi_+^* \sigma_+^\mu i \overleftrightarrow{\partial_\mu} \chi_+ - \frac{1}{2} m \chi_+ \sigma_2 \chi_+ - \frac{1}{2} m^* \chi_+^* \sigma_2 \chi_+^* . \tag{7.2.21}$$

Here we have chosen the scale of the field so as to have the kinetic energy term appear with a factor of $\frac{1}{2}$. The sign of this term, of course, cannot be altered by such a scaling. The positive sign is dictated by the positive norm of physical states: As will be clear later only after much work, with this sign choice, the functional integral produces a two-point function $\langle \chi_+^\dagger(x)\chi_+(y)\rangle$ of positive-definite form. The kinetic energy term is left invariant under the phase transformation

$$\chi_+ \to e^{i\theta}\chi_+ . \tag{7.2.22}$$

The angle θ may be chosen to make the mass parameter m real and positive, and we shall assume that this has been done,

$$m^* = m \geq 0 . \tag{7.2.23}$$

Note that the sign of the mass is purely a matter of convention — it could instead have been chosen to be negative. [In the massless case, $m = 0$, the theory is invariant under the phase transformation (7.2.22).] The field equations follow from the principle of stationary action

$$\delta \int (d^4x)\mathcal{L} = 0 . \tag{7.2.24}$$

Since on the one hand the χ_+^* are anticommuting variables, and on the other σ_2 is an antisymmetrical matrix,

$$\begin{aligned} \delta(\chi_+^* \sigma_2 \chi_+^*) &= \delta\chi_+^* \sigma_2 \chi_+^* + \chi_+^* \sigma_2 \delta\chi_+^* \\ &= 2\delta\chi_+^* \sigma_2 \chi_+^* . \end{aligned} \tag{7.2.25}$$

Hence the variation of χ_+^* in the action principle yields the field equation

$$\sigma_+^\mu \frac{1}{i}\partial_\mu \chi_+ + m\sigma_2 \chi_+^* = 0 . \tag{7.2.26}$$

The anticommutativity of the fields and Eq. (7.2.19) give

$$\chi_+^* \sigma_+^\mu \frac{i}{2} \overleftrightarrow{\partial_\mu} \chi_+ = \chi_+ \sigma_+^{\mu T} \frac{i}{2} \overleftrightarrow{\partial_\mu} \chi_+^* = \chi_+ \sigma_2 \sigma_-^\mu \sigma_2 \frac{i}{2} \overleftrightarrow{\partial_\mu} \chi_+^* . \tag{7.2.27}$$

Hence the result of varying χ_+ in the action principle gives the field equation

$$\sigma_-^\mu \frac{1}{i}\partial_\mu \sigma_2 \chi_+^* + m\chi_+ = 0 . \tag{7.2.28}$$

These equations are cumbersome in that they are a coupled pair. It is convenient to simplify the notation by introducing a four-component spinor

$$\psi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} . \tag{7.2.29}$$

If the two-spinor χ_+ is taken to be the independent field with χ_- given by the constraint

$$\chi = \sigma_2 \chi_+^* , \tag{7.2.30}$$

then ψ is a *Majorana* field which is equivalent to the χ_+ Weyl spinor. On the other hand, if the two spinor χ_- is taken to be the independent field with χ_+ given by the constraint

$$\chi_+ = -\sigma_2 \chi_-^* , \tag{7.2.31}$$

then ψ is a Majorana field which is equivalent to the χ_- Weyl spinor. Taking the Majorana field to be defined by the constraint (7.2.30), it is easy to see that the pair of coupled Weyl equations (7.2.26) and (7.2.28) is equivalent to the single four-spinor equation

$$\left[\alpha^\mu \frac{1}{i}\partial_\mu + \beta m\right]\psi(x) = 0 , \tag{7.2.32}$$

in which

$$\alpha^\mu = \begin{pmatrix} \sigma_+^\mu & 0 \\ 0 & \sigma_-^\mu \end{pmatrix} , \tag{7.2.33}$$

and

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{7.2.34}$$

If instead the constraint (7.2.31) had been used, then the Majorana field equation (7.2.32) would reproduce the Weyl equations for χ_-. To obtain the general case involving both χ_+ and χ_- including the coupling together of these two Weyl spinors, we need two separate Majorana fields. We will return to this general case after a rather lengthy discussion of the single Majorana field example which represents either the case of a single χ_+ or a single χ_- Weyl field.

The Weyl Lagrange function is equivalent to

$$\mathcal{L} = -\frac{1}{2}\psi^* \left[\alpha^\mu \frac{1}{2i}\overleftrightarrow{\partial_\mu} + \beta m\right]\psi . \tag{7.2.35}$$

The time component of the α-matrix is simply the identity matrix

$$\alpha^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.2.36}$$

while the spatial α-matrices are given by

$$\alpha^k = \begin{pmatrix} -\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}. \tag{7.2.37}$$

From the explicit matrix forms it is clear that α^μ and β are Hermitian matrices, $\alpha^{\mu\dagger} = \alpha^\mu$, $\beta^\dagger = \beta$. These matrices have the algebraic properties

$$\beta^2 = 1\,, \tag{7.2.38}$$

$$\left\{\alpha^k, \beta\right\} = 0\,, \tag{7.2.39}$$

$$\left\{\alpha^k, \alpha^l\right\} = 2\delta_{kl}\,, \tag{7.2.40}$$

and

$$\left[\alpha^k, \alpha^l\right] = 2i\epsilon_{klm}\,\sigma_m\,. \tag{7.2.41}$$

[Here it is implicit that the σ_m on the right multiplies a 2×2 identity matrix — we shall always use such notation.] Since the four-spinor (7.2.29) is composed of two Weyl spinors that transform as shown in Eq. (7.2.2), under a Lorentz transformation

$$\psi(x) \to \psi'(x') = L\psi(x)\,, \tag{7.2.42}$$

with

$$L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}. \tag{7.2.43}$$

Remembering the structure of $L_\pm$ given in Eq. (7.2.3), the parameter decomposition $\omega_k = \epsilon_{klm}\omega_{lm}/2$, $\nu_k = \omega^{0k} = -\omega^{k0}$, and the definition (7.2.37) of the α-matrices, we see that

$$L = \exp\left\{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right\}, \tag{7.2.44}$$

in which

$$\sigma_{kl} = \epsilon_{klm}\sigma_m = \frac{1}{2i}\left[\alpha_k, \alpha_l\right], \tag{7.2.45}$$

and

$$\sigma_{0k} = -\sigma_{k0} = -i\alpha_k\,. \tag{7.2.46}$$

In view of the development which has been made, the matrices $\sigma_{\mu\nu}/2$ must obey the Lorentz group commutator (7.1.24); this is easy to verify directly. In view of the skew nature of the β matrix (7.2.34), the Weyl relations $L_\pm^\dagger = L_\mp^{-1}$ have the Majorana matrix equivalence

$$\beta L^\dagger \beta = L^{-1}\,, \tag{7.2.47}$$

or

$$\beta L^\dagger = L^{-1}\beta\,. \tag{7.2.48}$$

This shows again that the mass term $\psi^*\beta\psi$ in the Lagrange function is Lorentz invariant. Remembering Eq. (7.2.15) and the definition (7.2.33) of α^μ, we see that

$$L^\dagger\alpha^\mu L = l^\mu{}_\nu\alpha^\nu\,, \tag{7.2.49}$$

which again implies that the kinetic energy term in the Lagrange function is Lorentz invariant.

3 Gamma matrices

The transformation law structure of the α^μ and β matrices in the form $L^\dagger\cdots L$ with $L^\dagger$ on the left rather than L^{-1}, is a reflection of the indefinite metric of the Lorentz group and the consequence that this group is not compact. However, the β-matrix in the four-spinor space plays a role that is akin to the the coordinate metric $g_{\mu\nu}$ in that, according to Eq. (7.2.47), it converts $L^\dagger$ into L^{-1}. This is analogous to the invariance of the metric tensor

$$l_\mu{}^\alpha l_\nu{}^\beta g_{\alpha\beta} = g_{\alpha\beta}\,, \tag{7.2.50}$$

which, written in matrix form, reads

$$lgl^T = g\,. \tag{7.2.51}$$

(Since l is real, $l^T = l^\dagger$.) Thus, multiplying Eq. (7.2.49) on the left by β and using Eq. (7.2.48) gives

$$L^{-1}\beta\alpha^\mu L = l^\mu{}_\nu\beta\alpha^\nu\,. \tag{7.2.52}$$

This is to say that the matrices

$$\gamma^\mu = \beta\alpha^\mu \tag{7.2.53}$$

transform as a four-vector under a similarity transformation $L^{-1}\cdots L$. It is a great advantage to have Lorentz transformations represented by similarity transformations because these preserve algebraic structures. Thus, with γ^μ transforming as a vector, $\gamma^\mu\gamma^\nu$ transforms as a second rank tensor, $\gamma^\mu\gamma^\nu\gamma^\lambda$ transforms as a third rank tensor and so forth. For this reason, except for the initial development that has been made, we shall work

mostly with the γ^μ matrices. Since $\alpha^0 = 1$, γ^0 is an Hermitian matrix which squares to the identity matrix, $\gamma^{0\dagger} = \gamma^0$, $\gamma^{0\,2} = 1$. On the other hand, since $\gamma^k = \beta\alpha^k$ is the product of two Hermitian matrices which anticommute, γ^k is skew-Hermitian, $\gamma^{k\dagger} = -\gamma^k$. In view of the definition of γ^μ, the previous algebraic properties (7.2.38), (7.2.39), and (7.2.40) of the α^k and β matrices combine to yield the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\,. \tag{7.2.54}$$

Multiplying Eq. (7.2.32) on the left by β gives

$$\Big[\gamma^\mu \frac{1}{i}\partial_\mu + m\Big]\psi(x) = 0\,. \tag{7.2.55}$$

This is the Dirac equation, but for the Majorana field ψ.

Products of the γ^μ matrices can be formed to produce a complete set of matrices that span the 4×4 spinor space. Since the symmetrical product of two γ matrices (7.2.54) is a multiple of the identity, independent matrices are obtained only with antisymmetrical products of γ matrices. Equations (7.2.45) and (7.2.46) imply that the antisymmetrical product of two γ matrices is the matrix $\sigma_{\mu\nu}$ defined previously,

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\,. \tag{7.2.56}$$

We can now give an example of the great utility of the γ-matrices. Using the basic anticommutator (7.2.54), we very quickly calculate

$$\begin{aligned} i\Big[\gamma_\mu, \frac{1}{2}\sigma_{\alpha\beta}\Big] &= -\frac{1}{2}[\gamma_\mu, \gamma_\alpha\gamma_\beta] \\ &= -\frac{1}{2}\{\gamma_\mu, \gamma_\alpha\}\gamma_\beta + \frac{1}{2}\gamma_\alpha\{\gamma_\mu, \gamma_\beta\} \\ &= g_{\mu\alpha}\gamma_\beta - g_{\mu\beta}\gamma_\alpha\,. \end{aligned} \tag{7.2.57}$$

This, of course, is essentially just the infinitesimal version of the Lorentz transformation (7.2.10). This derivation is clearly much faster than those previously used which entailed working out the space and time components separately. Next, we note that since the completely antisymmetrical symbol $\epsilon_{\alpha\beta\gamma\delta}$ is an invariant four-rank tensor,

$$\gamma_5 = -\frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \tag{7.2.58}$$

is a Lorentz scalar. However, all the γ matrices anticommute, and with $\epsilon^{0123} = -\epsilon_{0123} = +1$ and so this can be written simply as

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{7.2.59}$$

In view of this anticommutativity of the γ matrices and the previously listed properties, it is easy to see that γ_5 is a skew-Hermitian matrix which squares to the negative identity matrix,

$$\gamma_5{}^\dagger = -\gamma_5\,, \quad \gamma_5{}^2 = -1\,. \tag{7.2.60}$$

Moreover, the anticommutativity of the four γ^μ implies that γ_5 also anticommutes with each of them,

$$\{\gamma_5, \gamma_\mu\} = 0\,. \tag{7.2.61}$$

Accordingly, the simple product $\gamma_\mu\gamma_5$ is equivalent to an antisymmetrical commutator. We have now constructed 16 linearly independent matrices

$$1\,,\quad \gamma_5\,,\quad \gamma_\mu\,,\quad \gamma_\mu\gamma_5\,,\quad \sigma_{\mu\nu}$$

$$1 \quad +1 \quad +4 \quad +4 \quad +6 \quad = 16$$

Since there are only $4 \times 4 = 16$ independent matrices in the four-spinor space, these are a complete set of matrices. Notice, incidentally, that since there is now a complete set of matrices, $\sigma^{\mu\nu}\gamma_5$ must be a linear combination of these. Anticipating that γ_5 is in fact a *pseudo*scalar, $\sigma^{\mu\nu}\gamma_5$ must transform as a pseudotensor. There is only one such pseudotensor available and so it should be clear that

$$\sigma^{\mu\nu}\gamma_5 = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\,. \tag{7.2.62}$$

Explicit calculation verifies this and provides the numerical factor of $\frac{1}{2}$ shown.

We have given an explicit representation of the γ matrices in what is called the Weyl representation. This representation is characterized by writing the four-spinor in terms of Weyl spinors

$$\psi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}\,. \tag{7.2.63}$$

In this representation, the β matrix has the form

$$\beta = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho_1\,. \tag{7.2.64}$$

The matrix ρ_1 is identical in form to the Pauli matrix σ_1 except that its entries implicitly contain 2×2 identity matrices in the "up, down" space while the σ's act within these spaces. We shall later make use of the remaining independent matrices that act in the "up, down" space,

$$\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{7.2.65}$$

The β matrix interchanges the Weyl spinors,

$$\beta\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}\,. \tag{7.2.66}$$

Since these spinors are also interchanged by the parity operation, the β matrix represents the parity transformation. Moreover, using the explicit forms above,

$$\gamma_5 = -\alpha^1\alpha^2\alpha^3 = \sigma_1\sigma_2\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\rho_3 \,. \tag{7.2.67}$$

Therefore

$$i\gamma_5 \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} -\chi_+ \\ +\chi_- \end{pmatrix} . \tag{7.2.68}$$

This states that γ_5 measures the "chirality" of the field — it multiplies the left-handed component by -1, the right-handed component by $+1$, and thus the phase transformation (7.2.22) of the Weyl spinors is represented by the four-spinor as

$$\psi(x) \to \psi'(x) = \exp\{-\theta\gamma_5\}\,\psi(x) \,. \tag{7.2.69}$$

Since γ_5 commutes with α^μ, this chirality transformation is a symmetry of the Majorana field equation (7.2.32) when the mass vanishes, $m = 0$. In this massless case, the Majorana field may be replaced by its projection into chirality eigenstates,

$$\psi_+ = \frac{1}{2}(1 - i\gamma_5)\,\psi = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} , \tag{7.2.70}$$

or

$$\psi_- = \frac{1}{2}(1 + i\gamma_5)\,\psi = \begin{pmatrix} 0 \\ \chi_- \end{pmatrix} . \tag{7.2.71}$$

Of course, if the Majorana field represents χ_+, then $\chi_- = \sigma_2\chi_+^*$, and it is natural to use the ψ_+ projection, with the converse holding for the Majorana representation of χ_-. And, of course, these projections just give back the simple massless Weyl formulation since there is no mixing of χ_+ and χ_+^* in this case.

4 *Majorana gamma matrices*

The gamma matrices have been presented in terms of an explicit construction that arose from the examination of the Lorentz group representation content of spin one half fields. This explicit representation, which has its basis in the "up χ_+ components, down χ_- components", is accordingly called the *Weyl representation.* But a more natural representation is called for in the Majorana fields. Since these fields have either the constraint (7.2.30) or the constraint (7.2.31) which relates the lower two components to the complex conjugate of the upper two components, a representation must exist which makes the Majorana field real, with the

previous two independent complex variables replaced by four real variables. To obtain this real representation, we note that for the χ_+ case

$$\psi = \begin{pmatrix} \chi_+ \\ \sigma_2\chi_+^* \end{pmatrix} , \tag{7.2.72}$$

and so, since σ_2 is imaginary,

$$\psi^* = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \psi = i\rho_2\sigma_2\psi \,. \tag{7.2.73}$$

The same formula holds for the χ_- case. A transformation to real fields $\psi' = \psi'^*$ can be made by writing

$$\psi = U\psi' \,. \tag{7.2.74}$$

Inserting this decomposition into Eq. (7.2.73), it follows that ψ' will indeed be real if U obeys

$$U^* = i\rho_2\sigma_2 U \,. \tag{7.2.75}$$

To solve this equation, we note that $(\rho_2\sigma_2)^2 = 1$, and so

$$e^{i\rho_2\sigma_2\theta} = \cos\theta + i\rho_2\sigma_2 \sin\theta \,, \tag{7.2.76}$$

and, in particular,

$$e^{i\rho_2\sigma_2\pi/2} = i\rho_2\sigma_2 \,. \tag{7.2.77}$$

Hence, the condition (7.2.75) maybe written as

$$U^* = e^{i\rho_2\sigma_2\pi/2}U \,. \tag{7.2.78}$$

Since $(\rho_2\sigma_2)$ is real, a solution is given by

$$U = e^{-i\rho_2\sigma_2\pi/4} \,, \tag{7.2.79}$$

which defines a unitary matrix,

$$U^\dagger = U^{-1} \,. \tag{7.2.80}$$

With this field redefinition, the Majorana Lagrange function (7.2.35) becomes

$$\mathcal{L} = -\frac{1}{2}\psi' \left[\alpha'^\mu \frac{1}{2i}\overleftrightarrow{\partial}_\mu + \beta' m\right] \psi' \,, \tag{7.2.81}$$

in which

$$\alpha'^\mu = U^{-1}\alpha^\mu U \,, \quad \beta' = U^{-1}\beta U \,. \tag{7.2.82}$$

The change to the new basis is given by a unitary, similarity transformation which preserves the Hermitian character and algebraic structure of

the matrices. The explicit forms of the transformation is readily obtained. Since $\beta = \rho_1$ and ρ_2 anticommute,

$$\begin{aligned}\beta' &= e^{i\rho_2\sigma_2\pi/4}\beta e^{-i\rho_2\sigma_2\pi/4} = \rho_1 e^{-i\rho_2\sigma_2\pi/2} \\ &= -i\rho_1\rho_2\sigma_2 = \rho_3\sigma_2 \,. \end{aligned} \tag{7.2.83}$$

Since the matrix β is Hermitian, so is β'. The explicit construction (7.2.83) shows that β' is imaginary, $\beta^* = -\beta$. Hence β' is an antisymmetrical matrix, $\beta'^T = -\beta'$. Since $\gamma_5 = i\rho_3$ also anticommutes with $\rho_2\sigma_2$, a similar calculation gives

$$\gamma_5' = \rho_3\rho_2\sigma_2 = -i\rho_1\sigma_2 \,. \tag{7.2.84}$$

This defines a real, skew-Hermitian, antisymmetrical matrix, $\gamma_5'^* = \gamma_5'$, $\gamma_5'^\dagger = -\gamma_5'$, $\gamma_5'^T = -\gamma_5'$.

To construct the transformed α-matrices, we first note that, in view of Eqs. (7.2.37) and (7.2.67),

$$\alpha_k = i\sigma_k\gamma_5 \,. \tag{7.2.85}$$

Since we have already constructed the transformed γ_5 matrix, our task is reduced to transform the σ_k matrices. Again using the anticommutativity of different σ_k matrices and proceeding as before, we calculate

$$\begin{aligned}\begin{pmatrix}\sigma_1'\\ \sigma_2'\\ \sigma_3'\end{pmatrix} &= e^{i\rho_2\sigma_2\pi/4}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} e^{-i\rho_2\sigma_2\pi/4} \\ &= \begin{pmatrix}-i\rho_2\sigma_1\sigma_2\\ \sigma_2\\ -i\rho_2\sigma_3\sigma_2\end{pmatrix} = \begin{pmatrix}\rho_2\sigma_3\\ \sigma_2\\ -\rho_2\sigma_1\end{pmatrix} . \end{aligned} \tag{7.2.86}$$

These matrices are all antisymmetric and imaginary

$$\sigma_k'^T = -\sigma_k' \,, \quad \sigma_k'^* = -\sigma_k' \,. \tag{7.2.87}$$

Thus spatial rotations are represented in the real Majorana field by real, orthogonal transformations. Using now

$$\alpha_k' = i\sigma_k'\gamma_5' \,, \tag{7.2.88}$$

we see that the new α-matrices are real and symmetrical,

$$\alpha_k'^* = \alpha_k' \,, \quad \alpha_k'^T = \alpha_k' \,. \tag{7.2.89}$$

In view of Eq. (7.2.46), we have $\sigma_{0k}' = -i\alpha_k'$, which are imaginary but symmetrical matrices. The different character of the spatial rotation matrices σ_k' and boost matrices σ_{0k}' just corresponds the the fact that the spatial rotation subgroup is compact and so has a unitary — or orthogonal — representation, while the boost factor is not compact. Finally, we

note that since $\gamma_k = \beta\alpha_k$, the new spatial γ matrices are imaginary and symmetrical,

$$\gamma_k'^* = -\gamma_k' \,, \quad \gamma_k'^T = +\gamma_k' \,. \tag{7.2.90}$$

Note that all components of γ^μ are imaginary. This must be the case to keep the Dirac equation (7.2.55) real.

Let us summarize the properties of the matrices in the Majorana representation. Omitting the primes, and using the notation R = real, I = imaginary, S = symmetrical, A = antisymmetrical, gives the following table:

Majorana Matrices

matrix	reality	symmetry
1	R	S
$\gamma^0 = \beta$	I	A
γ_5	R	A
σ_k	I	A
σ_{0k}	I	S
α_k	R	S
γ_k	I	S

Since the Majorana field ψ is a Grassmann, anticommuting variable, a bilinear form of the structure $\psi M\psi$ is non-vanishing only if M is antisymmetrical. Moreover, since the complex conjugation operation interchanges the two Grassmann fields as well as taking the complex conjugate of the matrix M, requiring the structure $\psi M\psi$ to be real requires that the matrix M be Hermitian or, in this case, imaginary. It is worthwhile remarking how this fits in with the form of the Majorana Lagrange function

$$\mathcal{L} = -\frac{1}{2}\psi\left[\alpha^\mu\frac{1}{2i}\overleftrightarrow{\partial_\mu} + \beta m\right]\psi \,. \tag{7.2.91}$$

Here α^μ is a symmetrical matrix, but the derivative operator $\overleftrightarrow{\partial_\mu}$ has an antisymmetrical structure. The α-matrices are Hermitian as is the derivative operator $\frac{1}{i}\overleftrightarrow{\partial_\mu}$. Thus the kinetic term in the Lagrange function is real. The β-matrix is antisymmetrical and Hermitian. Thus the mass term is also real.

5 Dirac representation

As we shall see in the course of our work, almost all calculations can be carried out in ways that make no reference to a specific representation of the γ-matrices. There is, however, one more representation of these matrices that is sometimes useful. This is the Dirac representation. It is useful when treating the non-relativistic limit of the theory. To see how this goes, we write the Dirac equation in the form

$$i\partial_0\psi = \left[\vec{\alpha}\cdot\frac{1}{i}\nabla + \beta m\right]\psi\,. \tag{7.2.92}$$

In the non-relativistic limit, the field ψ has a slow spatial variation and $\mathbf{p}\psi = \frac{1}{i}\nabla\psi \approx 0$, and so Eq. (7.2.92) requires

$$i\partial_0\psi \approx \beta m\psi\,. \tag{7.2.93}$$

What is really meant here is that the field ψ appearing in a matrix element describing a particle at rest has this behavior. The non-relativistic limit is made clear by transforming to a representation in which β is a diagonal matrix. Since $\beta^2 = 1$ and β has a vanishing trace, in this new representation we must have

$$\beta'' = \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2.94}$$

With the usual conventions that will be made clear shortly, for a particle at rest $\psi \simeq e^{-imt}$, and so , according to Eq. (7.2.93), only the two "upper" components of ψ are non-vanishing, giving $\beta''\psi = +\psi$. The real, unitary (and thus orthogonal) transformation

$$V = e^{-i\rho_2\pi/4} \tag{7.2.95}$$

takes the original Weyl representation (which is denoted without a prime) into the Dirac representation (which we denote with double primes). This follows from the remark that in the original Weyl representation $\beta = \rho_1$ [Eq. (7.2.34)], and so

$$V^{-1}\beta V = \rho_1 e^{-i\rho_2\pi/2} = -i\rho_1\rho_2 = \rho_3\,. \tag{7.2.96}$$

Remembering that $\alpha_k = -\rho_3\sigma_k$ [Eq. (7.2.37)], we compute

$$\begin{aligned}\alpha_k'' &= V^{-1}\alpha_k V = -\rho_3\sigma_k e^{-i\rho_2\pi/2} = i\rho_3\rho_2\sigma_k \\ &= \rho_1\sigma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}.\end{aligned} \tag{7.2.97}$$

We see that $\alpha\cdot\mathbf{p}$ only couples the upper and lower components of ψ. For slowly varying fields, $\mathbf{p}$ is effectively a small quantity. Thus starting with a field that has only non-vanishing upper components, $\alpha\cdot\mathbf{p}$ produces small lower components, which in turn only give a small correction to the upper components when acted upon by $\alpha\cdot\mathbf{p}$. Thus in the non-relativistic limit the upper components of ψ are large while the lower components are small. This has lead to a terminology that is sometimes used in connection with the Dirac representation in which the upper components are called the "large" components, the lower components called "small". To complete the list of the significant matrices, we recall that $\gamma_5 = i\rho_3$ [Eq. (7.2.67)], and so in the Dirac representation

$$\gamma_5'' = V^{-1}\gamma_5 V = \rho_3\rho_2 = -i\rho_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.2.98}$$

6 Dirac field

The Dirac field represents Weyl fields of both handedness χ_+ and χ_- together with their complex conjugates. It is equivalent to a pair of Majorana fields ψ_1 and ψ_2, with ψ_1 the Majorana field representing χ_+ (together with its complex conjugate) while ψ_2 corresponds to χ_-. We shall first work in the Majorana representation so that these fields are real. We assemble the two Majorana fields into a two-component column

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{7.2.99}$$

Although the same symbol ψ is used here, it must be borne in mind that ψ now is an $2\times 4 = 8$ component object. Just as the four-component ψ was constructed in terms of the direct product of two component spinors — with the gamma matrices written as products of σ and ρ matrices — there is now a further doubling of the spinor space. We shall denote the set of matrices that act in the new "$1-2$" space which have the same form as the Pauli σ-matrices by τ_k. The most general form of the free-field Lagrange function for the 8-component field is given by

$$\mathcal{L} = -\frac{1}{2}\psi\left[\alpha^\mu\frac{1}{2i}\overleftrightarrow{\partial_\mu}\,\mathcal{K} + \beta\mathcal{M}\right]\psi\,. \tag{7.2.100}$$

Here $\mathcal{K}$ is a 2×2 matrix in the new space which may also contain the γ_5 matrix. It may be expressed as

$$\mathcal{K} = A + \gamma_5 B\,, \tag{7.2.101}$$

where A and B act in the $1-2$ space. The γ_5-matrix may appear since it is a scalar under Lorentz transformations. We have mentioned that γ_5 transforms as a pseudoscalar under spatial reflection, and so one might be tempted to omit it on the grounds of parity invariance. As should become clear, the precise definition of parity depends upon a proper choice of basis, and there is no need to make an *a priori* assumption about the structure of the free Lagrange function. Just as in the discussion of the

simple Majorana Lagrange function (7.2.91), A must be a symmetrical, Hermitian matrix and therefore real. This matrix is described by three real parameters. Since γ_5 is an antisymmetrical matrix which commutes with α^μ, the matrix product $\alpha^\mu\gamma_5$ is antisymmetrical. Hence the 2×2 matrix B must be antisymmetrical. Since γ_5 is anti-Hermitian while α^μ is Hermitian, $\alpha^\mu\gamma_5$ is anti-Hermitian. Hence B must be an anti-Hermitian and therefore a real, antisymmetrical matrix. This matrix is described by one real parameter. Thus the kinetic term in the Lagrange function contains $3 + 1 = 4$ real parameters. This, of course, is just the same number of real parameters that entered into the general kinetic term of the Weyl Lagrange function given in Eq. (7.2.20). The matrix $\mathcal{M}$ may also be expressed in terms of 2×2 matrices in the "$1 - 2$" space,

$$\mathcal{M} = M + \gamma_5 N\,. \tag{7.2.102}$$

Since β is an Hermitian, antisymmetrical matrix, M must be real, symmetrical and thus an Hermitian matrix. It contains three real parameters. The antisymmetrical matrices β and γ_5 anticommute, and so the product $\beta\gamma_5$ is an antisymmetrical matrix. Hence N must be a symmetrical matrix. The product $\beta\gamma_5$ is also Hermitian since β is Hermitian and γ_5 is anti-Hermitian and they anticommute. Hence N must also be a real, symmetrical and thus Hermitian matrix containing three real parameters. The mass term of the Lagrange function contains $3 + 3 = 6$ real parameters, the same number of real parameters as in the Weyl form of the mass term given in Eq. (7.2.13).

As we will now show, almost all of the arbitrary parameters in the Lagrange function can be removed by a redefinition of the Majorana field ψ, and the free-field Lagrange function can be put into a standard form. If there are additional interaction terms, then the the field redefinition will put these parameters into it, but the free field part of the Lagrange function will be in the standard form. Although we shall work with only a pair of Majorana fields, it should be clear that the general features of our development apply to the general situation of an arbitrary multiplet of Majorana fields. Let us first consider the kinetic term. In view of the properties listed above, it involves a real symmetrical matrix $\mathcal{K}$. Moreover, as we indicated above in connection with the Weyl Lagrange function (7.2.21), the kinetic term must have a positive-definite character so as to maintain a positive norm in the vector space of physical states. Without proving this in detail, we simply state that $\mathcal{K}$ must be a positive, symmetrical matrix which can therefore be written as the square of another such matrix,

$$\mathcal{K} = L^2\,, \quad L^T = L\,. \tag{7.2.103}$$

Accordingly, we may define a new real, Majorana field

$$\psi = L^{-1}\psi'\,, \tag{7.2.104}$$

which puts the Lagrange function in the form

$$\mathcal{L} = -\frac{1}{2}\psi'\left[\alpha^\mu \frac{1}{2i}\overleftrightarrow{\partial_\mu} + \beta\mathcal{M}'\right]\psi'\,, \tag{7.2.105}$$

in which

$$\beta\mathcal{M}' = L^{-1}\beta\mathcal{M}L^{-1} = \beta\left\{M' + \gamma_5 N'\right\}\,. \tag{7.2.106}$$

Here M' and N' remain real symmetrical matrices involving in total 6 real parameters. After having made this transformation, there remains the option of making real, orthogonal transformations R which are generated by the 4 imaginary, antisymmetrical matrices that act in the "$1-2$" space supplemented by the action of γ_5, namely the matrices $i\gamma_5\tau_1$, τ_2, $i\gamma_5\tau_3$, and $i\gamma_5$. All of these matrices commute with α^μ and they produce a real representation of the group $U(2) = SU(2) \otimes U(1)$, with the $SU(2)$ part a chiral, isospin group. The 4 parameters of the complete group may be chosen to remove 4 of the 6 parameters in $\mathcal{M}'$ with the whole matrix N transformed to vanish while the matrix M is put in diagonal form. This may be done as follows.

Dropping the primes of the previous transformation, we first note that

$$S = (\beta\mathcal{M})\,(\beta\mathcal{M})^\dagger = M^2 + N^2 + \gamma_5[M, N] \tag{7.2.107}$$

is a real, symmetrical matrix. Hence it may be placed in diagonal form by an orthogonal transformation,

$$R^T S R = D\,. \tag{7.2.108}$$

Here D is a diagonal matrix in the "$1 - 2$" space multiplying the unit matrix in the 4×4 spinor space in which γ_5 acts. The matrix γ_5 does not appear in D since γ_5 is antisymmetrical and not in diagonal form. Hence if we define

$$\beta\mathcal{M}' = R^T\beta\mathcal{M}R\,, \tag{7.2.109}$$

then

$$\mathcal{M}'\mathcal{M}'^\dagger = D\,. \tag{7.2.110}$$

Since the matrix $\beta\mathcal{M}'$ is related by an orthogonal transformation to the Hermitian, antisymmetric matrix $\beta\mathcal{M}$, it is also an Hermitian, antisymmetrical matrix with the structure

$$\mathcal{M}' = M' + \gamma_5 N'\,, \tag{7.2.111}$$

in which M' and N' are real, symmetrical matrices. Thus Eq. (7.2.110) tells us that

$$M'^2 + N'^2 - \gamma_5[M', N'] = D\,. \tag{7.2.112}$$

Since D does not contain the γ_5 matrix, we conclude that M' and N' commute. Hence they may be simultaneously diagonalized by a simple rotation generated by τ_2 which commutes with β. This additional rotation can be included in the original transformation, and so we have proven that a real, orthogonal matrix exists which produces

$$\beta\mathcal{M}' = R^T\beta\mathcal{M}R = \beta\begin{pmatrix} m_1' + \gamma_5 n_1' & 0 \\ 0 & m_2' + \gamma_5 n_2' \end{pmatrix}. \tag{7.2.113}$$

Finally, we note that

$$\gamma_{5\pm} = \frac{1}{2}(1 \pm \tau_3)\,\gamma_5 \tag{7.2.114}$$

are real, antisymmetrical matrices that act on the upper (+) or lower (−) parts of the "1−2" space. A further orthogonal transformation with these generators gives

$$\exp\{-\theta_+\gamma_{5+}/2 - \theta_-\gamma_{5-}/2\}\,\beta\mathcal{M}'\exp\{\theta_+\gamma_{5+}/2 + \theta_-\gamma_{5-}/2\}$$

$$= \beta\begin{pmatrix} (m_1' + \gamma_5 n_1')\exp\{\theta_+\gamma_5\} & 0 \\ 0 & (m_2' + \gamma_5 n_2')\exp\{\theta_-\gamma_5\} \end{pmatrix}. \tag{7.2.115}$$

The two parameters $\theta_\pm$ may be chosen to entirely remove the γ_5 matrices. We have proven that a real, orthogonal transformation exists which puts the mass term in a diagonal form with no γ_5 matrices. This is to say that there is no loss of generality in starting with a mass matrix of diagonal form with the Lagrange function appearing as

$$\mathcal{L} = -\frac{1}{2}\psi\left[\alpha^\mu\frac{1}{2i}\overleftrightarrow{\partial}_\mu + \beta\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\right]\psi\,. \tag{7.2.116}$$

This is not an unexpected result – the theory describes two free particles and the only parameters needed to describe these particles are their masses.

This is as far as we can go in placing the general Majorana Lagrange function in the standard form. To make a further simplification we must assume that the theory has a charge symmetry or, equivalently, assume that the mass matrix is is a multiple of the identity,

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = m\,. \tag{7.2.117}$$

In this case

$$\mathcal{L} = -\frac{1}{2}\psi\left[\alpha^\mu\frac{1}{2i}\overleftrightarrow{\partial}_\mu + \beta m\right]\psi\,, \tag{7.2.118}$$

and we have the charge rotation invariance

$$\psi \to e^{i\theta\tau_2}\psi\,. \tag{7.2.119}$$

The charge may be electric charge as in the case of the electron or baryonic charge when ψ is used for an effective field theory of a baryon such as the neutron. In any case, we now have, as we shall shortly see in detail, a pair of particles with equal masses but with opposite values of the charge. These are commonly referred to as particle and antiparticle, although which is which is a matter of convention. The Dirac field is the complex field

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \tag{7.2.120}$$

which diagonalizes the charge rotation,

$$\psi \to e^{-i\theta}\psi\,. \tag{7.2.121}$$

The previous 8-spinor Lagrange function is simply expressed in terms of the 4-spinor Dirac field

$$\mathcal{L} = -\psi^*\left[\alpha^\mu\frac{1}{2i}\overleftrightarrow{\partial}_\mu + \beta m\right]\psi\,, \tag{7.2.122}$$

Finally, to replace the α^μ and β matrices by the algebraically simpler γ-matrices, one defines

$$\bar{\psi} = \psi^*\beta = \psi^*\gamma^0 \tag{7.2.123}$$

so as to put the Lagrange function in the form

$$\mathcal{L} = -\bar{\psi}\left[\gamma^\mu\frac{1}{2i}\overleftrightarrow{\partial}_\mu + m\right]\psi\,. \tag{7.2.124}$$

7.3 Spin $\frac{1}{2}$ states

Since the Lagrange functions for the spin $\frac{1}{2}$ fields are of a canonical, first-order form, all of the Green's functions of the free-field theory (which describe the entire theory) are produced by functional integrals of the standard (Grassmann) structure. For the case of a single Majorana field, the generating functional of the Green's functions is given by

$$\begin{aligned} Z[\eta] &= \langle 0|T\exp\left\{i\int\psi\beta\eta\right\}|0\rangle \\ &= \int[d\psi]\exp\left\{i\int[\mathcal{L} + \psi\beta\eta]\right\}. \end{aligned} \tag{7.3.1}$$

A partial integration which does not affect the action may be made so as to put the Lagrange function in the form

$$\mathcal{L} = -\frac{1}{2}\psi\beta\left[\gamma^\mu \frac{1}{i}\partial_\mu + m\right]\psi\,. \tag{7.3.2}$$

Incidentally, note that

$$\psi\beta\eta = -\eta\beta^T\psi = +\eta\beta\psi\,. \tag{7.3.3}$$

Two independent sources η, $\bar{\eta}$ are needed to generate the matrix elements of the the Dirac fields $\bar{\psi}$, ψ, and for the Dirac field the functional representation is given by

$$\begin{aligned} Z[\eta,\bar{\eta}] &= \langle 0|T\exp\left\{i\int[\bar{\psi}\eta + \bar{\eta}\psi]\right\}|0\rangle \\ &= \int[d\psi][d\bar{\psi}]\exp\left\{i\int[\mathcal{L} + \bar{\psi}\eta + \bar{\eta}\psi]\right\}, \end{aligned} \tag{7.3.4}$$

in which

$$\mathcal{L} = -\bar{\psi}\left[\gamma^\mu \frac{1}{i}\partial_\mu + m\right]\psi\,. \tag{7.3.5}$$

In either case we need the Green's function defined by

$$\left[\gamma\frac{1}{i}\partial + m\right]G(x-x') = \delta(x-x')\,. \tag{7.3.6}$$

Taking the Fourier transform gives

$$G(x-x') = \int \frac{(d^4p)}{(2\pi)^4}\frac{e^{ip(x-x')}}{\gamma p + m - i\epsilon}\,. \tag{7.3.7}$$

Here the $i\epsilon$ prescription which gives the mass a small negative imaginary part ensures the convergence of the functional integral and produces the appropriate time-ordered products just as in the scalar case. Using this Green's function, the functional integrals are trivially calculated by the usual method of completing the square. For the Majorana case

$$Z[\eta] = \exp\left\{\frac{i}{2}\int\int \eta\beta G\eta\right\}, \tag{7.3.8}$$

while for the Dirac field

$$Z[\eta,\bar{\eta}] = \exp\left\{i\int\int \bar{\eta}G\eta\right\}. \tag{7.3.9}$$

Taking functional derivatives with respect to the sources and then taking the sources to vanish produces the vacuum expectation values of the time-ordered products of field operators in the usual way. In the Majorana case we obtain

$$\langle iT\psi(x)\psi(x')\beta\rangle = G(x-x')\,, \tag{7.3.10}$$

while for the Dirac field

$$\langle iT\psi(x)\bar{\psi}(x')\rangle = G(x-x')\,. \tag{7.3.11}$$

Clearly, both the Majorana and Dirac fields can be treated together. It should be borne in mind that the sources are Grassmann variables which anticommute, and so the time-ordered product of Fermi fields may have an extra minus sign if the position of the field differs from the standard order as indicated in*

$$\langle TA(t)B(t')\rangle = \begin{cases} \langle A(t)B(t')\rangle\,, & t > t'\,, \\ -\langle B(t')A(t)\rangle\,, & t' > t\,. \end{cases} \tag{7.3.12}$$

Just as in the scalar field theory, an examination of the two-point Green's function yields the physical interpretation of the action of the fields. To do this we need to first cast the Green's function in a perspicuous form. Noting that

$$-(\gamma p)^2 = -\frac{1}{2}\{\gamma_\mu,\gamma_\nu\}p^\mu p^\nu = p^2\,, \tag{7.3.13}$$

we may write

$$\frac{1}{\gamma p + m} = \frac{m-\gamma p}{(\gamma p + m)(m-\gamma p)} = \frac{m-\gamma p}{p^2 + m^2 - i\epsilon}\,, \tag{7.3.14}$$

where in the final form we have explicitly exhibited the $i\epsilon$ that is often left implicit. Inserting this decomposition in the Fourier representation (7.3.7) of the Green's function and performing the p^0-integral by the now familiar contour method yields

$$G(x-x') = i\int\frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{\pm ip(x-x')}(m \mp \gamma p)\,, \tag{7.3.15}$$

in which the plus and minus signs are assigned according to the rule

$$x^0 > x'^0 \;:\; \text{upper signs}\,, \quad x'^0 > x^0 \;:\; \text{lower signs}\,,$$

and where

$$p^0 = E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{7.3.16}$$

The γ-matrix structure in the Green's function is essentially in the form of a projection operator since

$$\left(\frac{m \mp \gamma p}{2m}\right)^2 = \left(\frac{m \mp \gamma p}{2m}\right), \tag{7.3.17}$$

* The sign change for the other ordering was previously discussed in the footnote to Problem 5 of Chapter 2.

and

$$\left(\frac{m+\gamma p}{2m}\right)\left(\frac{m-\gamma p}{2m}\right)=0\,. \tag{7.3.18}$$

This projection operator may be resolved into the outer product of eigenspinors defined by

$$(\gamma p+m)u_\lambda(p)=0\,, \tag{7.3.19}$$

and

$$(\gamma p-m)v_\lambda(p)=0\,. \tag{7.3.20}$$

Here the index λ is needed to label two different u's and two different v's so as to span the 4-dimensional spinor space. As we shall see, this is a spin label. In the non-relativistic limit we need to specify whether the spin is up or down relative to some axis. In the relativistic case the definition of the spin state is subtle, and various choices of the spinors can be made depending upon what sort of boost operations are used to define the spin relative to the particle at rest. For definiteness, we shall use λ to denote the helicity state.† This labeling is obtained by the remark that

$$[\vec{\sigma}\cdot\hat{\mathbf{p}},\gamma p]=0\,, \tag{7.3.21}$$

which implies that $\vec{\sigma}\cdot\hat{\mathbf{p}}$ and γp have simultaneous eigenvectors. Hence we may take

$$\vec{\sigma}\cdot\hat{\mathbf{p}}u_\lambda(p)=\lambda u_\lambda(p)\,, \tag{7.3.22}$$

and

$$(-\vec{\sigma}\cdot\hat{\mathbf{p}})v_\lambda(p)=\lambda v_\lambda(p)\,. \tag{7.3.23}$$

[The reason for the minus sign in Eq. (7.3.23) will be made clear below.] Since $(\vec{\sigma}\cdot\hat{\mathbf{p}})^2=1$, the helicity eigenvalues are given by $\lambda=\pm1$. The Dirac adjoint spinors are defined by

$$\bar{u}_\lambda(p)=u_\lambda(p)^\dagger\beta\,,\quad \bar{v}_\lambda(p)=v_\lambda(p)^\dagger\beta\,. \tag{7.3.24}$$

Since $\gamma^0=\beta$ is Hermitian while γ_k are skew-Hermitian, taking the adjoint of the spinor equations (7.3.19) and (7.3.20), and multiplying by β on the right and anticommuting it through γ_k through to produce the Dirac adjoint spinors yields

$$\bar{u}_\lambda(p)(\gamma p+m)=0\,,\quad \bar{v}_\lambda(p)(\gamma p-m)=0\,. \tag{7.3.25}$$

Since β and $\vec{\sigma}$ commute,

$$\bar{u}_\lambda(p)\vec{\sigma}\cdot\hat{\mathbf{p}}=\lambda u_\lambda(p)\,,\quad \bar{v}_\lambda(p)(-\vec{\sigma}\cdot\hat{\mathbf{p}})=\lambda\bar{v}_\lambda(p)\,. \tag{7.3.26}$$

† For more general spin states see Problem 2.

To understand the normalization which we shall impose on the spinors, we first note that the spinor Dirac equations imply that

$$\begin{aligned}\bar{u}_\lambda(p)\gamma^\mu u_{\lambda'}(p)&=\bar{u}_\lambda(p)\left(\frac{-\gamma p}{m}\right)\gamma^\mu u_{\lambda'}(p)=\bar{u}_\lambda(p)\gamma^\mu\left(\frac{-\gamma p}{m}\right)u_{\lambda'}(p)\\&=-\frac{1}{2m}\bar{u}_\lambda(p)\{\gamma^\mu,\gamma p\}u_{\lambda'}(p)=+\frac{p^\mu}{m}\bar{u}_\lambda(p)u_{\lambda'}(p)\,.\end{aligned} \tag{7.3.27}$$

A similar calculation gives

$$\bar{v}_\lambda(p)\gamma^\mu v_{\lambda'}(p)=-\frac{p^\mu}{m}\bar{v}_\lambda(p)v_{\lambda'}(p)\,. \tag{7.3.28}$$

Since

$$\bar{u}_\lambda\gamma^0u_\lambda=u_\lambda^\dagger u_\lambda>0\,, \tag{7.3.29}$$

and likewise

$$\bar{v}_\lambda\gamma^0v_\lambda>0\,, \tag{7.3.30}$$

a Lorentz invariant normalization requires the signs as shown:

$$\bar{u}_\lambda(p)u_{\lambda'}(p)=2m\delta_{\lambda,\lambda'}\,, \tag{7.3.31}$$

while

$$\bar{v}_\lambda(p)v_{\lambda'}(p)=-2m\delta_{\lambda,\lambda'}\,. \tag{7.3.32}$$

The factors of $2m$ appear for later convenience. Spinors with different eigenvalues of the helicity λ must be orthogonal since they are eigenspinors of the Hermitian matrix $\vec{\sigma}\cdot\hat{\mathbf{p}}$. In a similar way, the u and v spinors must be orthogonal since they are eigenspinors of γp with opposite signs of the eigenvalue m,

$$\bar{u}_\lambda(p)v_{\lambda'}(p)=0=\bar{v}_\lambda(p)u_{\lambda'}(p)\,. \tag{7.3.33}$$

In view of these orthonormality properties,

$$1=\frac{1}{2m}\sum_{\lambda=\pm1}\{u_\lambda(p)\bar{u}_\lambda(p)-v_\lambda(p)\bar{v}_\lambda(p)\}\,, \tag{7.3.34}$$

for this structure does indeed produce unity when it is applied to the complete basis set $u_{\pm1}(p)$, $v_{\pm1}(p)$. Using this resolution of the identity gives

$$m-\gamma p=\sum_{\lambda=\pm1}u_\lambda(p)\bar{u}_\lambda(p)\,, \tag{7.3.35}$$

and

$$m+\gamma p=-\sum_{\lambda=\pm1}v_\lambda(p)\bar{v}_\lambda(p)\,. \tag{7.3.36}$$

These are just the structures that are needed to give the physical interpretation to the Green's function (7.3.15).

Indeed, we now may write

$x^0 > x'^0$:

$$G(x-x') = i\int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{ip(x-x')}\sum_{\lambda=\pm1} u_\lambda(p)\bar{u}_\lambda(p)\,, \tag{7.3.37}$$

and

$x^0 < x'^0$:

$$G(x-x') = -i\int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{-ip(x-x')}\sum_{\lambda=\pm1} v_\lambda(p)\bar{v}_\lambda(p)\,. \tag{7.3.38}$$

These forms are to be compared to the field operator construction of the Green's functions,

$$x^0 > x'^0 : \qquad G(x-x') = i\langle 0|\psi(x)\bar{\psi}(x')|0\rangle\,, \tag{7.3.39}$$

and

$$x^0 < x'^0 : \qquad G(x-x') = -i\langle 0|\bar{\psi}(x')\psi(x)|0\rangle\,. \tag{7.3.40}$$

Let us first note that the result

$$\langle 0|\psi(x)\psi^\dagger(x')|0\rangle = \int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}\sum_{\lambda=\pm1}\left(u_\lambda(p)e^{ipx}\right)\left(u_\lambda(p)e^{ipx'}\right)^\dagger \tag{7.3.41}$$

expresses the operator matrix element in a positive form — it is in accord with the positive norm of the space of physical states. If the sign of the kinetic energy term in the Lagrange function were changed, this positivity would be upset. A generalization of this result leads to the requirement that the matrix $\mathcal{K}$ in the previous general form of the Lagrange be positive.

Introducing a complete set of intermediate states in the operator forms (7.3.39) and (7.3.40) of the Green's function, using the space-time translation or four-momentum operator P^μ to write

$$\psi(x) = e^{-iPx}\psi(0)e^{iPx}\,, \quad \bar{\psi}(x') = e^{-iPx'}\bar{\psi}(0)e^{iPx'}\,, \tag{7.3.42}$$

and comparing with the explicit Fourier integral representations (7.3.37) and (7.3.38) of the Green's functions, we see — just as we did for the scalar field theory - that only single particle states contribute with $-p^2 = m^2$. There are two types of single particle states that contribute, particle and antiparticle states. By convention the particle states are annihilated by ψ, antiparticle states are annihilated by $\bar{\psi}$. These states are labeled by the helicity λ as well as the momentum p, and we adopt the Lorentz invariant normalization

$$\langle p'\lambda'|p\lambda\rangle = (2\pi)^3 2p^0\delta_{\lambda',\lambda}\delta(\mathbf{p}-\mathbf{p}')\,, \tag{7.3.43}$$

with the same normalization for the antiparticle states which we denote by $|p\bar{\lambda}\rangle$. The particle and antiparticle states are orthogonal since, as we shall see, they have opposite eigenvalues of an Hermitian charge operator Q. With the invariant normalization (7.3.43), the resolution of the identity reads

$$1 = |0\rangle\langle 0| + \int \frac{(d^3p)}{(2\pi)^3}\frac{1}{2p^0}\sum_{\lambda=\pm1}|p\lambda\rangle\langle p\lambda| + \int \frac{(d^3p)}{(2\pi)^3}\frac{1}{2p^0}\sum_{\bar{\lambda}=\pm1}|p\bar{\lambda}\rangle\langle p\bar{\lambda}| + \ldots\,, \tag{7.3.44}$$

where the ellipsis stands for multiparticle state contributions. Inserting this resolution of the identity in the operator expressions (7.3.39) and (7.3.40) and comparing with the Fourier transform integrals (7.3.37) and (7.3.38), we conclude that, with a conventional phase choice,

$$\langle 0|\psi(x)|p\lambda\rangle = e^{ipx}u_\lambda(p)\,, \tag{7.3.45}$$

$$\langle p\lambda|\bar{\psi}(x)|0\rangle = e^{-ipx}\bar{u}_\lambda(p)\,, \tag{7.3.46}$$

$$\langle 0|\bar{\psi}(x)|p\bar{\lambda}\rangle = e^{ipx}\bar{v}_{\bar{\lambda}}(p)\,, \tag{7.3.47}$$

and

$$\langle p\bar{\lambda}|\psi(x)|0\rangle = e^{-ipx}v_{\bar{\lambda}}(p)\,. \tag{7.3.48}$$

When acting to the right, ψ destroys a particle of mass m which can have helicity $\lambda = +1$ or $\lambda = -1$, while $\bar{\psi}$ creates this particle. On the other hand, in the same manner, $\bar{\psi}$ destroys an antiparticle and ψ creates this particle.

All this has been written for the non-Hermitian Dirac field, and in this case any representation of the γ-matrices may be used. But the same formulae apply for the Hermitian Majorana field $\psi(x) = \psi^\dagger(x)$ if the Majorana representation for the γ-matrices is employed. In this Majorana Representation:

$$v_\lambda(p) = u_\lambda(p)^*\,, \tag{7.3.49}$$

since the γ- and σ-matrices are purely imaginary, and complex conjugation takes both of the equations (7.3.19) and (7.3.22) defining $u_\lambda(p)$ into the equations (7.3.20) and (7.3.23) that define $v_\lambda(p)$. In the Majorana case the particle and antiparticle particle states are identical there is only one particle state with two helicity values. In this case Eqs. (7.3.45) and (7.3.47) are identical as are Eqs. (7.3.48) and (7.3.46). Since the Majorana theory can be viewed as a special case of the Dirac theory, we shall concentrate on the latter.

The generating functional completely defines the theory. In particular, the equal-time anticommutation relations follow from the structure of the two-point Green's function. The $\delta(x^0 - x'^0)$ on the right-hand side of the Green's function equation (7.3.6) appears when the time derivative $\gamma^0\partial_0$ acts on the discontinuous $\theta(x^0 - x'^0)$, $\theta(x'^0 - x^0)$ terms in the time-ordered product that defines the operator matrix element. The remaining spatial coordinate δ-function requires that the resulting anticommutator be

$$\left\{\psi_\alpha(\mathbf{r},t), \psi^\dagger_\beta(\mathbf{r}',t)\right\} = \delta_{\alpha,\beta}\delta(\mathbf{r}-\mathbf{r}')\,, \tag{7.3.50}$$

where α , β label the spinor components. This equal-time anticommutator is a c-number, a multiple of the identity in the space of physical states. That there are no further operator terms on the right-hand side follows from the structure of the generating functional which expresses the many-point Green's functions as a sum of anti-symmetrized products of two-point functions. Since only the $\psi - \bar\psi$ combination appears in the two-point function, this general structure of the many-point function also implies the other equal-time anticommutators vanish,

$$\{\psi_\alpha(\mathbf{r},t), \psi_\beta(\mathbf{r}',t)\} = 0 = \{\psi^\dagger_\alpha(\mathbf{r},t), \psi^\dagger_\beta(\mathbf{r}',t)\}\,. \tag{7.3.51}$$

Incidentally, for the doublet of Hermitian, Majorana fields $\psi_{a,\alpha}(x)$, where $a = 1, 2$, similar considerations give

$$\{\psi_{a,\alpha}(\mathbf{r},t), \psi_{b,\beta}(\mathbf{r}',t)\} = \delta_{a,b}\delta_{\alpha,\beta}\delta(\mathbf{r}-\mathbf{r}')\,. \tag{7.3.52}$$

Writing $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$, $\psi^\dagger = (\psi_1 - i\psi_2)/\sqrt{2}$, it is easy to check that these produce the previous Dirac field anticommutators.‡

The explicit many-point Green's function construction implies that the Dirac field obeys the field equations

$$\left[\gamma\frac{1}{i}\partial + m\right]\psi(x) = 0 = \bar\psi(x)\left[-\gamma\frac{1}{i}\partial + m\right]\,. \tag{7.3.53}$$

These field equations show that the current operator

$$j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x) \tag{7.3.54}$$

is conserved,

$$\partial_\mu j^\mu(x) = 0\,. \tag{7.3.55}$$

‡ It should be admitted that we have gone around a logical circle here in deriving the anticommutation relations from the multi-point Green's functions. These Green's functions were produced by a functional integral whose derivation from the basic quantum mechanics of the system entailed just these anticommutation relations.

Hence, the total charge operator

$$Q = \int (d^3\mathbf{r}) j^0(x) = \int (d^3\mathbf{r})\psi^\dagger(x)\psi(x) \tag{7.3.56}$$

is time-independent. Since this operator is time-independent, the equal-time anticommutation relations may be used to evaluate

$$[\psi(x), Q] = +\psi(x)\,, \tag{7.3.57}$$

and

$$[\bar\psi(x), Q] = -\bar\psi(x)\,. \tag{7.3.58}$$

We may now use these result to confirm our interpretation of the spin $\frac{1}{2}$ states. First, we note that $|0\rangle$ is the unique vacuum state obeying $P^\mu|0\rangle = 0$. Since the total charge operator Q commutes with P^μ, we also have $P^\mu Q|0\rangle = 0$. This together with the uniqueness of the vacuum state implies that $Q|0\rangle$ is a constant multiple of $|0\rangle$. The constant must be real since Q is Hermitian. The current operator j^μ involves the product of two Fermi fields at a common space-time point. Such a product is potentially ill-defined since the operator product $\bar\psi(x)\psi(x')$ becomes singular when x approaches x' as is evident from the two-point Green's function whose Fourier transform does not vanish rapidly for large momentum. Thus, there may be some ambiguity in defining the total charge Q. This ambiguity is removed by requiring that it contains no constant term and thus annihilates the vacuum state,§

$$Q|0\rangle = 0\,. \tag{7.3.59}$$

Since Q annihilates the vacuum state, the commutation relation (7.3.57) informs us that

$$\langle 0|\psi Q|p\lambda\rangle = \langle 0|\psi|p\lambda\rangle\,. \tag{7.3.60}$$

Since, according to the structure of the two-point Green's function, the field ψ does not connect the vacuum to multi-particle states, this implies that

$$Q|p\lambda\rangle = +|p\lambda\rangle\,. \tag{7.3.61}$$

A similar calculation gives

$$Q|p\bar\lambda\rangle = -|p\bar\lambda\rangle\,. \tag{7.3.62}$$

Thus the particle and antiparticle states have opposite values of the (unit) charge.

§ In this regard, it should mentioned that the current operator in the Majorana representation appears as $j^\mu = \psi\alpha^\mu\tau_2\psi$. This form automatically gives a vanishing vacuum value $\langle 0|j^\mu|0\rangle$ since its calculation involves the vanishing trace of τ_2. This is an advantage of the Majorana formulation.

In a similar vein, we note that Eq. (7.3.42) gives

$$\frac{1}{i}\partial_\mu\psi(x) = [\psi(x), P_\mu]\,, \tag{7.3.63}$$

and using Eq. (7.3.45) we have

$$\frac{1}{i}\partial_\mu\langle 0|\psi(x)|p\lambda\rangle = \langle 0|\psi(x)P_\mu|p\lambda\rangle = \langle 0|\psi(x)|p\lambda\rangle p_\mu\,, \tag{7.3.64}$$

which implies that

$$P_\mu|p\lambda\rangle = |p\lambda\rangle p_\mu\,. \tag{7.3.65}$$

Similarly,

$$P_\mu|p\bar{\lambda}\rangle = |p\bar{\lambda}\rangle p_\mu\,. \tag{7.3.66}$$

This confirms that the states do carry the four-momentum p^μ.

The charge operator Q is the spatial volume integral of a density which is a bilinear form in the field operators $\psi^\dagger\cdots\psi$. The energy-momentum operator P^μ has a similar expression in terms of a bilinear field combination which is part of the energy-momentum tensor. The volume integral of a moment of this tensor produces the generator of spatial rotations, the angular momentum operator $\mathbf{J}$. We do not need this explicit construction, but only the fact (which follows from this explicit construction) that the field ψ transforms as spin $\frac{1}{2}$ under spatial rotations. This transformation property requires that

$$[\psi(x), J_k] = \mathcal{J}_k\psi(x)\,, \tag{7.3.67}$$

in which

$$\mathcal{J}_k = \left(\mathbf{r}\times\frac{1}{i}\nabla\right)_k + \frac{1}{2}\sigma_k\,. \tag{7.3.68}$$

There is no ambiguity in the sign in Eq. (7.3.67). It is fixed by the angular momentum commutation relations

$$[J_k, J_l] = i\epsilon_{klm}J_m\,, \tag{7.3.69}$$

which are also obeyed by the $\mathcal{J}_k$. This is shown by commuting Eq. (7.3.67) with J_l, subtracting the same result, but with the indices k, l interchanged, and using the Jacobi identity.¶ The sign is indeed fixed by the angular momentum commutation relations (7.3.69). The Hermitian adjoint of Eq. (7.3.67) gives

$$[\bar{\psi}(x), \mathbf{J}] = -\mathbf{r}\times\frac{1}{i}\nabla\bar{\psi}(x) - \bar{\psi}(x)\frac{1}{2}\vec{\sigma}\,. \tag{7.3.70}$$

¶ This is essentially the same as the calculation in Problem 1b of Chapter 2.

The vacuum state is rotationally invariant,

$$\langle 0|\mathbf{J} = 0\,. \tag{7.3.71}$$

Therefore, using Eqs. (7.3.67) and (7.3.70) in conjunction with the the wave functions (7.3.45) and (7.3.47), and taking the helicity component $\hat{\mathbf{p}}\cdot\mathbf{J}$ which makes the orbital term vanish [$\mathbf{p}\cdot\mathbf{r}\times\mathbf{p} = 0$], we compute

$$\langle 0|\psi(x)\hat{\mathbf{p}}\cdot\mathbf{J}|p\lambda\rangle = e^{ipx}\hat{\mathbf{p}}\cdot\frac{1}{2}\vec{\sigma}u_\lambda(p)\,, \tag{7.3.72}$$

and

$$\langle 0|\bar{\psi}(x)\hat{\mathbf{p}}\cdot\mathbf{J}|p\bar{\lambda}\rangle = e^{ipx}\bar{v}_{\bar{\lambda}}(p)\left(-\frac{1}{2}\hat{\mathbf{p}}\cdot\vec{\sigma}\right)\,. \tag{7.3.73}$$

Remembering the helicity definitions of the spinors, we see that

$$\hat{\mathbf{p}}\cdot\mathbf{J}|p\lambda\rangle = \frac{\lambda}{2}|p\lambda\rangle\,, \tag{7.3.74}$$

and

$$\hat{\mathbf{p}}\cdot\mathbf{J}|p\bar{\lambda}\rangle = \frac{\bar{\lambda}}{2}|p\bar{\lambda}\rangle\,. \tag{7.3.75}$$

This proves that the correct signs have been assigned to the helicity eigenvalues λ and $\bar{\lambda}$: Except for an additional factor of 2, these eigenvalues measure the component of angular momentum along the direction of propagation of the particle.

At long last the choice of "left"- and "right"-handed in the designation of the Weyl spinors can be justified. To do this, we consider massless particles so that $p^0 = |\mathbf{p}|$. In this case, multiplying the Dirac spinor equation (7.3.19) by β and dividing by p^0 gives

$$\vec{\alpha}\cdot\hat{\mathbf{p}}u_\lambda(p) = u_\lambda(p)\,, \tag{7.3.76}$$

or, remembering Eq. (7.2.85),

$$i\gamma_5\vec{\sigma}\cdot\hat{\mathbf{p}}u_\lambda(p) = u_\lambda(p)\,. \tag{7.3.77}$$

Hence, for the massless case,

$$i\gamma_5 u_\lambda(p) = \lambda u_\lambda(p)\,. \tag{7.3.78}$$

This implies that, in the massless limit, the application of $i\gamma_5$ to the wave function (7.3.45) also multiplies it by λ. Comparing this with Eq. (7.2.68), we see that χ_+ and χ_- do correspond to left- and right-handed fields. That is, χ_+ describes the destruction of a left-handed particle, a particle whose spin is opposed to it direction of motion; χ_- describes the destruction of a right-handed particle, a particle whose spin is orientated along its direction of motion. This connection between chirality — the eigenvalue of $i\gamma_5$ — and helicity — the eigenvalue of $\vec{\sigma}\cdot\hat{\mathbf{p}}$ — holds, of course, only in the massless limit.

Having the wave functions (7.3.45) – (7.3.48) in hand, we may now construct non-interacting particle and antiparticle states in terms of the action of the field operators on the vacuum state. First, we note that Eqs. (7.3.27), (7.3.28), (7.3.31), and (7.3.32) imply that

$$\bar{u}_{\lambda'}(p)\gamma^\mu u_\lambda(p) = \bar{v}_{\lambda'}(p)\gamma^\mu v_\lambda(p) = 2p^\mu \delta_{\lambda',\lambda}\,. \tag{7.3.79}$$

It is now easy to see that such free particle states are produced by

$$\langle p'\lambda'| = \int (d^3r) e^{-ip'x} \bar{u}_{\lambda'}(p')\gamma^0 \langle 0|\psi(x)\,. \tag{7.3.80}$$

Using the wave function (7.3.45), the scalar product of this state with $|p\lambda\rangle$ clearly gives the conventional state normalization (7.3.43). Similarly, the antiparticle state is given by

$$\langle p'\bar{\lambda}'| = \int (d^3r) e^{-ip'x} \langle 0|\bar{\psi}(x)\gamma^0 v_{\lambda'}(p')\,. \tag{7.3.81}$$

The Hermitian adjoint of these results produce the right (ket) vectors.

From this construction of states, one can derive "reduction formulae" for matrix elements of an interacting theory in exactly the same way as was done in Chapter 6, Section 2 for the scalar field. Since the argument is essentially identical to that given there, we shall simply write done the result. The matrix element of a time-ordered product of an arbitrary set of fields taken within an arbitrary final state and an arbitrary initial state and with with an extra particle of momentum p and helicity λ in the final state may be expressed as

$$\begin{aligned}&\langle \cdots p\lambda + |T\cdots|\cdots -\rangle \\ &\quad = \frac{i}{\sqrt{Z}} \int (d^4x) e^{-ipx} \bar{u}_\lambda(p)(\gamma p + m)\langle \cdots + |T\psi(x)\cdots|\cdots -\rangle\,.\end{aligned} \tag{7.3.82}$$

Here the four-momentum p^μ is initially taken to be slightly off mass shell, $-p^2 \neq m^2$, with the mass shell limit picking out the residue of the pole in the Fourier transform of the time-ordered product. The mass-shell wave function renormalization Z appears in an overall factor. The addition of an antiparticle to the final state is described by

$$\begin{aligned}&\langle \cdots p\bar{\lambda} + |T\cdots|\cdots -\rangle \\ &\quad = \frac{i}{\sqrt{Z}} \int (d^4x) e^{-ipx} \langle \cdots + |T\bar{\psi}(x)\cdots|\cdots -\rangle(\gamma p + m) v_\lambda(p)\,.\end{aligned} \tag{7.3.83}$$

The addition of a particle in the initial state is described by

$$\begin{aligned}&\langle \cdots + |T\cdots|p\lambda\cdots -\rangle \\ &\quad = \frac{i}{\sqrt{Z}} \int (d^4x) e^{ipx} \langle \cdots + |T\bar{\psi}(x)\cdots|\cdots -\rangle(\gamma p + m) u_\lambda(p)\,;\end{aligned} \tag{7.3.84}$$

the addition of an antiparticle to the initial state by

$$\begin{aligned}&\langle \cdots + |T\cdots|p\bar{\lambda}\cdots -\rangle \\ &\quad = \frac{i}{\sqrt{Z}} \int (d^4x) e^{ipx} \bar{v}_{\bar{\lambda}}(p)(\gamma p + m)\langle \cdots + |T\psi(x)\cdots|\cdots -\rangle\,.\end{aligned} \tag{7.3.85}$$

Problems

Neutrinos, they are very small.
They have no charge, they have no mass.
They do not interact at all.
The Earth is just a silly ball
  to them through which they simply pass
Like photons through a sheet of glass
Or dustmaids down a drafty hall.
They snub the most exquisite gas,
Insult the stallion in his stall,
Cold-shoulder steel and sounding brass
And pass, like tall and painless guillotines,
  through you and me into the grass.
At night they enter at Nepal
And pierce the lover and his lass
  from underneath the bed.
You call it wonderful? I call it crass.

John Updike

1. Spinor Calculus

It was shown in Eq. (7.2.9) that, except for a factor of σ_2, the complex conjugate of the left-handed spinor, χ_+^*, Lorentz transforms as the right-handed spinor χ_-. The factor of σ_2 produces a similarity transformation in relating L_+^* to L_- as shown in Eq. (7.2.10). Such similarity transformations give equivalent group representations - products of various L_+^* are the same as the corresponding products of L_- except for an overall similarity transformation. Thus, instead of working with right-handed spinors, one may equally well use spinors with a "dotted index" which transform under L_+^*. This remark forms the basis of a spinor calculus which is akin to ordinary tensor calculus. The position of indices in spinor calculus is important just as it is in tensor calculus. One writes

$$L^\alpha{}_\beta = (L_+)_{\alpha\beta}\,,$$

and uses superscripts for the spinors. The left-handed representation of the Lorentz group now reads

$$\chi^\alpha \to L^\alpha{}_\beta \chi^\beta\,.$$

The right-handed representation is equivalent to the "dotted index" spinor transformation

$$\chi^{\dot\alpha} \to L^{*\dot\alpha}{}_{\dot\beta}\chi^{\dot\beta}\,.$$

Higher rank spinors $\chi^{\alpha\beta\cdots\dot\rho\dot\sigma\cdots}$ transform as the direct product of simple spinors,

$$\chi^{\alpha\cdots\dot\rho\cdots} \to L^\alpha{}_\beta \cdots L^{*\dot\rho}{}_{\dot\sigma} \cdots \chi^{\beta\cdots\dot\sigma\cdots}\,.$$

Indices of both types are lowered by contraction with the antisymmetrical symbol $\epsilon_{\alpha\beta}$ as in

$$\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta}\,.$$

Lower indices are raised with $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$ so as to get back the original spinor with a plus sign.

a) Show that the contraction of an upper index with a lower index of the same type produces a lower rank spinor as illustrated in

$$\chi^{\alpha\dot\beta}{}_{\dot\beta} \to L^\alpha{}_\gamma \chi^{\gamma\dot\beta}{}_{\dot\beta}\,.$$

b) Symmetrical higher-rank spinors which are invariant under the interchange of any two dotted or any two undotted indices are equivalent to an irreducible $\chi^{(j^{(+)},j^{(-)})}$ representation. Show that such a symmetrical higher-rank spinor with $2j^{(+)}$ undotted indices and $2j^{(-)}$ dotted indices has $(2j^{(+)}+1)\times(2j^{(-)}+1)$ independent components.

c) Write the mass Lagrange function (7.2.13) in terms of dotted and undotted spinors. Write the kinetic energy Lagrange function (7.2.20) in terms of dotted and undotted spinors.

2. General Spin $\frac{1}{2}$ States

Helicity states were used in the text to describe the spin state of a spin $\frac{1}{2}$ particle. Our purpose here is to describe the spin states in a general fashion. The specification of the spin state for a particle at rest is trivial. Although the momentum and angular momentum operators do not commute, $[J_k, P_l] = i\epsilon_{klm}P_m$, eigenstates of one component of $\mathbf{J}$ do exist for states of a particle at rest, $\mathbf{p}=0$, since the commutator $[J_k, P_l]$ vanishes when applied to them. The spin of a particle in its rest frame is a clear and well-defined concept. We shall describe the spin quantization axis by the unit vector $\hat{\mathbf{s}}$, $\hat{\mathbf{s}}^2 = 1$, and anticipate that the spin is indeed $\frac{1}{2}$ (as will be verified in the course of the work), so that the states at rest may be labeled as $|\mathbf{p}=0,\, s\rangle$, with

$$2\hat{\mathbf{s}}\cdot\mathbf{J}|\mathbf{p}=0,\, s\rangle = |\mathbf{p}=0,\, s\rangle\,.$$

The state $|\mathbf{p} = 0,\, s\rangle$ describes "spin up" along the direction $\hat{\mathbf{s}}$. There is also a state with "spin down" along this direction with the eigenvalue -1 for $2\hat{\mathbf{s}}\cdot\mathbf{J}$. This state, however, may be viewed as an eigenstate of $2(-\hat{\mathbf{s}})\cdot\mathbf{J}$ with eigenvalue $+1$, and so there is no need to consider it separately — it corresponds to replacing $\hat{\mathbf{s}}$ by $-\hat{\mathbf{s}}$. As with the helicity eigenstates, we use a bar over the spin label to denote the antiparticle state, $|\mathbf{p} = 0,\, \bar{s}\rangle$, with again

$$2\hat{\mathbf{s}}\cdot\mathbf{J}|\mathbf{p} = 0,\, \bar{s}\rangle = |\mathbf{p} = 0,\, \bar{s}\rangle\,.$$

To describe the spin state of a moving particle, it is convenient to employ the Pauli-Lubanski operator

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}$$

introduced in Problem 7 of Chapter 3. With $\mathbf{p} \neq 0$, the previous spatial vector $\hat{\mathbf{s}}$ is Lorentz boosted to a unit space-like four-vector s^μ which is orthogonal to the particle's four-momentum p^μ,

$$s^\mu s_\mu = 1\,, \qquad s^\mu p_\mu = 0\,.$$

The previous rest frame spin operator is generalized to

$$\mathcal{S} = \frac{2}{m}s_\mu W^\mu\,,$$

and the moving particle and antiparticle states are characterized by

$$\mathcal{S}|p\,, s\rangle = |p\,, s\rangle\,, \qquad \mathcal{S}|p\,, \bar{s}\rangle = |p\,, \bar{s}\rangle\,.$$

The operators P^μ and $\mathcal{S}$ may be simultaneously diagonalized since, as shown in Problem 7 of Chapter 3, P^μ and W^ν commute.

a) Show that $\mathcal{S}$ reduces to $2\hat{\mathbf{s}}\cdot\mathbf{J}$ for states at rest.

b) Show that for

$$s^0 = \frac{|\mathbf{p}|}{m}\,, \qquad s_k = \frac{p^0}{m}\hat{p}_k\,,$$

$\mathcal{S}$ becomes the helicity operator $\hat{\mathbf{p}}\cdot\mathbf{J}$.

c) The relativistic generalization of Eq. (7.3.67) reads

$$\left[\psi(x),\, J_{\mu\nu}\right] = \left\{x_\mu\frac{1}{i}\partial_\nu - x_\nu\frac{1}{i}\partial_\mu + \frac{1}{2}\sigma_{\mu\nu}\right\}\psi(x)\,.$$

Use this commutator to show that

$$\langle 0|\psi(x)\mathcal{S}|p,\, s\rangle = e^{ipx}\mathcal{S}u_s(p)\,,$$

and

$$\langle 0|\bar{\psi}(x)\mathcal{S}|p,\, \bar{s}\rangle = e^{ipx}\bar{v}_{\bar{s}}(p)(-\mathcal{S})\,,$$

where $\mathcal{S}$ acting upon spinors of momentum p is given by*

$$\mathcal{S} = \frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}s_\mu p_\nu\sigma_{\alpha\beta}\,.$$

This is to say that the spinors that represent the particle and antiparticles must obey

$$\mathcal{S}u_s(p) = u_s(p)\,, \qquad \bar{v}_{\bar{s}}(p)\mathcal{S} = -\bar{v}_{\bar{s}}(p)\,.$$

The Dirac adjoints read

$$\bar{u}_s(p)\mathcal{S} = \bar{u}_s(p)\,, \qquad \mathcal{S}v_{\bar{s}}(p) = -v_{\bar{s}}(p)\,.$$

d) Since the spinors are simultaneous eigenvectors of γp and $\mathcal{S}$, the two matrices must commute, $[\gamma p,\, \mathcal{S}] = 0$. Show this by direct computation.

e) Show that

$$\mathcal{S} = \frac{1}{m}\gamma_5\, s_\mu\, p_\nu\, \sigma^{\mu\nu} = i\gamma_5\,\gamma s\,\gamma p/m\,.$$

Therefore

$$\mathcal{S}u_s(p) = \gamma s\, i\gamma_5\, u_s(p)\,, \qquad \mathcal{S}v_{\bar{s}}(p) = -\gamma s\, i\gamma_5\, v_{\bar{s}}(p)\,,$$

f) Show that $(\gamma s\, i\gamma_5)^2 = 1$. Hence

$$P_s = \frac{1}{2}(1 + \gamma s\, i\,\gamma_5)$$

is a projection operator, $P_s^2 = P_s$, and it projects into the $(\gamma s\, i\gamma_5)' = +1$ subspace.

g) Explain why

$$u_s(p)\bar{u}_s(p) = (m - \gamma p)P_s\,,$$

and

$$v_{\bar{s}}(p)\bar{v}_{\bar{s}}(p) = -(m + \gamma p)P_s\,.$$

These formulae are convenient for the calculation of processes involving polarized particles.

h) Suppose that $0 < s^\mu s_\mu < 1$. Explain how P_s represents a state of partial polarization — how P_s plays the role of a density matrix.

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* Here the same letter $\mathcal{S}$ is used to denote both the original abstract operator acting on physical states and a matrix in the Dirac spinor space because we have run out of different letters of the alphabet. The context should make clear which is which.

3. Gamma Matrices in $n = 2m$ Dimensions

We almost entirely use dimensional continuation to regulate and obtain a well-defined theory. With the inclusion of spin $\frac{1}{2}$, we need to use Dirac γ-matrices for a space of "arbitrary" dimensionality. The purpose of this problem is to show that γ-matrices can be constructed for a space-time of even integer dimensionality, $n = 2m$.

a) Consider first an abstract Clifford algebra defined by $2m$ generating elements ϵ_a, $a = 1, 2, \ldots, 2m$ that obey the anticommutation relation

$$\{\epsilon_a, \epsilon_b\} = 2\delta_{ab} .$$

The algebra consists of all linear combinations, with complex numerical coefficients, of all the products of the generators ϵ_a. A linear independent basis is formed by 1; ϵ_a, $a = 1, 2, \ldots, 2m$; $\epsilon_a\epsilon_b$, $a < b = 1, 2, \ldots, 2m$; $\epsilon_a\epsilon_b\epsilon_c$, $a < b < c = 1, 2, \ldots, 2m$; and so on. Explain why this process terminates with

$$\epsilon_{2m+1} = i^m \epsilon_1 \epsilon_2 \cdots \epsilon_{2m} .$$

Explain why the number of linearly independent elements in the basis is given by

$$\sum_{k=0}^{2m} \frac{(2m)!}{k!(2m-k)!} = 2^{2m} .$$

Prove that ϵ_{2m+1} anticommutes with each of the generating elements

$$\{\epsilon_a, \epsilon_{2m+1}\} = 0 .$$

Prove that

$$\epsilon_{2m+1}^2 = 1 .$$

[Hint:

$$\sum_{l=2m-1}^{1} l = m(2m-1) .]$$

Hence there are $2m + 1$ quantities which anticommute with one another and square to unity. Explain why this enumeration exhausts all such anticommuting quantities — why there are no more than $2m + 1$ of them.

Since the algebra has a basis of 2^{2m} linearly independent elements, it has a representation in terms of $2^m \times 2^m$ matrices. It is easy to explicitly construct this representation in a recursive fashion using direct products of two-dimensional matrices. For $m = 1$, we take the 2×2 Pauli matrix representation $\epsilon_1 = \sigma_1$ and $\epsilon_2 = \sigma_2$, which gives $\epsilon_3 = i\,\epsilon_1\,\epsilon_2 = -\sigma_3$. Thus, the initial stage of a recursive construction exists. Assuming that $2^m \times 2^m$ matrices exist for the ϵ_a, $a = 1, 2, \ldots, 2m$, the representation for $m + 1$ is made by adjoining a direct product with a new set of 2×2 matrices of the Pauli form, ρ_1, ρ_2, ρ_3. Namely, we have the $2m + 2$ anticommuting elements which square to the identity given by

$$\rho_1\{\epsilon_1, \epsilon_2, \cdots, \epsilon_{2m}\}\,,\ \rho_2\,,\rho_3 .$$

Since we start with a pair of Hermitian matrices $\sigma_1\,,\sigma_2$, and then adjoin at every stage a set of Hermitian matrices (ρ_1, ρ_2, ρ_3), this construction produces Hermitian generators, $\epsilon_a^\dagger = \epsilon_a$, $a = 1, 2, \ldots, 2m$. Moreover, since $\epsilon_{2m+1}^\dagger \epsilon_{2m+1} = 1 = \epsilon_{2m+1}^2$, ϵ_{2m+1} is also an Hermitian matrix.

We now have an explicit construction for the Dirac matrices γ^μ for a space-time of even dimensionality $n = 2m$. Namely, we take the $2m - 1$ spatial matrices to be the anti-Hermitian matrices $\gamma^k = i\epsilon_k\,, k = 1, 2, \ldots, 2m-1$, and take the time-component matrix to be the Hermitian matrix $\gamma^0 = \epsilon_{2m}$. This gives

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} .$$

The $2m$ generating matrices ϵ_a may also be used to produce Dirac matrices for a space-time of odd dimensionality $n = 2m + 1$. In this case one may take $\gamma^k = i\epsilon_k$, $k = 1, 2, \ldots, 2m$ and $\gamma^0 = \epsilon_{2m+1}$. This odd-dimensional representation, however, does not provide invariance under the space-time reflection $x^\mu \to -x^\mu$ as is illustrated by the non-vanishing trace in $n = 3\,,\ \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = -2i\epsilon^{\mu\nu\lambda}$. The reflection invariance is achieved if the $n = 2m+1$ dimensional theory is embedded in an $2m+2$ set of generators ϵ_a. It should be noted that although $2^m \times 2^m$ dimensional γ-matrices exist for the complex Dirac field in any even space-time dimension $n = 2m$, the Majorana representation suitable for a real field exists only in a certain subset of these even dimensions. This will be discussed in Problem 4.

b) For a space-time of even dimensionality n, show that

$$\mathrm{tr}\,1 = 2^{n/2} ,$$

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu = -2^{n/2} g^{\mu\nu} ,$$

and

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2^{n/2}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) .$$

Prove that

$$\mathrm{tr}\,\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_k} = 0$$

if k is odd. [Hint: Place $\gamma_{n+1}\gamma_{n+1} = \pm 1$ in the trace and use the cyclic symmetry of the trace to move one of the γ_{n+1} about.] Show that in four dimensions

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = -4\epsilon^{\mu\nu\rho\sigma} .$$

4. Majorana Dirac Matrices

Problem 3 showed that Dirac matrices may be constructed for a space-time of arbitrary even dimensionality $n = 2m$. The irreducible matrix representation — the representation of smallest dimension — consists of a set of $2^m \times 2^m$ matrices, giving 2^{2m} linearly independent matrices. In this previous problem, no attention was paid to the symmetry properties of these matrices, a property needed if they are to form a Majorana representation. In a space-time of dimensionality $n = 2m$, a Majorana representation exists if one can construct $\alpha_1, \alpha_2, \cdots, \alpha_{2m-1}$ real, symmetrical matrices and an imaginary, antisymmetrical matrix β. This is the representation needed to form a Lagrange function for a real, anticommuting Fermi field of the form $\psi[(1/i)\partial_0 + \alpha_k(1/i)\partial_k + \beta m]\psi$. The γ-matrices are then given by $\gamma^0 = \beta, \gamma^k = \beta\alpha_k$. The generalization of the four-dimensional γ_5 matrix is

$$\gamma_{2m+1} = \alpha_1\alpha_2\cdots\alpha_{2m-1}.$$

This is the generalized "chirality" matrix. It commutes with all the α_k and anticommutes with β.

In a two-dimensional space-time, we may take $\alpha_1 = \sigma_3, \quad \beta = \sigma_2$, which is a Majorana representation. In this case $\gamma_3 = \alpha_1 = \sigma_3$ is a real, symmetrical matrix which we have chosen to be diagonal. In the massless case, γ_3 commutes with the matrices in the Lagrange function so that the solutions to the field equations may be taken to be eigenspinors of γ_3 with eigenvalues $+1$ or -1. These are Weyl spinors. Thus, for $n = 2$, the irreducible representation of the Dirac algebra is both Majorana and Weyl.

The Majorana representation for four-dimensional space-time was explicitly constructed in the text in terms of the direct product of σ_a with another set of 2×2 matrices ρ_a which were again taken to have the Pauli form. This representation in terms of $2^2 \times 2^2$ matrices is irreducible. In this four-dimensional case, the matrix γ_5 is real but antisymmetrical. Hence it cannot be diagonalized by a real, orthogonal transformation. Thus the $n = 4$ irreducible representation is Majorana, but not Weyl.

The purpose of this problem is to generalize these results to arbitrary irreducible representations and show that there is a periodic structure mod 8: Majorana matrices exist for $n = 8k + 2$ and $n = 8k + 4$, but they do not exist for $n = 8k + 6$ and $n = 8k$. For $n = 8k + 2$ the matrices are also Weyl. It should be mentioned that the work in Problem 3 established that $n = 2m$ anticommuting generating elements ϵ_a which square to the identity have an irreducible representation in terms of $2^m \times 2^m$ matrices. This algebra is not altered by the similarity transformation $\epsilon_a \to \epsilon'_a = S^{-1}\epsilon_a S$. The question here is whether or not such a similarity transformation exists which makes $2m - 1$ of the transformed generators

symmetric, the remaining transformed generator antisymmetric. A formula useful for the counting involved in this problem is again the formula Gauss wrote in chalk on his slate in elementary school,

$$\sum_{l=1}^{N} l = \frac{1}{2}N(N+1).$$

a) Suppose that Majorana matrices exist for $n = 8k + 2$ and $n = 8k + 4$. Show that $n = 8k + 2$ is Weyl but $n = 8k + 4$ is not.

b) Show that the number of $2^m \times 2^m$ symmetrical matrices is given by

$$\mathcal{N} = 2^{m-1}(2^m + 1).$$

c) The matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

is a fourth root of unity, $M^4 = 1$. Consider

$$[1+M]^N = \begin{pmatrix} 2^N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1+i)^N & 0 \\ 0 & 0 & 0 & (1-i)^N \end{pmatrix}.$$

Expand the left-hand side of this equation using the binomial theorem. Write the right-hand side in the form $a + bM + cM^2 + dM^3$. Compare the two sides of the resulting equation to conclude that

$$\binom{N}{0} + \binom{N}{4} + \binom{N}{8} + \ldots = \frac{1}{2}\left(2^{N-1} + 2^{N/2}\cos\frac{N\pi}{4}\right),$$

$$\binom{N}{1} + \binom{N}{5} + \binom{N}{9} + \ldots = \frac{1}{2}\left(2^{N-1} + 2^{N/2}\sin\frac{N\pi}{4}\right),$$

and

$$\binom{N}{2} + \binom{N}{6} + \binom{N}{10} + \ldots = \frac{1}{2}\left(2^{N-1} - 2^{N/2}\cos\frac{N\pi}{4}\right),$$

where

$$\binom{N}{l} = \frac{N!}{l!(N-l)!}.$$

d) Suppose that there exist $2^m \times 2^m$ matrices $\alpha_1, \alpha_2, \ldots, \alpha_{2m-1}$, and β in which the α-matrices are symmetrical while the β-matrix is antisymmetrical. A complete basis for the space is provided by 1; α_a; $\alpha_a\alpha_b$, $a <$

b; $\alpha_a\alpha_b\alpha_c$, $a < b < c$; ... together with β; $\alpha_a\beta$; $\alpha_a\alpha_b\beta$; $\alpha_a\alpha_b\alpha_c\beta$; .... Show that this matrix basis is grouped into symmetrical (S) and antisymmetrical (A) matrices:

1	α_a	$\alpha_a\alpha_b$	$\alpha_a\alpha_b\alpha_c$	$\alpha_a\alpha_b\alpha_c\alpha_d$	$\alpha_a\alpha_b\alpha_c\alpha_d\alpha_e$	...
S	S	A	A	S	S	...

β	$\alpha_a\beta$	$\alpha_a\alpha_b\beta$	$\alpha_a\alpha_b\alpha_c\beta$	$\alpha_a\alpha_b\alpha_c\alpha_d\beta$	$\alpha_a\alpha_b\alpha_c\alpha_d\alpha_e\beta$	...
A	S	S	A	A	S	... .

Count the total number of symmetrical matrices and compare with the result given in part b) above. Show that there is a contradiction unless $n = 2m = 8k + 2$ or $n = 2m = 8k + 4$. That is, show that Majorana matrices do not exist in the irreducible representation for $n = 8k + 6$ and $n = 8k$.

e) A recursive construction can be made to show that the Majorana matrices do exist for $n = 8k + 2$ and $n = 8k + 4$. Suppose that the $2^{4k+1} \times 2^{4k+1}$ Majorana representation exists for $n = 8k + 2$. Then γ_{8k+3} is a real, symmetrical matrix which anticommutes with β and commutes with the α_a. The $n = 8k + 4$ representation is given by a direct product of the $n = 8k + 2$ representation with ρ_a:

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{8k+3}\} = \{(\alpha_1, \alpha_2, \ldots, \alpha_{8k+1})\rho_3, i\beta\gamma_{8k+3}\rho_3, \rho_1\},$$

and

$$\beta = \rho_2 .$$

Here it is understood that the matrices on the left-hand side of these two equations are the new $n = 8k+2$ matrices in the $2^{4k+2} \times 2^{4k+2}$ space while the matrices on the right-hand sides are the direct products of the old $2^{4k+1} \times 2^{4k+1}$ matrices with the ρ_a matrices. Show that this construction does indeed provide a Majorana representation for $n = 8k + 4$. With this construction in hand, we can go on to construct matrices for $n = 8k + 6$:

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{8k+5}\} = \{(\alpha_1, \alpha_2, \ldots, \alpha_{8k+3})\rho_3, \rho_1, \beta\rho_3\},$$

and

$$\beta = \rho_2 .$$

Again it is understood that the matrices on the left-hand sides are the new matrices, the matrices on the right-hand sides are the old matrices, and ρ_a now act in a new, additional 2×2 space. Show that the new matrices have the correct algebraic properties. Note, however, that while $\alpha_1, \ldots, \alpha_{8k+4}$ are symmetrical, α_{8k+5} as well as β are antisymmetrical. The construction for $n = 8k + 8$ is done in exactly the same fashion,

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{8k+7}\} = \{(\alpha_1, \alpha_2, \ldots, \alpha_{8k+4})\rho_3, \rho_1, \alpha_{8k+5}\rho_3, \beta\rho_3\},$$

$$\beta = \rho_2 .$$

Here $\alpha_1, \alpha_2, \ldots, \alpha_{8k+5}$ are symmetric while α_{8k+6}, α_{8k+7} and β are antisymmetric. This set of matrices defines

$$\gamma_{8k+9} = \alpha_1\alpha_2 \cdots \alpha_{8k+7},$$

which commutes with $\alpha_1, \ldots, \alpha_{8k+7}$, anticommutes with β, and which is real and antisymmetrical,

$$\gamma_{8k+9}^T = -\gamma_{8k+9} .$$

The recursive proof is now completed by proving that using the preceding construction, a Majorana representation exists at $n = 8k + 10$. Show that

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{8k+9}\} = \{(\alpha_1, \alpha_2, \ldots, \alpha_{8k+5})\rho_3, \alpha_{8k+6}\gamma_{8k+9}, \beta\alpha_{8k+7}\gamma_{8k+9}\rho_2, \alpha_{8k+7}\gamma_{8k+9}\rho_3, i\beta\alpha_{8k+6}\alpha_{8k+7}\},$$

and

$$\beta = \beta\alpha_{8k+7}\gamma_{8k+9}\rho_1$$

does indeed provide a Majorana representation at $n = 8k + 10$.

5. Stress-Energy Tensor

The stress-energy tensor for the free Dirac field is given by

$$T^{\mu\nu}(x) = \frac{1}{4i}\bar{\psi}(x)(\gamma^\mu\overleftrightarrow{\partial^\nu} + \gamma^\nu\overleftrightarrow{\partial^\mu})\psi(x),$$

where $\overleftrightarrow{\partial^\mu}$ is the derivative acting to the right minus the derivative acting to the left.

a) Prove that the stress tensor is conserved,

$$\partial_\nu T^{\mu\nu} = 0 .$$

b) Show that

$$T^{00} = \psi^\dagger\Big(\vec{\alpha}\cdot\frac{1}{2i}\overleftrightarrow{\nabla} + \beta m\Big)\psi,$$

and

$$T^{0k} = \psi^\dagger\Big(\frac{1}{2i}\overleftrightarrow{\nabla^k}\Big)\psi + \nabla_l\Big(\psi^\dagger\frac{1}{4}\sigma^{kl}\psi\Big).$$

c) The energy, momentum, and angular momentum operators are given by

$$P^0 = \int (d^3\mathbf{r})\, T^{00},$$

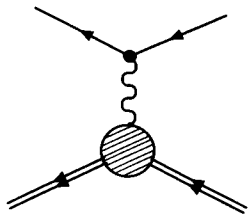

Fig. 7.1. Lowest-order electron-proton elastic scattering with a single photon exchange denoted by the wavy line. The top part of the graph describes the point interaction of the electron with the photon, the bottom part the extended structure of the proton-photon interaction.

$$P^k = \int (d^3\mathbf{r})\, T^{0k}\,,$$

and

$$J^k = \int (d^3\mathbf{r})\, \epsilon^{klm} x^l T^{0m}\,.$$

Using the Dirac field anticommutation relations, prove that the commutators $[\psi, P^0]$, $[\psi, P^k]$, and $[\psi, J^k]$ are those given in the text.

d) The Lorentz boost generator is defined by

$$J^{0k} = \int (d^3\mathbf{r})\, (x^0 T^{0k} - x^k T^{00})\,.$$

Compute $[\psi, J^{0k}]$ and show that it corresponds to an infinitesimal Lorentz transformation.

6. Form Factors

The proton (or neutron, neglecting the small weak interaction) is a stable, spin $\frac{1}{2}$ particle, and its physical state may be expressed as a helicity eigenstate $|p\lambda\rangle$. Although this state transforms as a simple spin $\frac{1}{2}$ particle under Lorentz transformations, the proton (or neutron) has an internal structure brought about by strong interactions. This structure is exhibited in the matrix element of the electromagnetic current between two proton (or neutron) states, $\langle p'\lambda'|j^\mu(x)|p\lambda\rangle$. It should be noted that this matrix element can be directly measured by electron-proton scattering. Since the electromagnetic coupling is small, the scattering amplitude is well-described by the lowest-order, single photon exchange graph of Fig. 7.1. The "blob" at the bottom of the graph is just the matrix element $\langle p'\lambda'|j^\mu|p\lambda\rangle$. The purpose of this problem is to show how simple principles restrict and explain the structure of this matrix element.

a) Use the translation character of field operators to prove that

$$\langle p'\lambda'|j^\mu(x)|p\lambda\rangle = e^{-ikx}\langle p'\lambda'|j^\mu(0)|p\lambda\rangle,$$

where

$$k = p' - p\,.$$

b) The two spin $\frac{1}{2}$ states and the four-vector j^μ must combine to make an object with the correct Lorentz transformation properties. Since this is just a kinematical problem, the proper structure is that which is given by treating the proton as if it were an elementary particle described by the field ψ and using the reduction technique described at the end of Section 3,

$$\langle p'\lambda'|j^\mu(0)|p\lambda\rangle = \bar{u}_{\lambda'}(p')\Gamma^\mu(p';p)u_\lambda(p).$$

Here $\Gamma^\mu(p',p)$ is, in general, a combination of Dirac matrices and the available four-vectors p'^α, p^β such that $\Gamma^\mu(p';p)$ transforms as a four-vector under Lorentz transformations. We shall neglect the weak interaction so that the matrix element is covariant under spatial reflections. That is, we shall assume that the theory contains a time-independent parity operator P which is unitary, $P^\dagger = P^{-1}$, which leaves that vacuum state invariant, and which transforms the current operators according to

$$P^{-1}j^0(\mathbf{r},t)P = j^0(-\mathbf{r},t), \qquad P^{-1}j^k(\mathbf{r},t)P = -j^k(-\mathbf{r},t).$$

The corresponding transformation law for the field ψ (which is used here only as a device for getting the kinematics right) is obtained by expressing the current as

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x).$$

The reflection character of the current is obtained if[†]

$$P^{-1}\psi(\mathbf{r},t)P = i\beta\psi(-\mathbf{r},t)\,,$$

and

$$P^{-1}\bar{\psi}(\mathbf{r},t)P = \bar{\psi}(-\mathbf{r},t)(-i\beta)\,.$$

According to the reduction formula, the current matrix element involves the Fourier transform of a structure of the form

$$\bar{u}_{\lambda'}(p')\cdots\langle 0|T\psi(x')j^\mu(0)\bar{\psi}(x)|0\rangle\cdots u_\lambda(p)\,.$$

The spatial reflection invariance of the vacuum state asserts that this is equal to

$$\bar{u}_{\lambda'}(p')\cdots\langle 0|TP^{-1}\psi(x')PP^{-1}j^\mu(0)PP^{-1}\bar{\psi}p|0\rangle\cdots u_\lambda(p)\,.$$

[†] The phase factor of i is put here so as to have the parity transformation represented by a real matrix $i\beta$. Thus this law also works for Hermitian fields. The phase factors could be omitted, but this corresponds to making a charge rotation in addition to a purely geometrical spatial inversion.

Hence parity invariance requires that

$$\Gamma^\mu(\mathbf{p}', p'^0; \mathbf{p}, p^0) = \pm\beta\Gamma^\mu(-\mathbf{p}', p'^0; -\mathbf{p}, p^0)\beta\,,$$

where the + sign holds for the time component $\mu = 0$, the − sign for the spatial components $\mu = k$.

The proof given of this result has been rather sketchy. Write down the details that make the proof clear. Explain clearly and in detail why the result coupled with Lorentz invariance implies that the most general (parity invariant) form of $\Gamma^\mu(p',\, p)$ is given by

$$\Gamma^\mu(p', p) = k^\mu F_0(k^2) + \gamma^\mu F_1(k^2) - \frac{i}{2M}\sigma^{\mu\nu}k_\nu F_2(k^2)$$
$$+\sigma^{\mu\nu}P_\nu F_3(k^2) + \frac{1}{2M}P^\mu F_4(k^2) + \frac{1}{4M^2}\epsilon^{\mu\nu\lambda\kappa}k_\nu P_\lambda\gamma_\kappa\gamma_5 F_5(k^2)\,,$$

where

$$P^\mu = p'^\mu + p^\mu\,,$$

$M^2 = -p'^2 = -p^2$ is the proton mass squared, and the various arbitrary functions, the "form factors", are only functions of the Lorentz invariant k^2. [Here we have omitted terms such as $k^\mu\gamma p$ which are obviously redundant when placed between the Dirac spinors $\bar{u}_{\lambda'}(p')\cdots u_\lambda(p)$.]

c) Although all of the vector and γ-matrix terms for $\Gamma^\mu(p', p)$ are linearly independent quantities, they are not independent when sandwiched between Dirac spinors in $\bar{u}_{\lambda'}(p')\Gamma^\mu(p', p)u_\lambda(p)$. Prove that within these spinors one has the equivalence

$$\Gamma \leftrightarrow -\frac{1}{4M}\Big(\{\gamma P, \Gamma\} + [\gamma k, \Gamma]\Big)\,,$$

where Γ is an arbitrary matrix. Use this formula to prove the equivalences

$$1 \leftrightarrow -\frac{1}{2M}\gamma P\,, \tag{1}$$

$$\gamma_5 \leftrightarrow -\frac{1}{2M}\gamma k\gamma_5\,, \tag{2}$$

$$\gamma^\mu \leftrightarrow \frac{1}{2M}P^\mu - \frac{i}{2M}\sigma^{\mu\nu}k_\nu\,, \tag{3}$$

$$\gamma^\mu\gamma_5 \leftrightarrow \frac{1}{2M}k^\mu\gamma_5 - \frac{i}{4M}\epsilon^{\mu\nu\lambda\kappa}P_\nu\sigma_{\lambda\kappa}\,, \tag{4}$$

and

$$\sigma^{\mu\nu} \leftrightarrow -\frac{i}{2M}(\gamma^\mu k^\nu - \gamma^\nu k^\mu) + \frac{i}{2M}\epsilon^{\mu\nu\lambda\kappa}\gamma_\lambda\gamma_5 P_\kappa\,. \tag{5}$$

Contract (5) with P_ν and use the spinor Dirac equation to prove that $F_3(k^2)$ is redundant — that its contribution is already included in $F_0(k^2)$. Contract (5) with k_ν to prove that $F_5(k^2)$ is redundant. Note that (3) implies that $F_4(k^2)$ is redundant. It is important to note that the elimination of $F_3(k^2)$, $F_4(k^2)$, and $F_5(k^2)$ brings forth no spurious, "kinematical singularities" — the replacements do not involve the introduction of denominators that could give one set of functions poles which another set does not have. Kinematical singularities will be discussed at the end of this problem.

d) We have now achieved the structure

$$\langle p'\lambda'|j^\mu(0)|p\lambda\rangle = \bar{u}_{\lambda'}(p')\Big\{k^\mu F_0(k^2) + \gamma^\mu F_1(k^2) - \frac{i}{2M}\sigma^{\mu\nu}k_\nu F_2(k^2)\Big\}u_\lambda(p)\,.$$

Prove that current conservation, $\partial_\mu j^\mu(x) = 0$, implies that $F_0(k^2) = 0$.

e) The current j^μ is an Hermitian operator. Prove that the form factors are real functions.

f) Prove that an equivalent description of the current matrix element is given by

$$\langle p'\lambda'|j^\mu(0)|p\lambda\rangle = \frac{1}{1 + k^2/4M^2}$$
$$\bar{u}_{\lambda'}(p')\Big\{\frac{1}{2M}P^\mu G_E(k^2) + \frac{1}{4M^2}\epsilon^{\mu\nu\lambda\kappa}k_\nu P_\lambda\gamma_\kappa\gamma_5 G_M(k^2)\Big\}u_\lambda(p)\,,$$

where

$$F_1(k^2) = \frac{G_E(k^2) + (k^2/4M^2)G_M(k^2)}{1 + k^2/4M^2}\,,$$

and

$$F_2(k^2) = \frac{G_M(k^2) - G_E(k^2)}{1 + k^2/4M^2}\,,$$

or

$$G_M(k^2) = F_1(k^2) + F_2(k^2)\,,$$

and

$$G_E(k^2) = F_1(k^2) - (k^2/4M^2)F_2(k^2)\,.$$

g) To understand the significance of the form factors $G_E(k^2)$ and $G_M(k^2)$, consider the matrix element in the Breit or brick-wall frame where $\mathbf{p}' = -\mathbf{p}$, which gives $p'^0 = p^0$. In this frame, the scattering of an electron on a moving proton just reverses the momentum of the proton, leaving its

energy unchanged. Show that in this frame

$$\langle -\mathbf{p}\lambda'|j^0(0)|\mathbf{p}\lambda\rangle = G_E(k^2)\frac{M}{p^0}\bar{u}_{\lambda'}(-\mathbf{p})u_\lambda(\mathbf{p}),$$

and

$$\langle -\mathbf{p}\lambda'|j_k(0)|\mathbf{p}\lambda\rangle = -G_M(k^2)\frac{i}{p^0}\bar{u}_{\lambda'}(-\mathbf{p})\gamma^0(\mathbf{p}\times\vec{\sigma})_k u_\lambda(\mathbf{p})\,.$$

Use the Dirac equation to show that

$$\bar{u}_{\lambda'}(-\mathbf{p})u_\lambda(\mathbf{p}) = \frac{p^0}{M}\bar{u}_{\lambda'}(-\mathbf{p})\gamma^0 u_\lambda(\mathbf{p})\,. \tag{1}$$

The spinor of momentum $\mathbf{p}$ is produced by a Lorentz transformation, a boost along the direction $\hat{\mathbf{p}}$, from a spinor at rest,

$$u_\lambda(\mathbf{p}) = L(\mathbf{p})u_\lambda(0)\,.$$

The boost $L(\mathbf{p})$ is constructed from the Dirac matrix $\vec{\alpha}\cdot\hat{\mathbf{p}}$. Prove that

$$\bar{u}_{\lambda'}(-\mathbf{p})\gamma^0 u_\lambda(\mathbf{p}) = \bar{u}_{\lambda'}(0)\gamma^0 u_\lambda(0) = u^\dagger_{\lambda'}(0)u_\lambda(0)\,.$$

Combining this with Eq. (1) gives

$$\bar{u}_{\lambda'}(-\mathbf{p})u_\lambda(\mathbf{p}) = \frac{p^0}{M}u^\dagger_{\lambda'}(0)u_\lambda(0)\,.$$

Explain why this implies that

$$u^\dagger_{\lambda'}(0)L(\mathbf{p})^2 u_\lambda(0) = \frac{p^0}{M}u^\dagger_{\lambda'}(0)u_\lambda(0)\,. \tag{2}$$

Since $(\vec{\alpha}\cdot\hat{\mathbf{p}})^2 = 1$,

$$L(\mathbf{p})^2 = A + B\vec{\alpha}\cdot\hat{\mathbf{p}}\,.$$

Use Eq. (2) to prove that

$$A = \frac{p^0}{M}\,.$$

Now prove that

$$\bar{u}_{\lambda'}(-\mathbf{p})\gamma^0(\mathbf{p}\times\vec{\sigma})u_\lambda(\mathbf{p}) = \frac{p^0}{M}u^\dagger_{\lambda'}(0)(\mathbf{p}\times\vec{\sigma})u_\lambda(0)\,.$$

Hence

$$\langle -\mathbf{p}\lambda'|j^0(0)|\mathbf{p}\lambda\rangle = G_E(k^2)2M\delta_{-\lambda',\lambda}\,,$$

and

$$\langle -\mathbf{p}\lambda'|\mathbf{j}(0)|\mathbf{p}\lambda\rangle = -G_M(k^2)2iu^\dagger_{\lambda'}(0)(\mathbf{p}\times\vec{\sigma})u_\lambda(0)/2M$$

The notation is a little tricky here. The index λ in $u_\lambda(0)$ refers to the eigenvalue of the spin in the direction $\hat{\mathbf{p}}$ while the index λ' in $u^\dagger_{\lambda'}(0)$ refers to the eigenvalue of the spin in the direction $\hat{\mathbf{p}}' = -\hat{\mathbf{p}}$. Hence, in the Breit frame, the matrix element of the charge density vanishes unless the helicity flips — which is to say that the charge density matrix element vanishes if the spin flips. Prove that the current density matrix element vanishes if the helicity flips.

h) Prove that the wave-packet state

$$|f\lambda\rangle = \int\frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}|p\lambda\rangle f(p)$$

has unit norm if

$$\int\frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}|f(p)|^2 = 1\,.$$

The electric charge operator, in units of e, is given by

$$Q = \int(d^3\mathbf{r})\,j^0(x)\,.$$

Prove that the condition that the particle carry unit charge,

$$\langle f\lambda|Q|f\lambda\rangle = 1\,,$$

requires that

$$G_E(0) = 1\,.$$

Suppose that the particle is placed in a uniform magnetic field $\mathbf{B}$. This gives an interaction Hamiltonian

$$H = -\frac{e}{2}\int(d^3\mathbf{r})\,\mathbf{j}(x)\cdot\mathbf{B}\times\mathbf{r}\,.$$

Show for a wave-packet state sharply peaked at $\mathbf{p} = 0$, which corresponds to a particle at rest, that

$$\langle f\lambda'|H|f\lambda\rangle = -\frac{e}{M}G_M(0)u^\dagger_{\lambda'}(0)\frac{1}{2}\mathbf{B}\cdot\vec{\sigma}u_\lambda(0)/2M.$$

This is the interaction energy of a particle at rest in a constant magnetic field. Since the spinor normalization gives $u^\dagger_\lambda(0)u_\lambda(0) = 2M$, this corresponds to the non-relativistic interaction

$$H = -\vec{\mu}\cdot\mathbf{B}\,,$$

with

$$\vec{\mu} = 2G_M(0)\frac{e}{2M}\frac{1}{2}\vec{\sigma}\,.$$

This identifies $G_M(0)$ in terms of the g-factor or magnetic moment factor $\mu = g/2$ as

$$G_M(0) = \mu = g/2\,.$$

Note that these normalizations translate into

$$F_1(0) = 1\,, \qquad F_2(0) = \mu - 1\,.$$

In particular, $F_2(0)$ is just the "anomaly" $(g-2)/2$.

i) Defining $\zeta(x)$ by

$$(\gamma\frac{1}{i}\partial + m)\psi(x) = \zeta(x)\,,$$

and using the reduction formula, it is not difficult to see that

$$\Gamma^\mu(p';p) = \int (d^4x')(d^4x)\, e^{-ip'x'+ipx}\langle 0|T\zeta(x')j^\mu(0)\bar{\zeta}(x)|0\rangle\,.$$

This form may be employed to investigate the consequences of assuming charge-conjugation invariance. This symmetry is most easily applied using the Majorana representation for the γ-matrices. In this representation, the charge-conjugation operator C produces

$$C^{-1}\psi(x)C = \psi^\dagger(x)\,.$$

With charge-conjugation invariance, C is a time-independent operator which leaves the vacuum state invariant, $C|0\rangle = |0\rangle$. Show that with

$$j^\mu(x) = \psi^\dagger(x)\alpha^\mu\psi(x)\,,$$

the charge conjugation operation gives

$$C^{-1}j^\mu(x)C = -j^\mu(x)\,.$$

Remembering the tricky extra minus sign that can appear in the time-ordered product of Fermi operators when they are permuted from their standard order, prove that charge-conjugation invariance requires that

$$\Gamma^\mu(p';p) = -(\beta\Gamma^\mu(-p;-p')\beta)^T\,.$$

Show that

$$(\gamma^0 1\gamma^0)^T = 1\,, \qquad (\gamma^0\gamma^\mu\gamma^0)^T = -\gamma^\mu\,, \qquad (\gamma^0\sigma^{\mu\nu}\gamma^0)^T = -\sigma^{\mu\nu}\,.$$

Hence, using the form of $\Gamma^\mu(p';p)$ written in part d) above, prove that charge-conjugation invariance implies that $F_0(k^2) = 0$, independently of current conservation.

Some final comments should be made. Using the reduction form written out in part i) above, it is easy to see that the same formula also applies to the proton-antiproton, current, vacuum matrix element

$$\langle p'\lambda'\bar{p}\bar{\lambda}|j^\mu(0)|0\rangle\,.$$

This entails replacing $p^\mu \to -\bar{p}^\mu$. It is an example of "crossing symmetry" in which a matrix element with some particle in the initial state is related to another matrix element in which the corresponding antiparticle appears in the final state. An antiparticle appears in the final state because the field $\bar{\psi}$ acting to the left creates an antiparticle, not a particle. This matrix element is measured (in the single-photon approximation) by crossing the elastic scattering process $e^- + p \to e^- + p$ to give the electron-positron reaction $e^- + e^+ \to p + \bar{p}$ which is mediated by a virtual photon. Complex conjugation of the matrix element above produces another matrix element $\langle 0|j^\mu(0)|p'\lambda'\bar{p}\bar{\lambda}\rangle$. Since the new matrix element is, in general, complex, so are the form factors $F(k^2)$ or $G(k^2)$. Originally, we had $k = p' - p$, which is a space-like four-vector, $k^2 > 0$. The replacement $p^\mu \to -\bar{p}^\mu$ sets $k = p' + \bar{p}$, which is a time-like four-vector, $k^2 < 0$. Since the form factors for $k^2 < 0$ are complex, an *analytic continuation* from $k^2 > 0$ to $k^2 < 0$ must be made, not a simple "replacement". The form factors have a cut[‡] on the real k^2 axis for $k^2 < 0$. Continuing to one side of this cut gives the outgoing state at $t = +\infty$, $\langle p'\lambda'\bar{p}\bar{\lambda} + |$; continuing to the other side produces the incoming state at $t = -\infty$, $\langle p'\lambda'\bar{p}\bar{\lambda} - |$. The appropriate "$i\epsilon$" prescription follows in a heuristic way from the derivation of the reduction formula which starts from the construction of an outgoing state in terms of a spatial, three-dimensional Fourier transform at a large time $t = +T$. The Fourier transform involves the phase factor $\exp\{ipx\}$ whose time dependence is given by $\exp\{-ip^0T\}$. This is made convergent by taking $p^0 \to p^0 - i\epsilon$. Hence the correct continuation to an outgoing proton-antiproton state is made by the continuation to $k^2 + k^0 i\epsilon$, which, since k^0 is positive, is the same as $k^2 + i\epsilon$.

The threshold of the proton-antiproton reaction appears at $k^2 = -4M^2$. There is a pole at this point in the equations that give the F-form factors in terms of the G-form factors presented in part f . On the other hand, the relations (1) to (5) shown in part c were used to eliminate all other possible structures in terms of $F_1(k^2)$ and $F_2(k^2)$ without dividing by any k^2-dependent quantity. We conclude that any singularities in $F_1(k^2)$ and $F_2(k^2)$ must represent true physical processes, and they are not an artifact of the construction of the form factors. Since there is no physical divergence at the threshold $k^2 = 4M^2$, the form factors $G_E(k^2)$ and $G_M(k^2)$ must obey the kinematical constraint

$$G_E(k^2 = -4M^2) = G_M(k^2 = -4M^2)\,.$$

This is an elementary example of the nature of kinematical singularities and kinematical constraints.

[‡] The cut extends, in fact, only for $k^2 < (2m_\pi)^2$, where m_π is the pion mass; the 2-pion state is the lowest mass intermediate state which can be put into this matrix element.

7. Electron-proton scattering

The cross section for electrons to elastically scatter on protons involves the square of the matrix element whose structure was described in the previous problem. If the proton is initially unpolarized and the spin state of the final scattered proton is not measured, then the cross section involves the tensor

$$t^{\mu\nu} = \frac{1}{2}\sum_{\lambda\lambda'}\langle p'\lambda'|j^\mu|p\lambda\rangle\langle p'\lambda'|j^\nu|p\lambda\rangle^* .$$

Using the notation of the previous problem, show that

$$t^{\mu\nu} = \frac{P^\mu P^\nu}{1+k^2/4M^2}\left[G_E(k)^2 + \frac{k^2}{4M^2}G_M(k)^2\right]$$

$$+ (g^{\mu\nu}k^2 - k^\mu k^\nu)G_M(k^2)^2 .$$

[Hint: The simplest way is to first prove that the result above contains the most general tensor structure consistent with current conservation and then evaluate the polarization sum in the brick-wall frame using the results of part g) of the previous problem.] Note that for an elementary Dirac particle with no internal structure this reduces to

$$t^{\mu\nu}(\text{point}) = P^\mu P^\nu + (g^{\mu\nu}k^2 - k^\mu k^\nu) .$$

8. Vacuum Charge

In this problem, a Dirac field will be examined which has no self interactions, but which does interact with an external potential. A charge density may be induced in the vacuum — the expectation value of the charge density operator in the lowest energy state may be non-vanishing. In some circumstances, the volume integral of this charge density, the total vacuum charge, may be non-vanishing.

The electromagnetic current is given by

$$j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x) = \psi^\dagger(x)\alpha^\mu\psi(x) .$$

It is convenient to employ a Majorana representation because then the charge conjugation operation becomes simply $\psi \to \psi^\dagger$, $\psi^\dagger \to \psi$. Thus, since the fields anticommute while α^μ is a symmetrical matrix, charge conjugation produces

$$j^\mu(x) \to \psi(x)\alpha^\mu\psi^\dagger(x) = -\psi^\dagger(x)\alpha^\mu\psi(x)$$
$$= -j^\mu(x) .$$

Here we have glossed over the fact that the anticommutator produces an ill-defined $\delta(0)$. This problem is eliminated by defining

$$j^\mu(x) = \frac{1}{2}\left[\psi^\dagger(x), \alpha^\mu\psi(x)\right]$$

$$= \frac{1}{2}\alpha^\mu_{\alpha\beta}\left(\psi^\dagger_\alpha(x)\,\psi_\beta(x) - \psi_\beta(x)\,\psi^\dagger_\alpha(x)\right) ,$$

which is automatically odd under the charge-conjugation operation.§ The vacuum current may be calculated from the Green's function. For this problem, it is convenient to use

$$\tilde G(x;x') = \langle iT\psi(x)\,\psi^\dagger(x')\rangle , \tag{1}$$

rather than the usual function $G(x;x')$ which has the Dirac adjoint field $\bar\psi(x')$.

a) Show that

$$\langle j^\mu(x)\rangle = \frac{i}{2}\left\{\lim_{t'\to t+0} + \lim_{t'\to t-0}\right\}\operatorname{tr}\alpha^\mu\tilde G(\mathbf{r},t\,;\,\mathbf{r},t') .$$

With time-independent external fields, the Green's function obeys

$$\left[H - i\frac{\partial}{\partial t}\right]\tilde G(x\,;x') = \delta(x-x') ,$$

where

$$H = \vec\alpha\cdot\frac{1}{i}\nabla + V(\mathbf{r}) ,$$

with $V(\mathbf{r})$ an external potential function that may involve the γ-matrices. Since this is a time-translation invariant system, we may write

$$\tilde G(\mathbf{r},t\,;\,\mathbf{r}',t') = \int\frac{dE}{2\pi}e^{-iE(t-t')}\tilde G(E\,;\,\mathbf{r},\mathbf{r}') , \tag{2}$$

with

$$\left[H - E\right]\tilde G(E\,;\,\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') .$$

To exhibit the structure of $\tilde G(E\,;\,\mathbf{r},\mathbf{r}')$, we introduce a quantum-mechanical bra-ket notation with

$$\frac{1}{i}\nabla\langle\,\mathbf{r}\,| = \langle\,\mathbf{r}\,|\mathbf{p}\,, \qquad \langle\,\mathbf{r}\,|\,\mathbf{r}'\,\rangle = \delta(\mathbf{r}-\mathbf{r}') .$$

(The spinor labels will be left implicit.) Thus

$$\tilde G(E\,;\,\mathbf{r},\mathbf{r}') = \left\langle\,\mathbf{r}\,\middle|\,\frac{1}{H-E}\,\middle|\,\mathbf{r}'\,\right\rangle ,$$

where now

$$H = \vec\alpha\cdot\mathbf{p} + V(\mathbf{r}) ,$$

§ This problem is avoided with the use of a doublet of Majorana field where the current appears as $j^\mu = \psi\alpha^\mu q\psi$ with $q = \tau_2$ an antisymmetrical matrix in the "1 - 2" charge space. However, we shall keep with the more conventional Dirac fields in this problem.

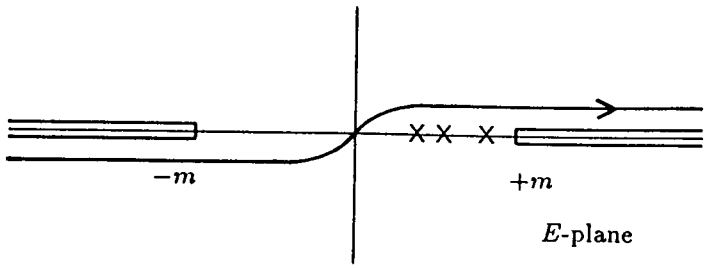

Fig. 7.2. Analytic structure of the Green's function and contour for the integration in Eq. (1).

with $\mathbf{r}$ considered to be an operator. Introducing a complete set

$$\sum_n |\,n\rangle\langle\, n\,| = 1$$

of energy eigenstates

$$H|\,n\rangle = |\,n\rangle E_n\,,$$

this becomes[¶]

$$\widetilde{G}(E\,;\,\mathbf{r},\mathbf{r}') = \sum_n \frac{\langle\,\mathbf{r}\,|\,n\,\rangle\langle\,n\,|\mathbf{r}'\,\rangle}{E_n - E}\,. \tag{3}$$

In general there is a continuum of energy eigenstates for $E_n > m$ and also for $E_n < -m$, and the sum becomes an integral in this region. There may be, however, additional discrete bound-states appearing for $E_n < m$. Thus the analytic structure of $\widetilde{G}(E\,;\,\mathbf{r},\mathbf{r}')$ is as shown in Fig. 7.2. The crosses denote the bound state contributions which give isolated poles in $\widetilde{G}(E\,;\,\mathbf{r},\mathbf{r}')$. The continuous spectrum of states produces the cuts in the energy plane which are shown.

b) Remember the discussion of Chapter 3, Section 2 and explain why the proper contour for the energy integration in Eq. (2) is that shown in Fig. 7.2. This assumes that we are working with the ground-state (vacuum) expectation value in Eq. (1).

c) Let us denote the lowest bound state energy by E_1 and write

$$\langle 0|\psi(x)|E_1\rangle = U_1(\mathbf{r})e^{-iE_1t}\,, \qquad \langle E_1|\psi^\dagger(x)|0\rangle = U_1^\dagger(\mathbf{r})e^{iE_1t}\,.$$

Use the Dirac equation obeyed by ψ to prove that

$$[H - E_1]U_1(\mathbf{r}) = 0\,.$$

[¶] It is important to keep in mind that the bra-ket notation is just a convenience in working with the Green's function – it must not be confused with the physical state space which defines the Green's function in Eq. (1).

Introduce a complete set of intermediate physical states in Eq. (1) and Fourier transform in time to prove that

$$\widetilde{G}(E;\mathbf{r},\mathbf{r}') = \frac{U_1(\mathbf{r})U_1^\dagger(\mathbf{r}')}{E_1 - E} + \widetilde{G}_1(E;\mathbf{r},\mathbf{r}')\,, \tag{4}$$

where $\widetilde{G}_1$ is regular when $E = E_1$. Thus $U_1(\mathbf{r}) = \langle\mathbf{r}|1\rangle$ in the notation of Eq. (3). With an operator notation,

$$\widetilde{G}(E)[H - E] = 1\,.$$

Multiply this equation on the right by U_1 to prove that

$$U_1^\dagger U_1 = 1\,,$$

and

$$\widetilde{G}_1(E)U_1 = 0\,. \tag{5}$$

Define

$$C_1^\dagger = \int (d^3\mathbf{r})\psi^\dagger(x)U_1(\mathbf{r})e^{-iE_1t}\,.$$

Prove that $C_1^\dagger$ is time independent. Using the contour of Fig. 7.2 and Eqs. (4) and (5), prove that

$$\langle 0|C_1^\dagger\psi(x)|0\rangle = 0\,.$$

Since, for this theory with no self-interactions, the Green's functions for an arbitrary number of fields factor into products of two-point Green's functions, this proves that all matrix elements $\langle 0|C_1^\dagger\cdots|0\rangle$ vanish. Hence $\langle 0|C_1^\dagger = 0$. Show by similar considerations that

$$\langle 0|\psi(x)C_1^\dagger|0\rangle = U_1(\mathbf{r})e^{-iE_1t}\,,$$

and again by examining the multipoint Green's functions that

$$|E_1\rangle = C_1^\dagger|0\rangle\,.$$

Use $C_1 = (C_1^\dagger)^\dagger$ and the field anticommutation relations to prove directly that

$$\{C_1, C_1^\dagger\} = 1\,.$$

Explain how this anticommutator confirms the proper normalization of the state $|E_1\rangle$.

d) Suppose that the energy-integration contour is instead routed as shown in Fig. 7.3. Explain why this gives the matrix element

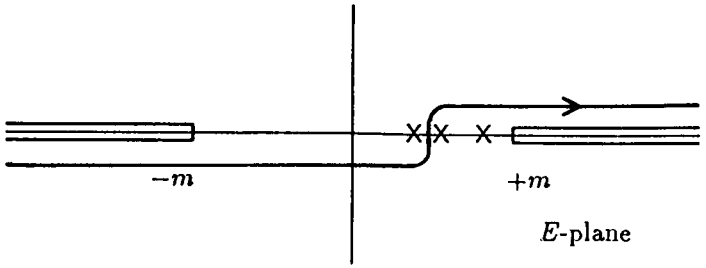

Fig. 7.3. Contour with one filled state.

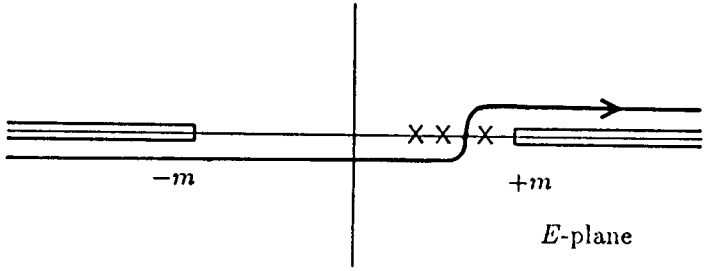

Fig. 7.4. Contour in question.

$$\langle E_1|iT\psi(x)\psi^\dagger(x')|E_1\rangle\,,$$

where $|E_1\rangle$ is the state with a particle bound in the lowest level. What matrix element does the contour in Fig. 7.4 give? For simplicity assume that there is a single lowest (non-degenerate) state. This is the case for a general, spin-dependent Hamiltonian. The lowest bound state for the Coulomb potential is a spin-degenerate doublet.

e) Assuming that the system is in the ground state, use the result in part a) above to prove that the expectation value of the charge density $\rho(x) = j^0(x)$ is given by

$$\langle\rho(x)\rangle = \frac{i}{2}\int_C \frac{dE}{2\pi}\,\mathrm{tr}\,G(E\,;\mathbf{r},\mathbf{r})\,,$$

with the contour C as shown in Fig. 7.5. Using the operator notation, we see that this result may be written as

$$\langle\rho(x)\rangle = \frac{i}{2}\int_{\bar{C}}\frac{dE}{2\pi}\mathrm{tr}\,\langle\,\mathbf{r}\,|\Big\{\frac{1}{H-E}+\frac{1}{H+E}\Big\}|\,\mathbf{r}\,\rangle$$

$$= i\int_{\bar{C}}\frac{dE}{2\pi}\,\mathrm{tr}\,\langle\,\mathbf{r}\,|\frac{H}{H^2-E^2}|\,\mathbf{r}\,\rangle\,,$$

where the contour $\bar{C}$ is as shown in Fig. 7.6. The second equality above

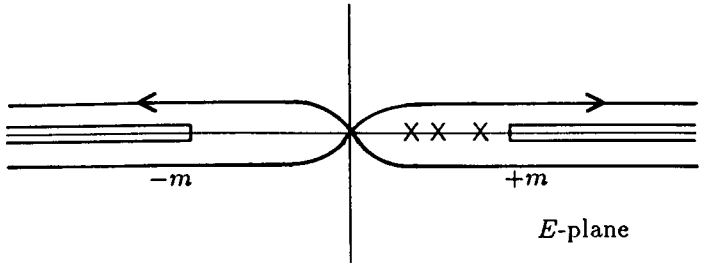

Fig. 7.5. Contour C.

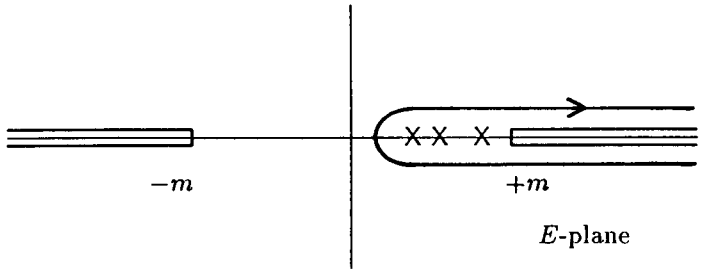

Fig. 7.6. Contour $\bar{C}$.

naively indicates that the integral behaves as $1/E^2$ for large $|E|$ so that quarter circles can be added at infinity to move the contour parallel to the imaginary energy axis as shown in Fig. 7.7. We said "naively" because the density of intermediate energy-eigenvalue states may be very rapidly increasing when E_n becomes large and so the H^2 in the denominator may not be negligible even though the E^2 is large. This is the case in three spatial dimensions where the vacuum charge density is ill-defined and requires renormalization as discussed in the next chapter. This complication will be neglected here.

We are now in the position to discuss an interesting phenomenon. Con-

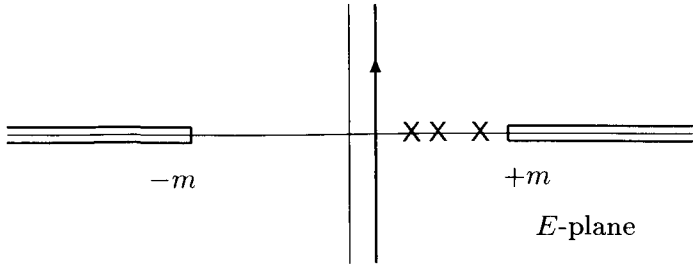

Fig. 7.7. Equivalent contour along imaginary axis.

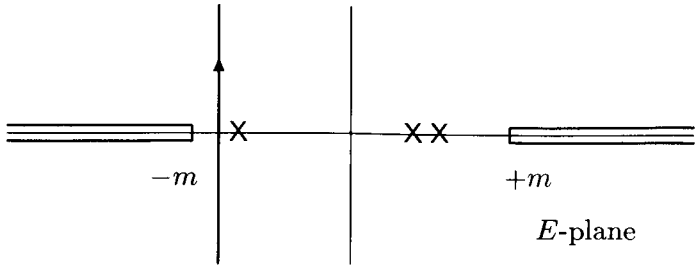

Fig. 7.8. Contour kept to the left of the lowest bound state.

sider a large bare nucleus of charge Z (with no atomic electrons). The vacuum state is perturbed by the electrostatic field of this nucleus, but the induced total vacuum charge vanishes, $\langle Q \rangle = 0$. In the approximation where the nucleus is treated as a point charge, the lowest energy eigenvalue of the Dirac equation in this Coulomb field is given by $E_1 = m\sqrt{1-(Z\alpha)^2}$. In the limit $Z \to \alpha^{-1} \simeq 137$, $E_1 \to 0$ and the Dirac wave function becomes singular at the origin. This singularity, however, is an artifice of the point-nucleus approximation. With a finite-size nucleus which has an extended charge distribution, the Dirac wave function does not become singular at the origin for large Z. However, for a nuclear charge Z a little larger than 137, the lowest bound state energy eigenvalue E_1 does pass below $E_1 = 0$. Here we are imagining that we have some sort of "knob" which we can turn to slowly alter the charge Z. This ideal adiabatic alteration may, perhaps, be crudely achieved experimentally in heavy ion collisions. When Z is increased such that E_1 becomes negative, the original energy integration contour along the imaginary axis which defines the vacuum charge must be moved to the left so that the pole does not cross it; such a crossing would be a discontinuous change. With $E_1 < 0$ this continuation of the original vacuum state no longer gives the state of lowest energy. That state is obtained by "bringing in an electron from infinity" to have a system which consists of the bare nucleus plus one electron. But this is not the state produced by the adiabatic increase of Z. As Z is increased further, the lowest energy level E_1 continues to move to the left and the integration contour must be kept to the left of this pole as shown in Fig. 7.8. There comes a point, however, when E_1 approaches the branch point of the left-hand cut, $E_1 \to -m$. (This happens when $Z \simeq 170$ for a charge distribution of characteristic heavy nuclear size.) When E_1 is only slightly greater than $-m$, the vacuum charge density $\langle \rho(x) \rangle$ is composed of two parts. One part, with total charge -1, is concentrated within about half an electron Compton wavelength $(1/2m)$ around the nucleus. The other part (a latent free positron) with

a total charge of $+1$ extends over a large distance. When E_1 approaches $-m$, the integration contour cannot be further continued to the left of the branch point at $E = -m$. When E_1 dives to the left of the branch point, the system is unstable: Its energy is lowered by electron—positron pair production in which the positron escapes to infinity while the electron is captured in the state with $E_1 < -m$. The positron, which initially was weakly bound and accounted for the extended charge distribution, now escapes with to infinity. The resulting "vacuum" state is now the state with an electron filling the lowest bound-state level, and the vacuum state now has a non-vanishing charge $\langle Q \rangle$. (On account of the spin degeneracy, two positrons will be emitted and two electrons will be captured giving $\langle Q \rangle = -2$.)

9. Irrational Vacuum Charge

The discussion of the vacuum charge is continued by investigating the behavior of Dirac field in a space of one dimension (plus one time dimension). The character of the induced vacuum charge may be strange. This happens if the field is coupled to an external scalar field — a mass-like term which varies with spatial position — if the scalar field has a "kink topology". In this case the total induced vacuum charge may be an irrational number.

a) Consider the scalar field theory in one space one time dimension described by the Lagrange function

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 + m^2 \cos(\phi/\lambda)\,.$$

What are the dimensions of ϕ, m, and λ. Why so? A classical vacuum — the lowest energy state — is obtained for $\phi = 0$. But $\phi = 2\pi n\lambda$ for arbitrary positive or negative integer n are equally valid classical vacua. There are topologically non-trivial kinks or solitons which connect the $\phi = 0$ vacuum with the $\phi = +2\pi\lambda$ or the $\phi = -2\pi\lambda$ vacua. These are the static fields

$$\phi_\pm(x) = 4\lambda \tan^{-1} e^{\pm mx/\lambda}$$

which interpolate between the vacua when x moves from $-\infty$ to $+\infty$. These solutions are topologically non-trivial because altering the boundary condition $\phi = 2\pi n\lambda$ as $|x| \to \infty$ requires an infinite amount of energy — a finite energy density extended over an infinite amount of space. Sketch $\phi_\pm(x)$. Derive the field equation for $\phi(x)$. Prove that $\phi_\pm(x)$ obey it.

For the remainder of this problem we shall work with a Dirac field in one space–one time dimension which interacts with such a scalar kink field. We shall neglect the reaction of the Dirac field back on the scalar

theory and consider $\phi(x)$ to be a prescribed external field. We shall make no restriction on the shape of $\phi(x)$ except to assume that it has a kink topology, namely that

$$x \to +\infty : \phi(x) \to \phi_+ ,$$

while

$$x \to -\infty : \phi(x) \to \phi_- ,$$

with ϕ_+ and ϕ_- differing in sign.

b) Consider the $1+1$ D Dirac field described by the Dirac Hamiltonian

$$H = \alpha p + \beta\phi(x) .$$

In this case there is a zero mode, a zero-energy, *square-integrable* eigenfunction which obeys

$$H\psi_0(x) = 0 .$$

Prove this by explicitly constructing $\psi_0(x)$ up to an overall normalization constant. [Hint: Multiply the equation by $i\alpha$ and consider spinors which diagonalize $i\alpha\beta$.]

c) If a theory has a zero-energy eigenvalue, the vacuum state is not well-defined — we do not know where to place the contour in the energy plane to define the Green's function $G(x, x')$. To remove this ambiguity, we alter the previous Dirac Hamiltonian and consider instead

$$H = \alpha p + \beta\phi - i\alpha\beta m' ,$$

where m' is a constant. The corresponding field energy density operator is given (essentially) by

$$T^{00} = \psi^\dagger H \psi .$$

Show that the m' term breaks the charge-conjugation invariance of the theory. Compute H^2 and show that

$$[\, i\alpha\beta\,,\, H^2\,] = 0 .$$

Hence, by inserting $(i\alpha\beta)^2 = 1$ in the spinor trace used in the expression for $\langle\rho(r)\rangle$ given in Problem 8e, prove that

$$\langle\rho(x)\rangle = i\int_C \frac{dE}{2\pi}\mathrm{tr}\langle x|\Big\{(-i\alpha\beta m')\frac{1}{H^2 - E^2}\Big\}|x\rangle .$$

Note that this changes sign when m' changes sign, reflecting the fact that it is this parameter which breaks the charge-conjugation invariance.

d) Evaluate the 2-dimensional spinor trace in the representation which diagonalizes $-i\alpha\beta \to +1, -1$, and prove that

$$\langle\rho(x)\rangle = -m'\int_{-\infty}^{\infty}\frac{d\mathcal{E}}{2\pi}\langle x|\Big\{\frac{1}{A^\dagger A + m'^2 + \mathcal{E}^2} - \frac{1}{A\,A^\dagger + m'^2 + \mathcal{E}^2}\Big\}|x\rangle ,$$

where

$$A = p + i\phi .$$

e) The spatial integral of the charge density gives the total charge. This expresses the charge as a diagonal coordinate sum, a trace which will be denoted by Tr. Exponentiate the denominators and perform the resulting $\mathcal{E}$-integral to prove that

$$\langle Q\rangle = \frac{m'}{2\sqrt{\pi}}\int_0^\infty \frac{ds}{\sqrt{s}}\, e^{-m'^2 s}\mathrm{Tr}\Big\{e^{-AA^\dagger s} - e^{-A^\dagger A s}\Big\} .$$

Prove that $\langle Q\rangle$ is a topological invariant. That is, consider local variations in $\phi(x)$, variations of $\phi(x)$ only in a limited region of the coordinate x. Such a variation gives a change δA. Noting that

$$A\,F(A^\dagger A) = F(AA^\dagger)\,A ,$$

and using the cyclic symmetry of the trace, prove that $\delta\langle Q\rangle = 0$.

f) To compute the induced vacuum charge $\langle Q\rangle$ we need to evaluate

$$w(s) = \mathrm{Tr}\Big\{e^{-AA^\dagger s} - e^{-A^\dagger A s}\Big\} .$$

To do this, first show that

$$\frac{dw(s)}{ds} = \frac{1}{2}\mathrm{Tr}\Big\{[A^\dagger,\, e^{-AA^\dagger s}A] - [A,\, A^\dagger e^{-AA^\dagger s}]\Big\} .$$

Note that $\mathrm{tr}[A,\, X] = 0$ for a finite matrix. Here we must be more careful. With $A = p + i\phi(x)$, we have, in fact,

$$\begin{aligned}\mathrm{Tr}\,[A,\, X] &= \int_{-\infty}^{\infty} dx\langle x|[A, X]|x\rangle = -i\int_{-\infty}^{\infty} dx\frac{d}{dx}\Big\{\langle x|\,X\,|x\rangle\Big\} \\ &= -i\Big\{\langle +\infty|X|+\infty\rangle - \langle -\infty|X|-\infty\rangle\Big\} .\end{aligned}$$

Show now that

$$\frac{dw(s)}{ds} = [\phi_+ e^{-s\phi_+^2} - \phi_- e^{-s\phi_-^2}]\langle x|e^{-p^2 s}|x\rangle .$$

Introduce a complete set of intermediate momentum eigenstates to prove that

$$\langle x|e^{-p^2 s}|x\rangle = \frac{1}{2\sqrt{\pi s}} .$$

A boundary condition is needed to calculate $w(s)$ from its derivative. Small s corresponds to a short (imaginary) time. At short times the "interaction" $\phi(x)$ is not important and so we have effectively $AA^\dagger \to A^\dagger A \to p^2$. Thus for small s we expect that the two terms in $w(s)$ cancel, giving $w(0) = 0$. Note that the result above for $\langle x|e^{-p^2 s}|x\rangle$ shows that this is a cancellation of singular terms. Assuming that $w(0) = 0$, we have

$$w(s) = \frac{1}{2\sqrt{\pi}} \int_0^s \frac{ds'}{\sqrt{s'}} \left[\phi_+ e^{-s'\phi_+^2} - \phi_- e^{-s'\phi_-^2}\right].$$

To confirm the boundary condition $w(0) = 0$, prove that

$$w(\infty) = \frac{1}{2}\left[\frac{\phi_+}{|\phi_+|} - \frac{\phi_-}{|\phi_-|}\right] = \pm 1 .$$

At first sight this appears to be a contradiction because $AA^\dagger$ and $A^\dagger A$ are positive operators, and so $w(s)$ should vanish when $s \to \infty$. The result is, however, correct and the boundary condition $w(0) = 0$ is verified. It is easy to see that if $w(\infty) = 1$, then the operator $A^\dagger$ has a normalizable zero mode, $A^\dagger f_+(x) = 0$, while if $w(\infty) = -1$, then a square-integrable $f_-(x)$ exists with $Af_-(x) = 0$. Writing out the trace in a basis that includes one or the other of these zero mode functions, it is clear that $w(s)$ contains a constant ± 1 that does not vanish for $s \to \infty$.

g) One can now see the light at the end of the tunnel. The integral representation in part e may now be written as

$$\langle Q\rangle = \frac{m'}{4\pi} \int_0^\infty \frac{ds}{\sqrt{s}}\, e^{-m'^2 s} \int_0^s \frac{ds'}{\sqrt{s'}} \left[\phi_+ e^{-s'\phi_+^2} - \phi_- e^{-s'\phi_-^2}\right].$$

Set $s' = \lambda^2 s$, perform the s integration and then the λ integral to prove that

$$\langle Q\rangle = \frac{1}{2\pi}\left[\frac{\phi_+ m'}{|\phi_+ m'|} \tan^{-1}\left|\frac{\phi_+}{m'}\right| - \frac{\phi_- m'}{|\phi_- m'|} \tan^{-1}\left|\frac{\phi_-}{m'}\right|\right].$$

This is a rather preculiar result, for in general the induced total vacuum charge $\langle Q\rangle$ is an irrational number. One would expect that the charge would take integer values. Note that the value of $\langle Q\rangle$ in the limit where the odd charge-conjugation parameter vanishes, $m' \to 0$, depends upon whether $m' \to 0^+$ or $m' \to 0^-$,

$$\langle Q\rangle \to \frac{1}{4}\frac{m'}{|m'|}\left[\frac{\phi_+}{|\phi_+|} - \frac{\phi_-}{|\phi_-|}\right] \to \pm\frac{1}{2} .$$

Upon further reflection, this phenomenon does not appear so odd. If the limits of the scalar field $\phi(x)$ at $\pm\infty$ are equal, $\phi_+ = \phi_-$, then there

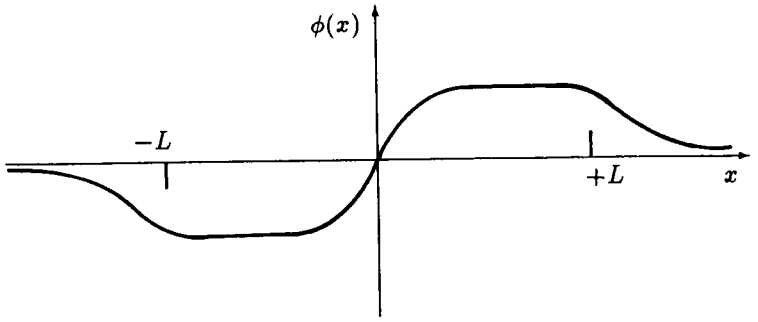

Fig. 7.9. Scalar field with kink topology.

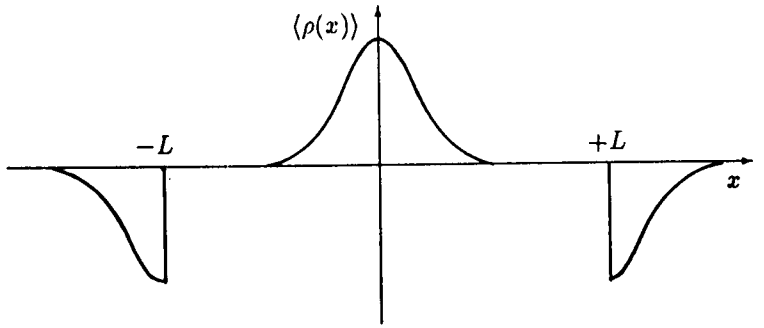

Fig. 7.10. Charge density induced by kink field.

is no charge induced in the vacuum, $\langle Q\rangle = 0$. But suppose that $\phi(x)$ has a kink-like behavior over the very large interval $-L < x < +L$, with $\phi(x)$ rapidly approaching $\phi(+L)$ or $\phi(-L)$ as x departs to the right or left from $x = 0$, with $\phi(+L) > 0$ and $\phi(-L) < 0$. And suppose that for $|x| > L, \phi(x)$ rapidly vanishes. For x small in comparison with L, the limit $L \to \infty$ can be taken, and so the amount of charge in this region may be an irrational number. This amount of charge, however, must be exactly compensated by charge of the opposite sign near $x = \pm L$. A plot of the scalar field appears as shown in Fig. 7.9, while the corresponding plot of the charge density is shown in Fig. 7.10. The total charge is zero. What we have considered corresponds to taking $L \to \infty$, with an irrational amount of charge escaping to infinity.

8
QED

Many of the aspects of relativistic quantum field theory have now been described, using the simple scalar theory to bring out the essential features. With the spin $\frac{1}{2}$ theory also explained, we can turn at last to describe the theory of quantum electrodynamics and several of the experimental tests that verify it. Since to low order, the non-Abelian character of the strong interaction QCD theory plays no important role, there are QCD analogs of QED processes, and these will also be discussed. We begin by explaining the quantization of the electromagnetic field. This will be done in a way which is easily generalized to the non-Abelian case. We shall not discuss many of the traditional topics in QED which are well-described in several texts, but we will illustrate the character of the theory with an example of modern interest and with a traditional topic treated with modern methods. Thus we shall not discuss the traditional topics of relativistic Coulomb and Compton scattering, but we shall instead compute the cross section for $e^+ + e^- \to \mu^+ + \mu^-$ which is essentially the same as the quark-antiquark production in high-energy electron-positron collisions. The effects of vacuum polarization will be treated with modern methods, and it will be related to the e^+e^- cross section. The lowest order electron self energy and vertex functions will be calculated and their properties examined — one cannot avoid computing the magnetic moment of the electron. The important infrared properties of QED will be examined by considering $e^+e^- \to \mu^+\mu^-\gamma$ whose hadronic analog will also be described. This survey of the basic ideas of QED will be concluded by a calculation of the Lamb shift using dimensional regularization which simplifies the work.

8.1 Field quantization

The Dirac Lagrange function

$$\mathcal{L}_D = -\bar{\psi}\Big[\gamma\frac{1}{i}\partial + m\Big]\psi \tag{8.1.1}$$

is invariant under the constant phase rotation $\psi \to e^{i\theta}\psi$, $\bar{\psi} \to e^{-i\theta}\bar{\psi}$. Since the field at all space-time points undergoes the same change, this is called a "global" invariance. The electromagnetic field may be introduced by promoting this global invariance to a local, gauge invariance. This invariance under a phase rotation which is independently performed at each space-time point,

$$\psi(x) \to \psi'(x) = e^{i\lambda(x)}\psi(x)\,, \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\lambda(x)}\bar{\psi}(x)\,, \tag{8.1.2}$$

requires the introduction of a new field, the vector potential $A_\mu(x)$. The invariance is achieved by replacing the ordinary derivative with a gauge-covariant derivative

$$D_\mu = \partial_\mu - iA_\mu(x) \tag{8.1.3}$$

which transforms as

$$D_\mu \to D'_\mu = e^{i\lambda(x)}D_\mu e^{-i\lambda(x)}\,. \tag{8.1.4}$$

This requires that the vector potential acquires a local gauge transformation,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x)\,. \tag{8.1.5}$$

In geometrical terms, the vector potential $A_\mu(x)$ appears as a "connection" whose addition to the ordinary derivative produces the covariant derivative D_μ. Different components of this derivative do not commute. In fact

$$i[D_\mu, D_\nu] = F_{\mu\nu}\,, \tag{8.1.6}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\,. \tag{8.1.7}$$

The field strength tensor $F_{\mu\nu}$ is unchanged by the gauge transformation (8.1.5). In the geometrical analogy, $F_{\mu\nu}(x)$ may be thought of as a curvature tensor.

The original Dirac Lagrange function plus the new coupling to the vector potential may be written as

$$\mathcal{L} = \mathcal{L}_D + j^\mu A_\mu\,, \tag{8.1.8}$$

with the current j^μ given by

$$j^\mu = \bar{\psi}\gamma^\mu\psi\,. \tag{8.1.9}$$

To obtain the gauge invariance, the vector potential must be a dynamical variable. A first-order formulation is needed to describe a quantum theory. Hence $F^{\mu\nu}$ and A_μ must be used as independent fields. The Maxwell Lagrange function for these fields is

$$\mathcal{L}_M = -\frac{1}{2e^2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu \,. \tag{8.1.10}$$

Only the combination $\partial_\mu A_\nu - \partial_\nu A_\mu$ can appear by virtue of the gauge invariance. Here J^μ denotes a generic current — it can represent an externally applied, classical current or the previous Dirac current or the sum of both currents. We shall return later to the interacting theory of coupled Maxwell and Dirac fields, but for now we concentrate on the quantum theory described by the Maxwell Lagrange function $\mathcal{L}_M$. Requiring that the action be stationary under field variations produces the field equations. The variation of $F^{\mu\nu}$ gives

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{8.1.11}$$

and the variation of A_μ yields

$$\partial_\nu F^{\mu\nu} = e^2 J^\mu \,. \tag{8.1.12}$$

These are just the Maxwell equations except that a somewhat unconventional normalization has been adopted. With this normalization, the conventional electric and magnetic fields are given by $E^k = F^{0k}/e$, $B^k = \frac{1}{2}\epsilon^{klm}F_{lm}/e$.

1 Radiation gauge

To derive the quantum theory, the canonical, "$p\dot{q}$" structure of the Lagrange function must be displayed. The time derivatives are exhibited in writing

$$\mathcal{L}_M = -\frac{1}{e^2}F^{0k}\partial_0 A_k + \dots \,. \tag{8.1.13}$$

We see that $\partial_0 A_0$ does not appear; A_0 is not a dynamical variable but rather a Lagrange multiplier field that enforces a constraint. The nature of this constraint depends upon a gauge choice. The freedom of making gauge transformations $A_\mu \to A_\mu + \partial_\mu \lambda$ implies that the field equations do not determine all the components of the vector potential. To obtain a well-defined theory, the gauge invariance must be broken by choosing some particular gauge. As will be seen, physical results will not depend upon this gauge choice. To make the canonical structure clear, we shall choose the radiation gauge,

$$\nabla \cdot \mathbf{A} = 0 \,. \tag{8.1.14}$$

With this gauge choice, the time component of the Maxwell equation (8.1.12) becomes

$$-\nabla^2 A^0 = e^2 J^0 \,. \tag{8.1.15}$$

This explicitly shows that A^0 is not a dynamical variable, but rather is determined by the charge density at the same time,

$$A^0(\mathbf{r},t) = \int (d^3\mathbf{r}')\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}e^2 J^0(\mathbf{r}',t) \,. \tag{8.1.16}$$

Writing the electric field in terms of transverse and longitudinal parts,

$$F^{0k} = {}^{(T)}F^{0k} + {}^{(L)}F^{0k} \,, \tag{8.1.17}$$

where

$$\partial_k {}^{(T)}F^{0k} = 0 \,, \tag{8.1.18}$$

we see that Eq. (8.1.11) determines the longitudinal piece

$${}^{(L)}F^{0k} = -\partial^k A^0 \,. \tag{8.1.19}$$

By Eq. (8.1.16), this longitudinal piece is completely determined in terms of the charge density J^0; it is determined by Coulomb's law. The spatial parts of Eq. (8.1.11),

$$F_{kl} = \partial_k A_l - \partial_l A_k \,, \tag{8.1.20}$$

simply define the magnetic field in terms of the vector potential. Discarding a total spatial divergence which does not contribute to the action,

$$\mathcal{L}_M = -\frac{1}{e^2}{}^{(T)}F^{0k}\partial_0 A_k - \frac{1}{2e^2}({}^{(T)}F^{0k})^2 - \frac{1}{4e^2}(F_{kl})^2 - \frac{1}{2}J^0 A^0 + J^k A_k \,, \tag{8.1.21}$$

where F_{kl} is defined by Eq. (8.1.20). Note that the charge density term gives the contribution to the Hamiltonian

$$\begin{aligned} H_C &= \int (d^3\mathbf{r})\frac{1}{2}J^0 A^0 \\ &= \frac{e^2}{2}\int (d^3\mathbf{r})(d^3\mathbf{r}')J^0(\mathbf{r},t)\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}J^0(\mathbf{r}',t) \,. \end{aligned} \tag{8.1.22}$$

This describes the Coulomb self interaction of the charge density and identifies e as the electric charge.

Having arrived at a canonical structure, we can simply write down the functional integral representation which produces the quantum Maxwell theory,

$$Z[J] = \int [d\,{}^{(T)}A_k][d\,{}^{(T)}F^{0k}] \exp\left\{ i\int \mathcal{L}_M \right\} \,. \tag{8.1.23}$$

Here the functional integration is over all transverse, 3-vector fields ${}^{(T)}A_k$, ${}^{(T)}F^{0k}$. The field ${}^{(T)}F^{0k}$ may be integrated out by the familiar process of completing the square. This gives

$$Z[J] = \int [d\,{}^{(T)}A_k] \exp\left\{ i \int \mathcal{L}_M \right\}, \tag{8.1.24}$$

where the Lagrange function now appears in the second-order form

$$\mathcal{L}_M = \frac{1}{2e^2}({}^{(T)}F^{0k})^2 - \frac{1}{4e^2}(F_{kl})^2 - \frac{1}{2}J^0A^0 + J^{k\,(T)}A^k, \tag{8.1.25}$$

in which the $F^{\mu\nu}$ is now just a short-hand notation for vector potential combinations,

$${}^{(T)}F^{0k} = -\partial_0\,{}^{(T)}A^k, \quad F_{kl} = \partial_k\,{}^{(T)}A_l - \partial_l\,{}^{(T)}A_k. \tag{8.1.26}$$

Omitting total derivatives which do not contribute to the action enables Eq. (8.1.25) to be written as

$$\mathcal{L}_M = -\frac{1}{2e^2}\,{}^{(T)}A_k(-\partial^2)\,{}^{(T)}A_k - \frac{1}{2}J^0A^0 + J^{k\,(T)}A^k. \tag{8.1.27}$$

Again we may complete the square to finish the calculation of the functional integral. Remembering that A^0 is defined by Eq. (8.1.16), we find that

$$Z[J] = \exp\left\{ \frac{ie^2}{2} \int (d^4x)(d^4x') J^\mu(x) D^{(R)}_{\mu\nu}(x-x') J^\nu(x') \right\}, \tag{8.1.28}$$

in which $D^{(R)}_{\mu\nu}(x-x')$ is the radiation-gauge propagator

$$D^{(R)}_{kl}(x-x') = \int \frac{(d^4k)}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 - i\epsilon}(\delta_{kl} - \hat{k}_k\hat{k}_l), \tag{8.1.29}$$

$$D^{(R)}_{00}(x-x') = -\int \frac{(d^4k)}{(2\pi)^4} \frac{e^{ik(x-x')}}{\mathbf{k}^2}, \tag{8.1.30}$$

with the other $(0,k)$ components vanishing. The transverse projection $(\delta_{kl} - \hat{k}_k\hat{k}_l)$ appears because only transverse functions ${}^{(T)}A_k$ are used in the measure of the functional integration. The k^0-integral in $D^{(R)}_{00}$ produces $\delta(x^0 - x'^0)$ while the remaining spatial yields the instantaneous Coulomb Green's function.

With $J^0 = 0$, Eq. (8.1.28) becomes the generating functional for the multipoint photon Green's functions,

$$Z[J] = \left\langle \exp\left\{ i \int JA \right\} \right\rangle. \tag{8.1.31}$$

In particular,

$$\langle iTA_k(x)A_l(x')\rangle = e^2 D^{(R)}_{kl}(x-x'). \tag{8.1.32}$$

As was previously fully described for spin 0 and $\frac{1}{2}$, the propagator structure describes the action of a field on the vacuum state. Inserting a complete set of intermediate states on the left-hand side of Eq. (8.1.32) and comparing with the explicit form (8.1.29) of the propagator shows that the vector potential only connects the vacuum state to a single-particle, massless state, the state of one photon. Since the vector A_k is transverse, it has two independent components. Hence there are two independent photon states $|k\epsilon\rangle$ of the same momentum, states with two different photon polarizations. The same symbol ϵ will be used both to label the polarization state and to be (a possibly complex) unit vector which is orthogonal to the photon momentum $\mathbf{k}$,

$$\vec{\epsilon}^{\,*} \cdot \vec{\epsilon} = 1, \qquad \vec{\epsilon} \cdot \mathbf{k} = 0. \tag{8.1.33}$$

Two orthogonal polarization vectors $\epsilon^{(1)}$ and $\epsilon^{(2)}$ span the space perpendicular to $\hat{\mathbf{k}}$, and so

$$\sum\nolimits_\epsilon \epsilon^*_l \epsilon_m = \epsilon^{(1)*}_l \epsilon^{(1)}_m + \epsilon^{(2)*}_l \epsilon^{(2)}_m = \delta_{lm} - \hat{k}_l\hat{k}_m. \tag{8.1.34}$$

As usual, we use the invariant normalization

$$\langle k\epsilon|k'\epsilon'\rangle = \vec{\epsilon}^{\,*} \cdot \vec{\epsilon}'\, 2k^0(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'). \tag{8.1.35}$$

Then, examining the sum over intermediate states in the propagator, we conclude that the photon wave functions are given by

$$\langle 0|A_l(x)|k\epsilon\rangle = e\epsilon_l\, e^{ikx}. \tag{8.1.36}$$

Consider a photon moving in the positive z-direction. Then $\vec{\epsilon}^{\,(1)} = \hat{x}$ and $\vec{\epsilon}^{\,(2)} = \hat{y}$ describe two independent states of linear polarization. Orthogonal combinations of these states describe other possible states of polarization. In particular, the complex polarization vectors $\vec{\epsilon}^{\,(\pm)} = (\hat{x} \pm i\hat{y})/\sqrt{2}$ describe two orthogonal states of circular polarization. These are helicity eigenstates that have unit angular momentum along or opposed to the propagation direction. This is easily verified. Just as in the discussion of the helicity eigenstates of a spin $\frac{1}{2}$ particle, it is easy to check that the angular momentum operator $\mathbf{J}$ generates rotations according to

$$[A_k(x),\, J_l] = \left(\mathbf{r} \times \frac{1}{i}\nabla\right)_l A_k(x) + i\epsilon_{klm}\, A_m(x), \tag{8.1.37}$$

which gives

$$\langle 0|A_k(0)\,\hat{\mathbf{k}} \cdot \mathbf{J}|k\epsilon\rangle = e\, i\epsilon_{klm}\, \hat{k}_l\, \epsilon_m. \tag{8.1.38}$$

For propagation along the z-axis, and for circular polarization, this entails $i\hat{z} \times \vec{\epsilon}^{\,(\pm)} = \pm\vec{\epsilon}^{\,(\pm)}$. Hence

$$\hat{\mathbf{k}} \cdot \mathbf{J}|k\epsilon^{(\pm)}\rangle = \pm|k\epsilon^{(\pm)}\rangle. \tag{8.1.39}$$

Clearly a similar construction can be made for any direction of propagation. We see that $\epsilon^{(+)}$ describes a right-handed photon of positive helicity — unit spin aligned along the direction of motion — while $\epsilon^{(-)}$ describes a left-handed photon of negative helicity.

The radiation field has only transverse spatial components. We may write, however,

$$\langle 0|A_\mu(x)|k\epsilon\rangle = e\epsilon_\mu\, e^{ikx}\,, \tag{8.1.40}$$

with $\epsilon_0 = 0$. Introducing a unit vector in the time direction, $n^\mu = (1,0)$, and noting that $k^0 = |\mathbf{k}| = -nk$, the previous polarization sum (8.1.34) is the same as

$$\sum\nolimits_\epsilon \epsilon^*_\mu \epsilon_\nu = g_{\mu\nu} - \frac{k_\mu k_\nu}{(nk)^2} - \frac{n_\mu k_\nu + k_\mu n_\nu}{nk}\,. \tag{8.1.41}$$

It is straightforward to check that this expression has no time-time or time-space components while the space-space components are those given in Eq. (8.1.34). The vector potential always appears contracted with a conserved current; it always appears in the form $A_\mu J^\mu$, with $\partial_\mu J^\mu = 0$. Therefore, when summing over photon polarizations as in computing a total cross section, we may make the replacement

$$\sum\nolimits_\epsilon \epsilon^*_\mu \epsilon_\nu \to g_{\mu\nu}\,, \tag{8.1.42}$$

for the omitted terms contain a factor of k_μ or k_ν which vanishes when contracted with (the Fourier transform of) a conserved current.

2 *Covariant gauges*

An example has just been given of how gauge invariance may be used to make the Lorentz invariance of quantum electrodynamics explicit. The Lorentz invariance of the theory was apparently broken by the radiation gauge choice $\nabla \cdot \mathbf{A} = 0$ which is imposed in some particular coordinate frame. But, as will be seen, the gauge invariance of the original theory implies that the Lorentz invariance still holds for physically measurable quantities. The Lorentz invariance can be made manifest by exploiting the gauge invariance of the functional integral formulation of the theory. To see how this goes, we return to representation given by Eqs. (8.1.24), (8.1.25), and (8.1.26). First we note that the Coulomb law constraint (8.1.16) may be expressed by

$$\exp\left\{\frac{ie^2}{2}\int\int J^0 D^{(R)}_{00} J^0\right\} = \int [dA^0] \exp\left\{i\int\left[\frac{1}{2e^2}A^0(-\nabla^2)A^0 - J^0A^0\right]\right\}\,. \tag{8.1.43}$$

This follows as usual by completing the square and then performing the Gaussian functional integral. In view of Eq. (8.1.19),

$$\int A^0(-\nabla^2)A^0 = \int ({}^{(L)}F^{0k})^2\,, \tag{8.1.44}$$

and using the representation (8.1.43), the second order Lagrange function (8.1.25) is altered to read

$$\begin{aligned}\mathcal{L}_M \to \frac{1}{2e^2}({}^{(L)}F^{0k})^2 + \frac{1}{2e^2}({}^{(T)}F^{0k})^2 - \frac{1}{4e^2}(F_{kl})^2 - J^0A^0 + J^kA^k \\ = -\frac{1}{4e^2}(F_{\mu\nu})^2 + J^\mu A_\mu\,,\end{aligned} \tag{8.1.45}$$

where the field strength is considered to be defined in terms of the vector potential,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu\,. \tag{8.1.46}$$

Thus, with this new definition of $\mathcal{L}_M$,

$$Z[J] = \int [d^{(T)}A^k][dA^0] \exp\left\{i\int \mathcal{L}_M\right\}\,. \tag{8.1.47}$$

A general 3-vector may be written in terms of its transverse and longitudinal parts,*

$$A_k = {}^{(T)}A_k + \partial_k\phi\,. \tag{8.1.48}$$

Noting that

$$\int [d\phi]\delta[\nabla\cdot\mathbf{A}] = \int [d\phi]\delta[\nabla^2\phi] = \frac{1}{\mathrm{Det}[-\nabla^2]}\,, \tag{8.1.49}$$

the functional integration over transverse vector fields may be written as an integral over all vector fields with a constraint,

$$\int [d^{(T)}A^k]\cdots = \int [dA^k]\mathrm{Det}[-\nabla^2]\delta[\nabla\cdot\mathbf{A}]\cdots\,. \tag{8.1.50}$$

The factor $\mathrm{Det}[-\nabla^2]$ is the functional determinant the functional Jacobian — in changing variables from ϕ to $\nabla^2\phi$. It is not a well-defined quantity. Since it is simply an overall, numerical factor, we could hide $\mathrm{Det}[-\nabla^2]$ in the definition of the measure $[dA^k]$. The overall normalization of the functional integral is, of course, prescribed by the condition that $Z[0] = 1$. As we shall see, there is, however, a formal advantage of displaying this albeit ill-defined determinant. Writing simply

* The heavy-handed treatment which follows could be avoided. It is presented because a similar treatment is needed for the non-Abelian case.

$[dA] = [dA^k][dA^0]$, Eq. (8.1.47) now appears as a functional integral over all the components of the 4-vector potential A_μ,

$$Z[J] = \int [dA] \mathrm{Det}[-\nabla^2] \delta[\nabla \cdot \mathbf{A}] \exp\left\{ i \int \mathcal{L}_M \right\} . \tag{8.1.51}$$

Gauge invariance may be exploited to put this expression in a form which is manifestly Lorentz invariant. With a conserved current, $\partial_\mu J^\mu = 0$, the action is invariant under the gauge transformation

$$A \to A_\mu^{(\lambda)} = A_\mu + \partial_\mu \lambda . \tag{8.1.52}$$

The gauge invariance can be utilized by observing that

$$\int [d\lambda] \mathrm{Det}[-\partial^2] \delta[\partial^\mu A_\mu^{(\lambda)} - \chi] = 1 \tag{8.1.53}$$

for an arbitrary function $\chi(x)$. Since the functional integration variable is a dummy variable, the integration in Eq. (8.1.51) may be done over $A_\mu^{(-\lambda)} = A_\mu - \partial_\mu \lambda$ rather than A_μ, which removes the gauge transformation in Eq. (8.1.52). Hence

$$Z[J] = \int [d\lambda] \int [dA^{(-\lambda)}] \mathrm{Det}[-\partial^2] \delta[\partial A - \chi]$$
$$\mathrm{Det}[-\nabla^2] \delta[\nabla \cdot \mathbf{A}^{(-\lambda)}] \exp\left\{ i \int \mathcal{L}_M(A^{(-\lambda)}) \right\} . \tag{8.1.54}$$

A functional integration measure is not altered by a translation and so $[dA^{(-\lambda)}] = [dA]$. Moreover, the Lagrange function is gauge invariant, $\mathcal{L}_M(A^{(-\lambda)}) = \mathcal{L}_M(A)$. Thus the only λ dependence appears in $\delta[\nabla \cdot A^{(-\lambda)}]$, and the $[d\lambda]$ integral is trivial,

$$\int [d\lambda] \mathrm{Det}[-\nabla^2] \delta[\nabla \cdot \mathbf{A} - \nabla^2 \lambda] = 1 . \tag{8.1.55}$$

Accordingly,

$$Z[J] = \int [dA] \mathrm{Det}[-\partial^2] \delta[\partial A - \chi] \exp\left\{ i \int \mathcal{L}_M \right\} . \tag{8.1.56}$$

The gauge-fixing constraint $\delta[\partial A - \chi]$ in the measure is necessary to secure a well-defined functional integral. The invariance of the remainder of the integrand under the gauge transformation $A \to A^{(\lambda)}$ means that it provides no damping for the infinite-dimensional functional integration region that corresponds to gauge transformations. Without the gauge-fixing constraint, the functional integral would diverge badly. To get an intuitive understanding of this, consider replacing the huge, infinite-dimensional function space of the vector potentials $A_\mu(x)$ by a simple two-dimensional space as shown in Fig. 8.1. A line running upward represents the "gauge orbit" of the vector potential $A_\mu^{(\lambda)}$ as the gauge transformation

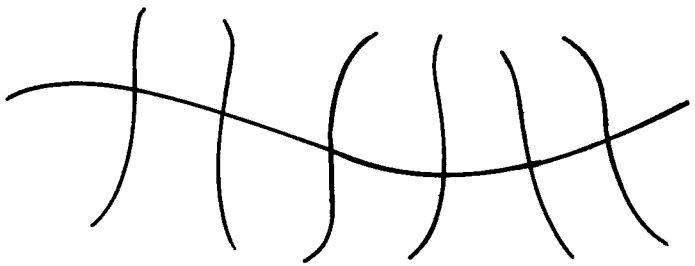

Fig. 8.1. Two-dimensional analog of gauge orbits of various vector potential fields intersected by the gauge-fixing surface.

parameter λ varies. By gauge invariance, all points along these lines are equivalent — each point on a given line gives the same physical result. Therefore, rather than dealing with all vector potential functions, one need use only some representative point along each gauge orbit. In the two-dimensional picture, such a set of points is obtained at the intersection where the gauge orbits cross the curve running from left to right. This curve corresponds to the surface in function space that is described by the constraint $\partial^\mu A_\mu(x) = \chi(x)$. The functional integration volume in this restricted surface, weighted by the action exponential, is finite.

If we were to set $\chi = 0$, we would have a manifestly Lorentz invariant form. This, however, is a cumbersome form since it involves the functional integration with the constraint $\delta[\partial A]$. We shall instead exploit the fact that the integral is independent of the arbitrary function $\chi(x)$ and integrate over this function with a convenient weight. We shall take this weight to be a Gaussian with

$$\int [d\chi] \exp\left\{ \frac{i}{2\xi} \int \chi^2 \right\} = 1 , \tag{8.1.57}$$

where, as usual, we are cavalier about overall numerical factors. Therefore, with the same neglect of $\mathrm{Det}[-\partial^2]$,

$$Z[J] = \int [d\chi] \int [dA] \delta[\partial A - \chi] \exp\left\{ \frac{i}{2\xi} \int \chi^2 \right\} \exp\left\{ i \int \mathcal{L}_M \right\}$$
$$= \int [dA] \exp\left\{ i \int \mathcal{L}_{M,F} \right\} , \tag{8.1.58}$$

where

$$\mathcal{L}_{M,F} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + J^\mu A_\mu . \tag{8.1.59}$$

This is a "gauge-fixed" Lagrange function and ξ is the gauge-fixing parameter.

Writing the vector potentials in the action in terms of Fourier transforms, the differential operator structure in Eq. (8.1.59) becomes proportional to

$$D^{-1}(k)_{\mu\nu} = g_{\mu\nu}k^2 - \left(1 + \frac{e^2}{\xi}\right) k_\mu k_\nu \,. \tag{8.1.60}$$

This identifies the Fourier transform $D_{\mu\nu}(k)$ of the propagator,

$$D^{-1}(k)_{\mu\lambda} D^{\lambda\nu}(k) = \delta^\nu_\mu \,. \tag{8.1.61}$$

A little calculation shows that

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left[g_{\mu\nu} - \left(1 + \frac{\xi}{e^2}\right)\frac{k_\mu k_\nu}{k^2}\right] . \tag{8.1.62}$$

Note how the introduction of the gauge fixing is indeed needed to obtain a well-defined photon propagator. The limit $\xi \to \infty$ removes the gauge fixing. But in this limit, Eq. (8.1.60) becomes proportional to a projection operator that has no inverse as is evidenced by the divergence of Eq. (8.1.62). The by now familiar computation expresses the generating functional as

$$Z[J] = \exp\left\{\frac{ie^2}{2}\int\int J^\mu D_{\mu\nu} J^\nu\right\} , \tag{8.1.63}$$

and identifies the photon propagator as

$$\langle iT A_\mu(x) A_\nu(x')\rangle = e^2 D_{\mu\nu}(x - x') \,. \tag{8.1.64}$$

Although we use an operator and vector space notation here, it should be borne in mind that we have gone outside the canonical framework in the manipulations that have been made with the functional integral representation. The bracket notation $\langle X\rangle$ is simply a short-hand notation for the functional integral with an extra factor of X in the integrand (with the sources usually set to zero). That a true operator representation is not present in Eq. (8.1.64) is evident from the fact that it does not define a positive-definite function. Nonetheless, since photons are always detected by the coupling of A_μ to a conserved current J^μ, we may use this formulation to construct single-photon states in the same manner that was used for the physical radiation gauge.

We are finally at the stage where we can include the coupling to the Dirac field which was our original motivation for the introduction of the photon field. Since our previous work is valid for any sort of current coupling to the photon field, we can simply write down the functional integral representation for the coupled Maxwell-Dirac theory, the theory of quantum electrodynamics. The generating functional is given by

$$Z[J,\,\eta,\,\bar\eta] = \int [dA][d\psi][d\bar\psi] \exp\left\{i\int \mathcal{L}_{QED}\right\} , \tag{8.1.65}$$

where

$$\mathcal{L}_{QED} = -\frac{1}{4e_0^2}F_{\mu\nu}^2 + \frac{1}{2\xi_0}(\partial_\mu A^\mu)^2 - \bar\psi\left[\gamma\left(\frac{1}{i}\partial - A\right) + m_0\right]\psi + J^\mu A_\mu + \bar\psi\eta_0 + \bar\eta_0\psi \,. \tag{8.1.66}$$

We have shown that the non-covariant, canonical form may be transformed to Lorentz invariant structure which we have written here. This requires that the current to which A_μ is coupled be conserved. This, in fact, is not the case here because the external sources of the Dirac field $\eta_0\,, \bar\eta_0$ break the phase symmetry. This, however, causes no real problem. The only ultimate role of the sources is to generate physically measurable amplitudes. These quantities are gauge invariant and so our method applies to their calculation. The ingredients of which the physical quantities are composed may, however, not be gauge invariant. The overall gauge invariance of a measurable amplitude often can serve as a useful check that the calculation has been correctly performed. In particular, physical amplitudes must be independent of the gauge fixing parameter ξ_0. Since this is an interacting theory, the coupling parameters (including the sources) must be renormalized, and so they are written with a 0 subscript to emphasize that these are the bare quantities. An exception is the external current J^μ which, as we shall see, is not renormalized. The gauge-fixing parameter ξ also is not renormalized, but it carries a mass dimension away from $n = 4$, and so we need to distinguish the dimensional ξ_0 from the dimensionless ξ.

3 *Ward identity*

The underlying gauge invariance of the theory is expressed by the generalized Ward identity which we now derive. So as not to encumber the notation, we shall for the moment drop the 0-subscript notation for bare quantities and write the theory in $n = 4$ dimensions rather than the $n \neq 4$ dimensions needed to regulate the theory. The functional integration fields are dummy variables and they may be replaced by their gauge transforms

$$\begin{aligned} A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x)\,, \\ \psi(x) &\to \psi'(x) = e^{i\lambda(x)}\psi(x)\,, \\ \bar\psi(x) &\to \bar\psi'(x) = e^{-i\lambda(x)}\bar\psi(x)\,. \end{aligned} \tag{8.1.67}$$

Since $A_\mu(x)$ and $A'_\mu(x)$ differ only by a functional translation, $[dA] = [dA']$. Since ψ and $\bar\psi$ undergo opposite phase rotations, $[d\psi][d\bar\psi] = [d\psi'][d\bar\psi']$. Except for the source terms and the gauge-fixing term, the

QED action is not altered by this transformation. Therefore

$$Z[J^\mu, \eta, \bar\eta] = \int [dA][d\psi][d\bar\psi] \exp\Big\{ i \int \Big[-\frac{1}{4e^2} F^2 + \frac{1}{2\xi}(\partial A)^2 - \bar\psi\Big(\gamma\Big(\frac{1}{i}\partial - A\Big) + m\Big)\psi + \frac{1}{\xi}\partial A \partial^2\lambda + \frac{1}{2\xi}(\partial^2\lambda)^2 + JA + J\partial\lambda + \bar\psi\eta e^{-i\lambda} + e^{i\lambda}\bar\eta\psi\Big]\Big\}$$
$$= Z\Big[J^\mu - \frac{1}{\xi}\partial^\mu\partial^2\lambda, \eta e^{-i\lambda}, \bar\eta e^{i\lambda}\Big] \exp\Big\{ i \int \Big[\frac{1}{2\xi}(\partial^2\lambda)^2 + J\partial\lambda\Big]\Big\}. \tag{8.1.68}$$

It is convenient to express this result in terms of the effective action $W[J, \eta, \bar\eta]$, the generating functional of connected graphs given by $Z = \exp\{iW\}$. Thus

$$W[J, \eta, \bar\eta] = W[J', \eta', \bar\eta'] + \int \Big[\frac{1}{2\xi}(\partial^2\lambda)^2 + J\partial\lambda\Big], \tag{8.1.69}$$

where

$$J' = J - \frac{1}{\xi}\partial\partial^2\lambda, \tag{8.1.70}$$

and

$$\eta' = \eta e^{-i\lambda}, \qquad \bar\eta' = \bar\eta e^{i\lambda}. \tag{8.1.71}$$

Variational derivatives of $W[J, \eta, \bar\eta]$ with respect to the sources produce field expectation values in the presence of the sources,

$$\frac{\delta W}{\delta J^\mu(x)} = \langle A_\mu(x)\rangle, \tag{8.1.72}$$

and

$$\frac{\delta_r W}{\delta\eta(x)} = \langle\bar\psi(x)\rangle, \qquad \frac{\delta_l W}{\delta\bar\eta(x)} = \langle\psi(x)\rangle. \tag{8.1.73}$$

The derivatives with the anticommuting, Grassmann sources η and $\bar\eta$ need an ordering specification. The right derivative $\delta_r/\delta\eta$ entails the variation with respect to η, with $\delta\eta$ anticommuted past all Fermionic variables so as to appear on the extreme right; it is then deleted. The left derivative $\delta_l/\delta\bar\eta$ is defined with the opposite convention with the variation $\delta\bar\eta$ moved to the far left. The same derivatives of $W[J', \eta', \bar\eta']$ with respect to J', η', and $\bar\eta'$ produce $\langle A\rangle'$, $\langle\bar\psi\rangle'$, and $\langle\psi\rangle'$. The identity (8.1.69) informs us that

$$\langle A\rangle = \langle A\rangle' + \partial\lambda, \tag{8.1.74}$$

and

$$\langle\bar\psi\rangle = \langle\bar\psi\rangle' e^{-i\lambda}, \qquad \langle\psi\rangle = \langle\psi\rangle' e^{i\lambda}. \tag{8.1.75}$$

The invariance is concisely put in terms of the Legendre transform

$$\Gamma[\langle A\rangle, \langle\bar\psi\rangle, \langle\psi\rangle] = W[J, \eta, \bar\eta] - \int [J\langle A\rangle + \langle\bar\psi\rangle\eta + \bar\eta\langle\psi\rangle]. \tag{8.1.76}$$

It obeys

$$\frac{\delta\Gamma}{\delta\langle A\rangle} = -J, \tag{8.1.77}$$

and

$$\frac{\delta_l\Gamma}{\delta\langle\bar\psi\rangle} = -\eta, \qquad \frac{\delta_r\Gamma}{\delta\langle\psi\rangle} = -\bar\eta. \tag{8.1.78}$$

The same construction for $\Gamma[\langle A\rangle', \langle\bar\psi\rangle', \langle\psi\rangle']$ in terms of $W[J', \eta', \bar\eta']$, using the relation between the primed and unprimed sources and fields, gives

$$\Gamma[\langle A\rangle', \langle\bar\psi\rangle', \langle\psi\rangle'] = W[J, \eta, \bar\eta] - \int \Big[\frac{1}{2\xi}(\partial^2\lambda)^2 + J\partial\lambda\Big] - \int \Big[\Big(J - \frac{1}{\xi}\partial\partial^2\lambda\Big)\Big(\langle A\rangle - \partial\lambda\Big) + \langle\bar\psi\rangle\eta + \bar\eta\langle\psi\rangle\Big]$$
$$= \Gamma[\langle A\rangle, \langle\bar\psi\rangle, \langle\psi\rangle] + \frac{1}{\xi}\int \Big[\frac{1}{2}(\partial^2\lambda)^2 - (\partial^2\lambda)\partial\langle A\rangle\Big]. \tag{8.1.79}$$

This result becomes very simple for the functional

$$\bar\Gamma[\langle A\rangle, \langle\bar\psi\rangle, \langle\psi\rangle] = \Gamma[\langle A\rangle, \langle\bar\psi\rangle, \langle\psi\rangle] - \frac{1}{2\xi}\int (\partial\langle A\rangle)^2. \tag{8.1.80}$$

It is invariant under the gauge transformations listed in Eqs. (8.1.74) and (8.1.75),

$$\bar\Gamma[\langle A\rangle', \langle\bar\psi\rangle', \langle\psi\rangle'] = \bar\Gamma[\langle A\rangle, \langle\bar\psi\rangle, \langle\psi\rangle]. \tag{8.1.81}$$

The properties of Γ were discussed at some length in Chapter 6, Section 5. It is the generating functional for connected, single-particle irreducible graphs. The double variational derivative of Γ with respect to $\langle A\rangle$, with all the fields then taken to vanish, produces the inverse photon's Green's function,

$$\mathcal{G}^{-1}(x - x')^{\mu\nu} = -\frac{\delta}{\delta\langle A_\mu(x)\rangle}\frac{\delta}{\delta\langle A_\nu(x')\rangle}\Gamma\Big|_0. \tag{8.1.82}$$

This is the inverse of the fully interacting propagator. Since $\bar\Gamma$ is gauge invariant, its double functional derivative must produce a transverse result. Only the additional term with the factor $1/2\xi$ in Eq. (8.1.80) gives a longitudinal piece. Passing to momentum space, this Green's function

has the structure

$$\mathcal{G}^{-1}(k)_{\mu\nu} = (g_{\mu\nu}k^2 - k_\mu k_\nu)F(k^2) - \frac{1}{\xi}k_\mu k_\nu \,. \tag{8.1.83}$$

Equation (8.1.83) shows that the interaction does not alter the longitudinal part of the propagator. Hence the gauge-fixing parameter ξ needs no renormalization. To first order, the external current J produces a field expectation value schematically indicated by $\langle A\rangle = \mathcal{G}J$. Since the longitudinal part of $\mathcal{G}$ is finite, the longitudinal piece of this relation informs us that the source J^μ needs no renormalization.† The transverse part of $\mathcal{G}_{\mu\nu}$ is made finite by the renormalization of the charge which appears in it.

The inverse electron Green's function is given by

$$G^{-1}(x-y) = -\frac{\delta_l}{\delta\langle\bar\psi(x)\rangle}\,\frac{\delta_r}{\delta\langle\psi(y)\rangle}\Gamma\Big|_0 \,. \tag{8.1.84}$$

Since Γ and $\bar\Gamma$ differ by a term which is quadratic in $\langle A\rangle$, Γ may be replaced by $\bar\Gamma$ here. The photon-electron vertex is defined by

$$\Gamma^\mu(x,y,z) = \frac{\delta}{\delta\langle A_\mu(z)\rangle}\,\frac{\delta_l}{\delta\langle\bar\psi(x)\rangle}\,\frac{\delta_r}{\delta\langle\psi(y)\rangle}\Gamma\Big|_0 \,. \tag{8.1.85}$$

Again Γ may be replaced by $\bar\Gamma$. Taking the gauge parameter $\lambda(x)$ to be infinitesimal in the gauge transformations (8.1.74) and (8.1.75), the gauge invariance (8.1.81) of Γ informs us that

$$\int (d^4z)\lambda(z)\left\{\partial_\mu\frac{\delta}{\delta\langle A_\mu(z)\rangle} + i\langle\bar\psi(z)\rangle\frac{\delta_l}{\delta\langle\bar\psi(z)\rangle} - i\frac{\delta_r}{\delta\langle\psi(z)\rangle}\langle\psi(z)\rangle\right\}\bar\Gamma = 0\,. \tag{8.1.86}$$

Here the last variational derivative with respect to $\langle\psi\rangle$ is assumed to act only on $\bar\Gamma$. Taking functional derivatives of Eq. (8.1.86) with respect to $\langle\bar\psi(x)\rangle$ and $\langle\psi(y)\rangle$ and then taking all the fields to vanish gives

$$\frac{1}{i}\frac{\partial}{\partial z^\mu}\Gamma^\mu(x,y,z) = G^{-1}(x-y)\left[\delta(z-x) - \delta(z-y)\right] . \tag{8.1.87}$$

We Fourier transform with the convention

$$\Gamma^\mu(x,y,z) = \int \frac{(d^4p')}{(2\pi)^4}\,\frac{(d^4p)}{(2\pi)^4}\Gamma^\mu(p',p)e^{ip'(x-z)}e^{-ip(y-z)} \,. \tag{8.1.88}$$

This Fourier transform of Eq. (8.1.87) gives

$$(p'-p)_\mu\Gamma^\mu(p',p) = G^{-1}(p') - G^{-1}(p)\,. \tag{8.1.89}$$

† Renormalization counter terms correspond to local interactions — the space-time longitudinal and transverse parts of J^μ cannot be separately renormalized.

This is the identity usually referred to as the "Ward identity".

We have just gone through some rather formal manipulations. To shed some light on the Ward identity, let us consider the lowest order form of Γ. This is just the classical action,

$$\Gamma_{(0)}[\langle A\rangle, \langle\bar\psi\rangle, \langle\psi\rangle] = \int\left\{\frac{1}{2}\langle A_\mu\rangle\left[\frac{1}{e^2}(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) - \frac{1}{\xi}\partial^\mu\partial^\nu\right]\langle A_\nu\rangle \right.$$
$$\left. -\langle\bar\psi\rangle\left[\gamma\Big(\frac{1}{i}\partial - \langle A\rangle\Big) + m\right]\langle\psi\rangle\right\} . \tag{8.1.90}$$

The definition (8.1.85) of $\Gamma^\mu(x,y,z)$ in this lowest order yields

$$\Gamma^\mu_{(0)}(x,y,z) = \gamma^\mu\delta(x-z)\delta(y-z)\,, \tag{8.1.91}$$

whose Fourier transform is

$$\Gamma^\mu_{(0)}(p',p) = \gamma^\mu \,. \tag{8.1.92}$$

In lowest order

$$G^{-1}_{(0)}(p) = \gamma p + m\,, \tag{8.1.93}$$

and clearly

$$(p'-p)_\mu\Gamma^\mu_{(0)}(p',p) = G^{-1}_{(0)}(p') - G^{-1}_{(0)}(p)\,. \tag{8.1.94}$$

The interactions "dress" γ^μ to produce the single-particle irreducible vertex $\Gamma^\mu(p',p)$. They also "dress" $\gamma p+m$ to produce the inverse propagator $G^{-1}(p)$. The Ward identity, which is obvious for free fields, is maintained by the "dressing process" of the interactions.

To relate the vertex Γ^μ to vacuum matrix elements of field operators,‡ we note that its definition (8.1.85) may be expressed as

$$\Gamma^\mu(x,y,z) = -\frac{\delta}{\delta\langle A_\mu(z)\rangle}G^{-1}(x,y)\Big|_0 \,, \tag{8.1.95}$$

where now G^{-1} is the inverse Green's function in the presence of a non-vanishing $\langle A\rangle$. In an operator notation,

$$\delta G^{-1} = -G^{-1}\delta G G^{-1}\,. \tag{8.1.96}$$

and the functional chain rule reads

$$\frac{\delta}{\delta\langle A\rangle} = \frac{\delta J}{\delta\langle A\rangle}\,\frac{\delta}{\delta J} = \mathcal{G}^{-1}\frac{\delta}{\delta J}\,. \tag{8.1.97}$$

‡ As was noted before, the manipulations made with the functional integral are such that it no longer represents the original operator theory, and by "field matrix elements" we really mean the corresponding functional integral averages.

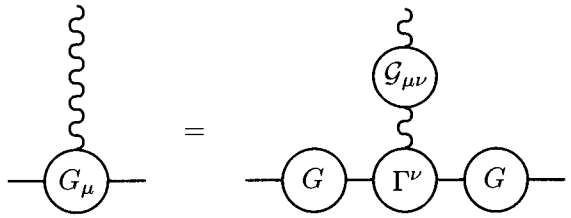

Fig. 8.2. Graphical description of the decomposition of the vector potential–Dirac field Green's function into single-particle irreducible parts.

Using these relations, we encounter

$$\frac{\delta}{\delta J^\mu(z')}G(x',y')\Big|_0 = \langle -T\psi(x')\bar{\psi}(y')A_\mu(z')\rangle \equiv G_\mu(x',y',z')\,. \tag{8.1.98}$$

Putting together the pieces yields

$$\Gamma^\mu(x,y,z) = \int (d^4x')(d^4y')(d^4z')\mathcal{G}^{-1}(z-z')^{\mu\nu} G^{-1}(x-x')G_\nu(x',y',z')G^{-1}(y'-y)\,, \tag{8.1.99}$$

where now we have reverted to the notation in which G^{-1} refers to the function with $\langle A\rangle = 0$. The convolutions with the inverse propagators removes the self-energy contributions and constructs the single-particle irreducible function as is clear from the graphical description in Fig. 8.2.

The vector-potential field equation gives

$$\left\{\frac{1}{e^2}\left[g^{\mu\nu}(-\partial_z^2)+\partial_z^\mu\partial_z^\nu\right]+\frac{1}{\xi}\partial_z^\mu\partial_z^\nu\right\}G_\nu(x,y,z) = C^\mu(x,y,z)\,, \tag{8.1.100}$$

where

$$C^\mu(x,y,z) = \langle -T\psi(x)\bar{\psi}(y)j^\mu(z)\rangle \tag{8.1.101}$$

is the three-point, current field-field, "improper vertex". The field equation above is undone by its convolution with the free photon propagator $\mathcal{G}^{\mu\nu}_{(0)}(z'-z)$. The result of these operations is best expressed in terms of the resulting Fourier transform,

$$\Gamma^\mu(p',p) = \mathcal{G}^{-1}(k)^\mu{}_\nu\mathcal{G}^{\nu\sigma}_{(0)}(k)G^{-1}(p')C_\sigma(p',p)G^{-1}(p)\,, \tag{8.1.102}$$

where

$$k = p'-p\,. \tag{8.1.103}$$

The multiplications by $G^{-1}(p')$ and $G^{-1}(p)$ "truncate" the external electron propagation — they remove the self-energy insertions from the electron lines. This is as it should be to produce a single-particle irreducible function. But it should also be noted that the irreducible, "proper" vertex $\Gamma^\mu(p',p)$ does not directly measure the current operator $j^\mu(z)$; it measures this operator with a vacuum polarization correction removed by the $\mathcal{G}^{-1}\mathcal{G}_{(0)}$ operation. Since only the transverse part of $\mathcal{G}_{\mu\nu}(k)$ is altered by interactions,

$$k_\mu\mathcal{G}^{-1}(k)^\mu{}_\nu\mathcal{G}^{\nu\sigma}_{(0)}(k) = k^\sigma\,. \tag{8.1.104}$$

Thus the Ward identity involves

$$\partial_z^\mu C_\mu(x,y,z) = \partial_z^\mu\langle -T\psi(x)\bar{\psi}(y)j_\mu(z)\rangle\,. \tag{8.1.105}$$

Since the current is conserved, $\partial^\mu j_\mu = 0$, only the equal-time commutators of $j^0(z)$ with the fields which result from the time-derivative of the discontinuous nature of the time-ordered product enter here. These produce

$$\frac{1}{i}\partial_z^\mu C_\mu(x,y,z) = G(x-y)[\delta(z-x)-\delta(z-y)]\,, \tag{8.1.106}$$

which just gives the Ward identity (8.1.87).

The generalized Ward identities which have been presented are extremely helpful in understanding the nature of the renormalization of the theory. They are also quite useful tools in checking the validity of a calculation. One can also derive an identity for the response to a variation of the gauge-fixing parameter ξ. Going back to the functional integral representation, we see that such a variation produces an insertion of $(\partial_\mu A^\mu)^2$. The contribution of the longitudinal part of the vector potential, $\partial_\mu A^\mu$, is determined by the generalized Ward identity, and thus the response may be calculated. We shall not do this here since the result is of less utility.

8.2 $e^+ + e^- \to \mu^+ + \mu^-$

Much of the increase of our knowledge of elementary particle physics has come from experiments on high energy electron-positron collisions. The simplest process to compute is the production of a muon pair in high-energy electron-positron annihilation. The muon, just as the electron, is an elementary particle, and the experimental study of this reaction tests and confirms the validity of quantum electrodynamics at high energy. Moreover, the leading order calculation of the muon pair production is — up to a trivial overall numerical factor — the same as the leading order calculation for a quark pair production process $e^+ + e^- \to q + \bar{q}$. The coupling of the strong-interaction, QCD theory is asymptotically free — it decreases as the energy increases. Hence the strong interactions become

weak at sufficiently high energies, and it is a good approximation to neglect their effects, with the result that the quark production is simply the same electrodynamic process as is the $\mu\bar{\mu}$ production. It turns out that the basic scale $\Lambda_{\rm QCD}$ for the running of the strong coupling is on the order of 100 MeV. However, since the coupling decreases only logarithmically, the strong coupling becomes weak $[\alpha_S(E)/\pi \approx 1/10]$ only for energies E on the order of 10 GeV. At these and higher energies, quark production may be computed to a good approximation from the lowest-order QED theory. Free quarks are not observed — they are always bound into hadronic particles. Nonetheless, the basic quark production process is experimentally observed in the production of "jets", narrow cones of particles about the original quark directions that carry the energy and momentum of the quarks. Only small transverse momenta are involved in the branching and binding of the quarks to form the observed hadrons, the "quark fragmentation". These fragmentation effects themselves become small at high energies if suitably defined quantities are measured which make no reference to individual hadrons, quantities such as the angular distribution of the total energy in the final state hadrons or the angular correlations of this hadronic energy. We shall discuss the nature of the quark production in e^+e^- collisions and indicate how it can be measured in Section 6. It has been mentioned here in order to further motivate the calculation of muon pair production to which we now turn.

We label the momenta and helicities of the initial electron and positron by $p\lambda$ and $\bar{p}\bar{\lambda}$; the momenta and helicities of the final produced muon and antimuon by $p'\lambda'$ and $\bar{p}'\bar{\lambda}'$. The functional integral contains the photon-electron and photon-muon interaction

$$\mathcal{L}_I = {}^{(e)}j^\mu A_\mu + {}^{(\mu)}j^\mu A_\mu \,. \tag{8.2.1}$$

Expanding to first order in ${}^{(e)}j^\mu$ and first order in ${}^{(\mu)}j^\mu$, the remaining functional integral represents a non-interacting theory, and to leading order the production amplitude is given by

$$\begin{aligned}
&\langle p'\lambda'\bar{p}'\bar{\lambda}' + |p\lambda\bar{p}\bar{\lambda}-\rangle \\
&= i^2 \int (d^4x)(d^4x')\langle p'\lambda'\bar{p}'\bar{\lambda}'|T\,{}^{(e)}j^\mu(x)A_\mu(x){}^{(\mu)}j^\nu(x')A_\nu(x')|p\lambda\bar{p}\bar{\lambda}\rangle \\
&= ie^2 \int (d^4x)(d^4x')D_{\mu\nu}(x-x')\langle p'\lambda'\bar{p}'\bar{\lambda}'|T{}^{(e)}j^\mu(x){}^{(\mu)}j^\nu(x')|p\lambda\bar{p}\bar{\lambda}\rangle .
\end{aligned} \tag{8.2.2}$$

Since no loop corrections are involved here, we work directly at $n = 4$ with the observed charge e. The current operators are of the form

$$j^\mu = \bar{\psi}\gamma^\mu\psi \,, \tag{8.2.3}$$

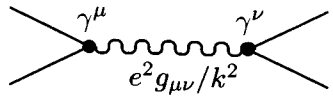

Fig. 8.3. Lowest-order graph for electron-positron annihilation into a muon pair.

and Eq. (8.2.2) involves the free theory matrix elements

$$\langle 0|\bar{\psi}(x)\gamma^\mu\psi(x)|p\lambda\bar{p}\bar{\lambda}\rangle = e^{i(p+\bar{p})x}\,\bar{v}_{\bar{\lambda}}(\bar{p})\gamma^\mu u_\lambda(p) \,, \tag{8.2.4}$$

and

$$\langle p'\lambda'\bar{p}'\bar{\lambda}'|\bar{\psi}(x')\gamma^\nu\psi(x')|0\rangle = e^{-i(p'+\bar{p}')x'}\,\bar{u}_{\lambda'}(p')\gamma^\nu v_{\bar{\lambda}'}(\bar{p}') \,. \tag{8.2.5}$$

Therefore, using the Fourier transform

$$D_{\mu\nu}(x-x') = \int \frac{(d^4k)}{(2\pi)^4} D_{\mu\nu}(k)e^{ik(x-x')} \,, \tag{8.2.6}$$

the space-time integrations in Eq. (8.2.2) produce δ-functions and give

$$\langle p'\lambda'\bar{p}'\bar{\lambda}' + |p\lambda\bar{p}\bar{\lambda}-\rangle = i(2\pi)^4\delta^{(4)}(p'+\bar{p}'-p-\bar{p})\,T \,, \tag{8.2.7}$$

where

$$T = e^2\bar{u}_{\lambda'}(p')\gamma^\nu v_{\bar{\lambda}'}(\bar{p}')\,D_{\mu\nu}(k)\,\bar{v}_{\bar{\lambda}}(\bar{p})\gamma^\mu u_\lambda(p) \,, \tag{8.2.8}$$

in which

$$k = p + \bar{p} = p' + \bar{p}' \,. \tag{8.2.9}$$

The Dirac equation for the spinor factors tells us that

$$k_\mu\bar{v}_{\bar{\lambda}}(\bar{p})\gamma^\mu u_\lambda(p) = 0 = k_\nu\bar{u}_{\lambda'}(p')\gamma^\nu v_{\bar{\lambda}'}(\bar{p}') \,. \tag{8.2.10}$$

This is just a reflection of the conservation of the current. Hence the longitudinal components of the photon propagator may be omitted and

$$T = e^2\bar{u}_{\lambda'}(p')\gamma^\mu v_{\bar{\lambda}'}(\bar{p}')\,\frac{1}{k^2}\,\bar{v}_{\bar{\lambda}}(\bar{p})\gamma_\mu u_\lambda(p) \,. \tag{8.2.11}$$

This amplitude is in precise correspondence with the Feynman graph in Fig. 8.3.

The calculation of the cross section for the production of a muon pair in the final momentum interval Δ is the same as that discussed at length in Chapter 3, Section 4. Since our normalization conventions for the spin $\frac{1}{2}$ particles are identical to those used for spin 0 particles, we may simply take over Eq. (3.4.22) of Chapter 3, which reads, with the present labels,

$$\Delta\sigma = \int_\Delta \frac{(d\mathbf{p}')}{(2\pi)^3}\frac{1}{2p'^0}\int \frac{(d\bar{\mathbf{p}}')}{(2\pi)^3}\frac{1}{2\bar{p}'^0}(2\pi)^4\delta^{(4)}(p'+\bar{p}'-p-\bar{p})|T|^2\frac{1}{4p^0\bar{p}^0 v_{\rm rel}} \,. \tag{8.2.12}$$

As usual, the square of the center-of-mass energy is denoted by s, with

$$s = -(p+\bar{p})^2 = -(p'+\bar{p}')^2 = -k^2 . \tag{8.2.13}$$

The flux factor for the equal-mass (m_e) initial electron state is given by [Eq. (3.4.27)]

$$4p^0\bar{p}^0 v_{\rm rel} = 2\sqrt{s(s-4m_e^2)} . \tag{8.2.14}$$

We shall work in the center-of-mass frame where $\bar{\mathbf{p}} = -\mathbf{p}$ and the electrons and positrons have the same energy, $p^0 = \bar{p}^0 = E$. This is also the laboratory frame for the e^+e^- colliding machines. In this frame, $s - 4m_e^2 = 4p^2$ where p is magnitude of the momentum of one of the initial particles and so

$$4p^0\bar{p}^0 v_{\rm rel} = 4p\sqrt{s} = 8pE . \tag{8.2.15}$$

Note, incidentally, that in the high-energy limit $p^0 \gg m_e$, this formula tells us that $v_{\rm rel} \to 2$; the two particles are approaching each other at the velocity of light.

Since $\gamma^0\gamma^\mu$ is an Hermitian matrix,

$$[\bar{v}_{\bar{\lambda}}(\bar{p})\gamma^\mu u_\lambda(p)]^* = [v^\dagger_{\bar{\lambda}}(\bar{p})\gamma^0\gamma^\mu u_\lambda(p)]^* = \bar{u}_\lambda(p)\gamma^\mu v_{\bar{\lambda}}(\bar{p}) . \tag{8.2.16}$$

This is to say that complex conjugation just reverses the role of the spinor factors. We shall assume that the initial electron and positron beams are not polarized. Hence in computing the cross section we sum over the spins of the electron and positron and divide by $2\times 2 = 4$. We shall also assume that the spin states of the produced muons are not measured. Hence we must also sum over their spins and compute

$$\sum_{\lambda'\bar{\lambda}'}\frac{1}{4}\sum_{\lambda\bar{\lambda}}|T|^2 = \frac{1}{4}\left(\frac{e^2}{s}\right)^2 \sum[\bar{u}_{\lambda'}(p')\gamma^\mu v_{\bar{\lambda}'}(\bar{p}')]\bar{v}_{\bar{\lambda}}(\bar{p})\gamma_\mu u_\lambda(p)$$
$$\bar{u}_\lambda(p)\gamma_\nu v_{\bar{\lambda}}(\bar{p})\bar{v}_{\bar{\lambda}'}(\bar{p}')\gamma^\nu u_{\lambda'}(p') . \tag{8.2.17}$$

Placing the factor in the square brackets on the far right-hand side, this expression has the form

$$\sum_{\lambda'\lambda'}\frac{1}{4}\sum_{\lambda\bar{\lambda}}|T|^2 = \frac{1}{4}\left(\frac{e^2}{s}\right)^2 {}^{(e)}t_{\mu\nu}(\bar{p},p)\,{}^{(\mu)}t^{\nu\mu}(\bar{p}',p') , \tag{8.2.18}$$

where

$${}^{(e)}t_{\mu\nu}(\bar{p},p) = \sum_{\lambda\bar{\lambda}}\bar{v}_{\bar{\lambda}}(\bar{p})\gamma_\mu u_\lambda(p)\bar{u}_\lambda(p)\gamma_\nu v_{\bar{\lambda}}(\bar{p}) , \tag{8.2.19}$$

with a similar expression for ${}^{(\mu)}t^{\nu\mu}(\bar{p}',p')$. The tensor $t_{\mu\nu}$ may be expressed in terms of a trace,

$$t_{\mu\nu}(\bar{p},p) = \mathrm{tr}\,\gamma_\mu\sum_\lambda u_\lambda(p)\bar{u}_\lambda(p)\gamma_\nu\sum_{\bar{\lambda}}v_{\bar{\lambda}}(\bar{p})\bar{v}_{\bar{\lambda}}(\bar{p})$$
$$= -\mathrm{tr}\,\gamma_\mu(m-\gamma p)\gamma_\nu(m+\gamma\bar{p}) . \tag{8.2.20}$$

Using (see Problem 3 of Chapter 7)

$$\mathrm{tr}\,\gamma_\mu\gamma_\nu = -4g_{\mu\nu} , \tag{8.2.21}$$

and

$$\mathrm{tr}\,\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta = 4[g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\alpha\nu}] , \tag{8.2.22}$$

a short calculation shows that

$$t_{\mu\nu}(\bar{p},p) = 4[(m^2-\bar{p}p)g_{\mu\nu} + p_\mu\bar{p}_\nu + \bar{p}_\mu p_\nu]$$
$$= t_{\nu\mu}(\bar{p},p) = t_{\mu\nu}(p,\bar{p}) . \tag{8.2.23}$$

Having performed a calculation it is always useful to check it to get some confidence that it is correct. Such a test is to see if the tensor corresponds to a conserved current,

$$k^\mu t_{\mu\nu}(\bar{p},p) = (p+\bar{p})^\mu t_{\mu\nu}(\bar{p},p) = 0 . \tag{8.2.24}$$

A small amount of algebra confirms this.

The tensor contraction in Eq. (8.2.18) is a Lorentz invariant. It is most easily computed in the center-of-mass frame that we use for our final evaluation of the cross section. In this frame, k^μ has only a time component and so the current conservation condition (8.2.24) informs us that $t_{0\nu} = t_{\nu 0} = 0$. Thus in this frame, the tensor $t_{\mu\nu}$ has only the spatial components t_{kl}. With $\bar{\mathbf{p}} = -\mathbf{p}$, writing $p^0 = \bar{p}^0 = E$, and noting that $\bar{p}p = -E^2 - \mathbf{p}^2$, these spatial components read

$$t_{kl}(\bar{p},p) = 8[E^2\delta_{kl} - p_k p_l] . \tag{8.2.25}$$

The final muons have the same energy as the initial electrons, $p'^0 = \bar{p}'^0 = E$. Hence

$$\frac{1}{4}t_{\mu\nu}(\bar{p},p)t^{\nu\mu}(\bar{p}'p') = 16[E^2\delta_{kl} - p_k p_l][E^2\delta_{kl} - p'_k p'_l]$$
$$= 16[E^2(3E^2 - \mathbf{p}^2 - \mathbf{p}'^2) + (\mathbf{p}\cdot\mathbf{p}')^2] . \tag{8.2.26}$$

Here $\mathbf{p}^2 = E^2 - m_e^2$, $\mathbf{p}'^2 = E^2 - m_\mu^2$, and $s = 4E^2$. The production angle θ may be defined as the angle between the direction of the produced μ^- (or μ^+) and the direction of the initial e^- (or e^+) as illustrated in Fig. 8.4. With this notation, we find that

$$\sum_{\lambda'\bar{\lambda}'}\frac{1}{4}\sum_{\lambda\bar{\lambda}}|T|^2 = e^4\left[\left(1+\frac{m_e^2+m_\mu^2}{E^2}\right)+\left(1-\frac{m_e^2}{E^2}\right)\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] . \tag{8.2.27}$$

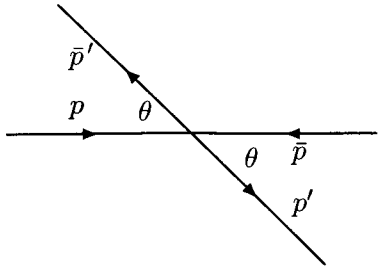

Fig. 8.4. Definition of production angle θ.

Using the flux factor (8.2.15), the cross section formula (8.2.12) becomes

$$\Delta\sigma = \int_\Delta \frac{(d\mathbf{p}')}{(2\pi)^3}\frac{1}{2p'^0}\int \frac{(d\bar{\mathbf{p}}')}{(2\pi)^3}\frac{1}{2\bar{p}'^0} (2\pi)^4\delta^{(4)}(p'+\bar{p}'-p-\bar{p})\frac{1}{8pE}\sum_{\lambda'\bar{\lambda}'}\frac{1}{4}\sum_{\lambda\bar{\lambda}}|T|^2 . \quad (8.2.28)$$

The $\bar{\mathbf{p}}'$-integration may be performed against the δ-function of spatial momentum conservation which, in the center-of-mass frame, is $\delta(\mathbf{p}'+\bar{\mathbf{p}}')$. There remains

$$\frac{1}{(2\pi)^2}\int_\Delta (d\mathbf{p}')\frac{1}{4p'^{0\,2}}\delta(2p'^0-2E) = \frac{1}{(2\pi)^2}\int_\Delta d\Omega'\int p'^2dp'\frac{1}{4p'^{0\,2}}\delta(2p'^0-2E) = \frac{1}{(2\pi)^2}\frac{p'}{8E}\int_\Delta d\Omega' , \quad (8.2.29)$$

since $p'dp' = p'^0dp'^0$. Remembering that the fine structure constant α is given, in the rationalized units used here, by

$$\alpha = \frac{e^2}{4\pi} , \quad (8.2.30)$$

the cross section for the production of a muon into the solid angle $d\Omega$ is given, according to Eqs. (8.2.27), (8.2.28), and (8.2.29), by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\frac{p'}{p}\Big[1+\cos^2\theta + \frac{m_e^2+m_\mu^2}{E^2}\sin^2\theta + \frac{m_e^2m_\mu^2}{E^4}\cos^2\theta\Big] . \quad (8.2.31)$$

This yields the total cross section*

$$\sigma_{\mu\bar{\mu}} = \int d\Omega\frac{d\sigma}{d\Omega} = \frac{4\pi\alpha^2}{3s}\frac{p'}{p}\Big(1+\frac{m_e^2}{2E^2}\Big)\Big(1+\frac{m_\mu^2}{2E^2}\Big) . \quad (8.2.32)$$

Terms involving the lepton masses m_e and m_μ have been kept for the sake of completeness. But in the experiments of interest where the energies are 10's of GeV, these mass terms are completely negligible. Neglecting the masses, $p'/p = 1$ and

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}[1+\cos^2\theta] . \quad (8.2.33)$$

The same formula applies to the production of light quarks except for two things: The charge of a quark of type ("flavor") f is eQ_f, where $Q_f = \frac{2}{3}$ (up type) or $Q_f = -\frac{1}{3}$ (down type). Each quark flavor comes in 3 "colors" — there is an additional triplet multiplicity associated with the SU(3) gauge group of the strong interactions. Hence the differential cross section for the production of quarks into the solid angle $d\Omega$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}3\sum\nolimits_f Q_f^2[1+\cos^2\theta] . \quad (8.2.34)$$

As we indicated at the start of this section, this is to be interpreted as giving the angular distribution of jets. The total cross section has a sharper interpretation. The conservation of probability — unitarity — implies that this is also the total cross section for the production of all the hadrons which are made by the fragmentation of the quarks. This is strictly true only for sufficiently high energies where the strong coupling becomes weak, as we shall explain in the next section. Since the quarks have the same basic point-like cross section as the muons, the ratio of the total hadronic cross section to the muon pair cross section,

$$R = \frac{\sigma_{\text{tot hadrons}}}{\sigma_{\mu\bar{\mu}}} , \quad (8.2.35)$$

should become a constant at high energy, with

$$R = 3\sum\nolimits_f Q_f^2 . \quad (8.2.36)$$

The experimental results for this ratio appears in Fig. 8.5. At low energies there are the prominent vector meson resonances ρ, ω, and ϕ. Outside of the resonance region $R \approx 2$, in accord with the count for u, d, and

* The electron mass mass may be neglected if the energy exceeds a few MeV, and the total cross section then involves the factor $(p'/E)(1+m_\mu^2/2E^2)$. At energies large in comparison with the muon mass, the terms of order m_μ^2/E^2 cancel, and this factor becomes approximately $(1-3m_\mu^4/8E^4)$. This remark is significant for the analogous quark production discussed below in that it shows that once the energy exceeds a quark threshold, the mass of the quark rapidly becomes unimportant.

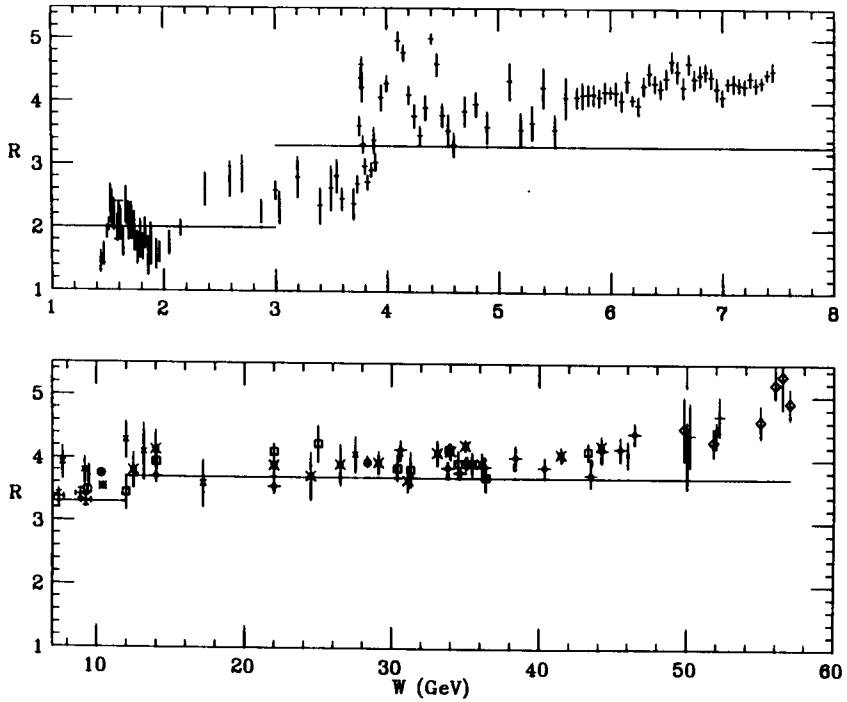

Fig. 8.5. Experimental results for the ratio R as a function of the total electron-positron energy $s^{1/2} = W$ in GeV (courtesy of M. Barnett, Particle Data Group). The sharp resonant states mentioned in the text are not displayed here.

s quarks which may be produced in this energy region, $R = 3(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}) = 2$. As the energy is further increased, further quark production thresholds are passed and new channels of quark production open up. The $\bar{c}c$ charm quark ψ states appear just above $\sqrt{s} = 3$ GeV. Apart from these resonances, $R \approx 3$ in accord with the count including the $3 \cdot \frac{4}{9}$ contribution of the c-quark, $R = 2 + \frac{4}{3}$. The ratio remains roughly at 3 until an energy of greater than 10 GeV is reached and the $\bar{b}b$ Υ resonances are excited. The b-quark contributes $3 \cdot \frac{1}{9}$ giving $R = \frac{11}{3}$ in rough agreement with the experimental value $R \approx 4$ above 15 GeV. A detailed comparison of theory and experiment needs, of course, much more work. Even away from the resonance regions, strong interaction corrections to the cross section are not negligible†, and at higher energies the effect of intermediate virtual Z^0 particles in addition to the intermediate photons cannot be neglected.

8.3 Vacuum polarization

The generating functional for quantum electrodynamics is given by

$$Z[J, \eta, \bar{\eta}] = \int [dA][d\psi][d\bar{\psi}] \exp\left\{ i \int \mathcal{L}_{QED} \right\}, \tag{8.3.1}$$

in which

$$\mathcal{L}_{QED} = -\frac{1}{4e_0^2} F_{\mu\nu}^2 + \frac{1}{2\xi_0} (\partial_\mu A^\mu)^2 + J^\mu A_\mu - \bar{\psi}\left[\gamma\left(\frac{1}{i}\partial - A\right) + m_0\right]\psi + \bar{\psi}\eta_0 + \bar{\eta}_0\psi. \tag{8.3.2}$$

Since we are now dealing with the full theory, we must use the bare coupling parameters; the theory will be regulated by working with a space-time dimensionality $n \neq 4$. Previously, we discussed the electrodynamic part by integrating out the vector potential. Writing

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{8.3.3}$$

this produces

$$Z[J, \eta, \bar{\eta}] = \int [d\psi][d\bar{\psi}] \exp\{iS_{\text{eff}}\}, \tag{8.3.4}$$

where S_{eff} is the effective action

$$S_{\text{eff}} = \int \left\{ -\bar{\psi}\left(\gamma\frac{1}{i}\partial + m_0\right)\psi + \bar{\psi}\eta_0 + \bar{\eta}_0\psi \right\} + \frac{e_0^2}{2} \int\int (J^\mu + j^\mu) D_{\mu\nu} (J^\nu + j^\nu). \tag{8.3.5}$$

† The correction to the R ratio to first order in the strong coupling constant α_S is derived in Section 6, with the result appearing in Eq. (8.6.93).

This effective action involves the interaction of the total current $J^\mu + j^\nu$ with itself mediated by the non-local photon propagator $D_{\mu\nu}(x-x')$.

Incidentally, this form could be used to obtain a simpler derivation of the transformation from the radiation gauge to the covariant gauge which we now use. Namely, working in the radiation gauge would give the same result but with $D_{\mu\nu}$ replaced by $D^{(R)}_{\mu\nu}$. Since the two propagators differ by terms involving ∂_μ or ∂_ν, $D^{(R)}_{\mu\nu}$ may be changed to $D_{\mu\nu}$ if the total current $J^\mu + j^\mu$ is conserved. As we discussed before, this is ultimately the case for physically observable amplitudes. We may then work backwards and note that the propagator $D_{\mu\nu}$ results from the functional integration (8.3.1) over a four-vector potential with the the covariant Lagrange function (8.3.2). As we discussed before, we went through a more lengthly procedure because it shows how the gauge must be fixed to provide a finite measure for the functional integration over the vector potentials and because the procedure is easily generalized to the case of non-Abelian gauge fields.

1 *Integrating out Fermions*

Instead of integrating out the photon field to obtain a non-local Fermion interaction, we may do the opposite. To integrate out the Fermi fields, we first define the Green's function $G[A]$ by

$$\left\{\gamma\left(\frac{1}{i}\partial - A\right) + m_0\right\} G[A;\, x, x'] = \delta(x-x')\,. \tag{8.3.6}$$

Then using an operator notation, the Fermi field functional integration becomes, by the usual method of completing the square,

$$\int [d\psi][d\bar\psi] \exp\{i[-\bar\psi G^{-1}[A]\psi + \bar\psi\eta_0 + \bar\eta_0\psi]\} = \mathrm{Det}G^{-1}[A] \exp\{i\bar\eta_0 G[A]\eta_0\}. \tag{8.3.7}$$

As it appears, the infinite determinant is not well-defined. However, in the absence of external fields and sources, the functional integral must be normalized so as to produce unity. Hence we must divide this formal result by $\mathrm{Det}G^{-1}[0]$. Writing simply $G^{-1}[0] = G^{-1}$, this entails

$$\frac{\mathrm{Det}G^{-1}[A]}{\mathrm{Det}G^{-1}} = \mathrm{Det}G^{-1}[A]G = \mathrm{Det}[1-\gamma AG]\,. \tag{8.3.8}$$

Writing the determinant in the trace-log form gives

$$Z[J,\,\eta,\,\bar\eta] = \int [dA] \exp\{iS_{\mathrm{eff}}\}\,, \tag{8.3.9}$$

in which now

$$S_{\mathrm{eff}} = \int \left[-\frac{1}{4e_0^2}F_{\mu\nu}^2 + \frac{1}{2\xi_0}(\partial_\mu A^\mu)^2 + J^\mu A_\mu\right]$$

$$-i\mathrm{Tr}\ln(1-\gamma AG) + \int\int \bar\eta_0 G[A]\eta_0\,. \tag{8.3.10}$$

We now show that only even powers of A_μ are present in $\mathrm{Tr}\ln(1-\gamma AG)$. To do this, we note that if a doublet of Majorana fields ψ_1, ψ_2 had been used rather than the Dirac field ψ, with $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$, the Fermi current would have the form

$$j^\mu = \psi\alpha^\mu q\psi\,, \tag{8.3.11}$$

where q is a matrix in the "1-2" space,

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{8.3.12}$$

The current, expressed in terms of Majorana fields ψ, must involve this antisymmetrical matrix q because α^μ is symmetrical in the Majorana representation and the anticommutativity of the ψ requires that a bilinear $\psi M\psi$ is non-vanishing only for an antisymmetrical M. Of course, writing out the Dirac field current $\bar\psi\gamma^\mu\psi$ in terms of the Majorana fields would produce the structure that we have written. Working with Majorana fields we would encounter $\mathrm{Det}^{1/2}[1-q\gamma AG]$. Writing this determinant in the trace-log form and expanding in powers of A_μ involves the trace in the "1-2" space of increasing powers of q, $\mathrm{tr}\,q^k$. Since $\mathrm{tr}\,q^{2l+1} = \mathrm{tr}\,q = 0$, while $\mathrm{tr}\,q^{2l} = \mathrm{tr}\,1 = 2$, odd powers of A_μ do not appear in this sum, while the even powers are multiplied by a factor of 2 if the trace in the "1-2" space is omitted. This latter factor of two just corresponds to replacing $\mathrm{Det}^{1/2}[1-q\gamma AG]$ by $\mathrm{Det}[1-\gamma AG]$, but we also see that the terms with odd powers of A_μ must vanish in the expansion of this determinant. We have shown that the determinant is charge conjugation invariant which, in the Majorana representation, corresponds to a field replacement (such as $\psi_1 \to \psi_1$, $\psi_2 \to -\psi_2$) that is equivalent to the transformation $q \to -q$ $(= \tau_3 q \tau_3)$.

Since only even powers of A contribute,

$$\begin{aligned}\mathrm{Tr}\ln(1-\gamma AG) &= -\frac{1}{2}\mathrm{Tr}\gamma AG\gamma AG - \frac{1}{4}\mathrm{Tr}\gamma AG\gamma AG\gamma AG\gamma AG - \cdots \\ &= \frac{i}{2}\int (d^n x)(d^n x') K^{\mu\nu}(x-x') A_\mu(x) A_\nu(x') \\ &\quad + \frac{i}{4!}\int (d^n x)\cdots(d^n x''') K^{\mu\nu\rho\sigma}(x,x',x'',x''') \\ &\qquad A_\mu(x)A_\nu(x')A_\rho(x'')A_\sigma(x''') + \ldots\,.\end{aligned} \tag{8.3.13}$$

Here

$$K^{\mu\nu}(x-x') = i\,\mathrm{tr}\,\gamma^\mu G(x-x')\gamma^\nu G(x'-x)\,, \tag{8.3.14}$$

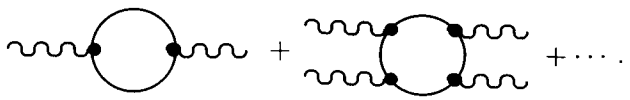

Fig. 8.6. Graphical expansion of the single-loop vacuum polarization.

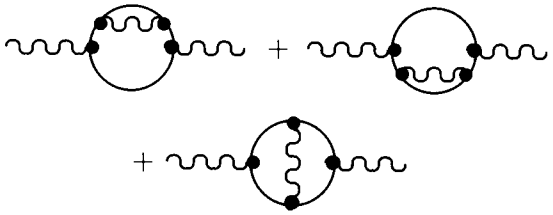

Fig. 8.7. First-order corrections to the photon self-energy.

where now the trace is only over the Dirac γ-matrix space. Similarly,

$$K^{\mu\nu\rho\sigma}(x,x',x'',x''') = i\,\mathrm{tr}\,\gamma^\mu G(x-x')\gamma^\nu G(x'-x'')\gamma^\rho G(x''-x''')\gamma^\sigma G(x'''-x) + \text{ perms}. \tag{8.3.15}$$

A symmetrical combination of all the possible permutations appears here since the functional expansion entails a $K^{\mu_1\mu_2\mu_3\mu_4}(x_1,x_2,x_3,x_4)$ which is completely symmetrical under the interchange of any index-coordinate pair, $(\mu_a\,,x_a)\leftrightarrow(\mu_b\,,x_b)$. The cyclic symmetry of the trace in Eq. (8.3.14) ensures that $K^{\nu\mu}(x'-x)=K^{\mu\nu}(x-x')$, so no explicit symmetrization is needed here. The expansion of the trace-log corresponds to the series of Feynman graphs indicated in Fig. 8.6. It corresponds to a sum of ring or single-loop graphs attached to an increasing number of external photons. The internal lines correspond to electron propagators G. The first term describes a virtual photon converting to a virtual electron-positron pair which reconverts to a virtual photon. This term is a photon self-energy correction. The graph with four external photons, when restricted to real photons, gives light-by-light scattering in the vacuum. These are the lowest-order forms. Higher-order corrections result from the functional integral over the photon field which inserts additional photon lines into the lowest-order graphs together with photon lines connected to other electron loops. Thus the vacuum polarization graphs have further corrections. These are illustrated by first correction to the photon self-energy shown in Fig. 8.7.

The Fermi functional integral which produces this sum of ring graphs defines the non-interacting generating functional for the current correlation functions

$$\exp\{\mathrm{Tr}\ln(1-\gamma AG)\} = \int[d\psi][d\bar\psi]\exp\left\{-i\int\bar\psi\left[\gamma\left(\frac{1}{i}\partial - A\right)+m_0\right]\psi\right\} = \left\langle\exp\left\{i\int j^\mu A_\mu\right\}\right\rangle^{(0)}. \tag{8.3.16}$$

Expanding in powers of A_μ identifies

$$K_{\mu\nu}(x-x') = \langle iTj_\mu(x)j_\nu(x')\rangle^{(0)}, \tag{8.3.17}$$

$$K_{\mu\nu\rho\sigma}(x,x',x'',x''') = \langle -iTj_\mu(x)j_\nu(x')j_\rho(x'')j_\sigma(x''')\rangle_C^{(0)}, \tag{8.3.18}$$

and so forth. The subscript C denotes the connected part with, for example,

$$\langle T(ABCD)\rangle_C = \langle T(ABCD)\rangle - \langle T(AB)\rangle\langle T(CD)\rangle - \langle T(AC)\rangle\langle T(BD)\rangle - \langle T(AD)\rangle\langle T(BC)\rangle. \tag{8.3.19}$$

2 Photon propagator

To exhibit the effects of the lowest-order, two-point current correlation function $K_{\mu\nu}$, we take the Fermi field sources to vanish, $\eta_0 = 0 = \bar\eta_0$, and truncate the trace-log expansion, keeping only the leading $K_{\mu\nu}$ term. Using an operator notation for the action, this approximation reads

$$Z[J,0,0] = \int[dA]\exp\left\{-\frac{i}{2e_0^2}A^\mu[D^{-1}_{\mu\nu}-e_0^2K_{\mu\nu}]A^\nu + iJA\right\}. \tag{8.3.20}$$

In general, we write the interacting photon propagator as

$$\mathcal{G}_{\mu\nu}(x-x') = \langle iTA_\mu(x)A_\nu(x')\rangle = \int\frac{(d^nk)}{(2\pi)^n}\mathcal{G}_{\mu\nu}(k)e^{ik(x-x')}. \tag{8.3.21}$$

In our approximation,

$$\mathcal{G}^{-1}(k)_{\mu\nu} = \frac{1}{e_0^2}D^{-1}(k)_{\mu\nu} - K_{\mu\nu}(k). \tag{8.3.22}$$

Taking the inverse to compose $\mathcal{G}_{\mu\nu}(k)$, this approximation corresponds to an infinite sum of products of the loop graph shown in Fig. 8.8. This interacting propagator involves the photon self-energy or vacuum polarization function

$$\Pi_{\mu\nu}(k) = -e_0^2K_{\mu\nu}(k). \tag{8.3.23}$$

This function would appear, of course, if we were to compute the first correction to the photon propagator by using the interaction representation

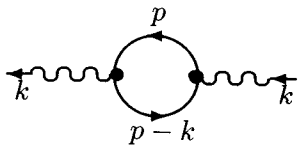

Fig. 8.8. Graph with momentum flow for the photon self-energy.

expansion as we did in computing the scalar-field propagator in Chapter 4.

We turn now to compute $K^{\mu\nu}(k)$. The Fourier transform of Eq. (8.3.14) gives

$$K^{\mu\nu}(k) = i\int \frac{(d^n p)}{(2\pi)^n} \mathrm{tr}\,\gamma^\mu G(p)\gamma^\nu G(p-k)\,. \tag{8.3.24}$$

Going into Euclidean space and rationalizing the electron propagators, this becomes

$$K^{\mu\nu}(k) = -\int \frac{(d_E^n p)}{(2\pi)^n} \frac{\mathrm{tr}\{\gamma^\mu(m-\gamma p)\gamma^\nu(m-\gamma(p-k))\}}{(p^2+m^2)((p-k)^2+m^2)}\,, \tag{8.3.25}$$

corresponding to the Feynman graph of Fig. 8.8.

Simple power counting shows that the separate terms in Eq. (8.3.25) diverge quadratically when $n = 4$. However, the sum of terms diverges only logarithmically. This lessening of the degree of divergence arises from the fact that the current is conserved, which requires that that the current correlation function $K^{\mu\nu}(k)$ be proportional to $(g_{\mu\nu}k^2 - k_\mu k_\nu)$. Therefore, it follows from dimensional analysis that the integrand for the sum of terms behaves as $1/(p^2)^2$ for large p, not as $1/p^2$. The spinor trace is easily evaluated using the results of Problem 3, Chapter 7,

$$\mathrm{tr}\{\cdots\} = \left\{p^\mu(p-k)^\nu + (p-k)^\mu p^\nu - [m^2 + p(p-k)]g^{\mu\nu}\right\}\mathrm{tr}\,1\,. \tag{8.3.26}$$

For the irreducible representation of the gamma matrices used in that problem,

$$\mathrm{tr}\,1 = 2^{n/2}\,. \tag{8.3.27}$$

As we shall see, the precise n-dependence of tr 1 will not be needed here except that $\mathrm{tr}\,1 = 4$ for $n = 4$. Exponentiating the denominators in the usual fashion,

$$K^{\mu\nu}(k) = -\int \frac{(d_E^n p)}{(2\pi)^n}\int_0^\infty dsdt e^{-s(p^2+m^2)}e^{-t((p-k)^2+m^2)}\mathrm{tr}\{\cdots\}\,, \tag{8.3.28}$$

and completing the square with

$$p' = p - \frac{tk}{s+t}\,, \tag{8.3.29}$$

yields

$$K^{\mu\nu}(k) = -(\mathrm{tr}\,1)\int_0^\infty dsdt e^{-(s+t)m^2}e^{-\frac{st}{s+t}k^2}\int\frac{(d_E^n p')}{(2\pi)^n}$$
$$\left\{2p'^\mu p'^\nu - 2\frac{st}{(s+t)^2}k^\mu k^\nu - g^{\mu\nu}\left[m^2 + p'^2 - \frac{st}{(s+t)^2}k^2\right]\right\}e^{-(s+t)p'^2}\,. \tag{8.3.30}$$

The momentum integration here entails

$$\int\frac{(d_E^n p)}{(2\pi)^n}e^{-Ap^2} = \frac{1}{(4\pi A)^{n/2}}\,, \tag{8.3.31}$$

from which, with

$$p^2 e^{-Ap^2} = -\frac{\partial}{\partial A}e^{-Ap^2}\,, \tag{8.3.32}$$

also yields

$$\int\frac{(d_E^n p)}{(2\pi)^n}p^2 e^{-Ap^2} = \frac{n}{2A}\,\frac{1}{(4\pi A)^{n/2}}\,. \tag{8.3.33}$$

Since $g^\mu{}_\mu = n$, the last integral implies that

$$\int\frac{(d_E^n p)}{(2\pi)^n}p^\mu p^\nu e^{-Ap^2} = g^{\mu\nu}\frac{1}{2A}\,\frac{1}{(4\pi A)^{n/2}}\,. \tag{8.3.34}$$

Therefore,

$$K^{\mu\nu}(k) = -\frac{(\mathrm{tr}\,1)}{(4\pi)^{n/2}}\int_0^\infty\frac{dsdt}{(s+t)^{n/2}}e^{-(s+t)m^2}\exp\left\{-\frac{st}{s+t}k^2\right\}$$
$$\left\{\frac{1}{s+t}g^{\mu\nu} - 2\frac{st}{(s+t)^2}k^\mu k^\nu - \left[m^2 + \frac{n}{2}\frac{1}{s+t} - \frac{st}{(s+t)^2}k^2\right]g^{\mu\nu}\right\}\,. \tag{8.3.35}$$

The calculation is completed following, as usual, the standard method explained at length in Chapter 4. One introduces

$$1 = \int_0^\infty dS\delta(S-s-t) \tag{8.3.36}$$

into the integral and changes variables by writing $s = \alpha S$, $t = \beta S$. This produces the structure

$$\int_0^\infty dS S^{1-n/2}e^{-(m^2+\alpha\beta k^2)S}\left\{g^{\mu\nu}\left[(m^2-\alpha\beta k^2)+\left(\frac{n}{2}-1\right)\frac{1}{S}\right]+2\alpha\beta k^\mu k^\nu\right\}. \tag{8.3.37}$$

The terms without the additional factor of $1/S$ give a result proportional

to $\Gamma\left(2-\frac{n}{2}\right)$. Relative to these terms, that involving $1/S$ gives

$$\left(\frac{n}{2}-1\right)\Gamma\left(1-\frac{n}{2}\right)(m^2+\alpha\beta k^2) = -\Gamma\left(2-\frac{n}{2}\right)(m^2+\alpha\beta k^2)\,. \quad (8.3.38)$$

Thus the m^2 terms cancel and one finds that

$$K^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)K(k)\,, \quad (8.3.39)$$

in which

$$K(k) = -2\frac{(\mathrm{tr}\,1)}{(4\pi)^{n/2}}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 d\alpha d\beta\delta(1-\alpha-\beta)\alpha\beta[m^2+\alpha\beta k^2]^{\frac{n}{2}-2}\,. \quad (8.3.40)$$

It is significant to note that the method of dimensional regularization produces a result that obeys current conservation for arbitrary dimensionality n,

$$k_\mu K^{\mu\nu}(k) = 0\,. \quad (8.3.41)$$

The reason is that the dimensional regularization corresponds simply to writing the QED theory in n-dimensions with all the symmetries of the theory left intact. In particular, the dimensional regularization maintains gauge invariance and thus the conservation of the current.

Under the dimensional continuation, the vector potential must maintain its dimensionality $[A_\mu]_d = M$ since it appears together with the derivative to make up the covariant derivative $D_\mu = \partial_\mu - iA_\mu$. Hence the dimensional count for the Maxwell part of the action appears as $M^{-n}(1/e_0^2)M^4$, and requiring this to be dimensionless requires that the bare charge carry the dimension

$$[e_0^2]_d = M^{4-n}\,. \quad (8.3.42)$$

The dimensional regularization with minimal subtraction defines a dimensionless running, finite, renormalized charge e^2 according to

$$e_0^2 = e^2\mu^{4-n}\left\{1+\frac{\#e^2}{n-4}+\cdots\right\}. \quad (8.3.43)$$

Remembering the structure of the bare photon propagator and using Eq. (8.3.39) in the interacting propagator expression (8.3.22), we get

$$\mathcal{G}^{-1}(k)_{\mu\nu} = (g_{\mu\nu}k^2 - k_\mu k_\nu)\left(\frac{1}{e_0^2}-K(k^2)\right) - \frac{1}{\xi_0}k_\mu k_\nu\,. \quad (8.3.44)$$

The charge renormalization is determined by making this interacting propagator finite. The gauge-fixing parameter ξ_0 in a dimension apart from $n=4$ carries the same dimension as does e_0^2. Since the gauge-fixing parameter is not renormalized,

$$\xi_0 = \xi\mu^{4-n}\,. \quad (8.3.45)$$

To perform the charge renormalization, we write

$$\frac{1}{e_0^2} - K(k^2) = \frac{\mu^{n-4}}{e^2} - K_{\mathrm{ren}}(k^2) \quad (8.3.46)$$

where $K_{\mathrm{ren}}(k^2)$ is finite at $n=4$. The minimal subtraction scheme requires that the same mass unit μ be used to exhibit the dimensionality of $K(k^2)$, and so

$$K(k^2) = -2\frac{(\mathrm{tr}\,1)}{(4\pi)^{n/2}}\Gamma\left(2-\frac{n}{2}\right)\mu^{n-4} \int_0^1 d\alpha d\beta\delta(1-\alpha-\beta)\alpha\beta\left[\frac{m^2+\alpha\beta k^2}{\mu^2}\right]^{\frac{n}{2}-2}\,. \quad (8.3.47)$$

Remembering that

$$n\to 4: \qquad \Gamma\left(2-\frac{n}{2}\right) \simeq -\frac{2}{n-4}-\gamma\,, \quad (8.3.48)$$

minimal subtraction isolates the divergent part

$$K_{\mathrm{div}}(k^2) = \frac{16}{(4\pi)^2}\frac{\mu^{n-4}}{n-4}\int_0^1 d\alpha\,\alpha(1-\alpha) = \frac{8}{3}\frac{1}{(4\pi)^2}\frac{\mu^{n-4}}{n-4}\,. \quad (8.3.49)$$

Thus, in our approximation, the relation between the bare and renormalized charge is given by

$$\frac{1}{e_0^2} = \mu^{n-4}\left[\frac{1}{e^2}+\frac{8}{3}\frac{1}{(4\pi)^2}\frac{1}{n-4}\right]. \quad (8.3.50)$$

This defines the running coupling at the scale μ, $e^2 = e^2(\mu)$. Since e_0^2 is independent of μ,

$$0 = \frac{n-4}{e^2}+\frac{8}{3}\frac{1}{(4\pi)^2}+\mu\frac{d}{d\mu}\frac{1}{e^2}\,, \quad (8.3.51)$$

which implies that

$$\mu\frac{de^2}{d\mu} = (n-4)e^2+\beta(e^2)\,, \quad (8.3.52)$$

with

$$\beta(e^2) = \frac{8}{3}\frac{e^4}{(4\pi)^2}\,. \quad (8.3.53)$$

Passing to the physical limit $n=4$ and solving

$$\mu\frac{de^2}{d\mu} = \beta(e^2) \quad (8.3.54)$$

gives

$$\frac{1}{e^2(\mu_2)} - \frac{1}{e^2(\mu_1)} = \frac{8}{3(4\pi)^2}\ln(\mu_1/\mu_2)\,, \tag{8.3.55}$$

or

$$e^2(\mu_2) = \frac{e^2(\mu_1)}{1 + \frac{8e^2(\mu_1)}{3(4\pi)^2}\ln(\mu_1/\mu_2)}\,. \tag{8.3.56}$$

This solution is also obtained, of course, by using Eq. (8.3.50) to write $(1/e_0^2)$ in terms of $1/e^2(\mu_1)$ and $1/e^2(\mu_2)$, equating the results, and taking the limit $n \to 4$. The point is that our initial approximation corresponds to a "bubble sum", and such a sum is reproduced in the relation between $e^2(\mu_2)$ and $e^2(\mu_1)$.

The renormalized current correlation function may be written as

$$\begin{aligned} K_{\rm ren}(k^2) = &-2\frac{(\mathrm{tr}\,1)}{(4\pi)^{n/2}}\Gamma\left(2-\frac{n}{2}\right)\mu^{n-4} \\ &\int_0^1 d\alpha\, d\beta\delta(1-\alpha-\beta)\alpha\beta\left\{\left[\frac{m^2+\alpha\beta k^2}{\mu^2}\right]^{\frac{n}{2}-2} - 1\right\} \\ &-\frac{4}{3}\frac{1}{(4\pi)^2}\frac{\mu^{n-4}}{2-\frac{n}{2}}\left[\frac{(\mathrm{tr}\,1)/4}{(4\pi)^{\frac{n}{2}-2}}\Gamma\left(3-\frac{n}{2}\right) - 1\right]. \end{aligned} \tag{8.3.57}$$

The physical limit $n \to 4$ of the second term produces a finite constant C whose precise value depends upon how the γ-matrix trace $\mathrm{tr}\,1$ is extrapolated away from its value $\mathrm{tr}\,1 = 4$ at $n = 4$. In the physical limit,

$$\begin{aligned} n \to 4: \qquad K_{\rm ren}(k^2) = &\frac{8}{(4\pi)^2}\int_0^1 d\alpha d\beta\delta(1-\alpha-\beta) \\ &\alpha\beta\ln\left[\frac{m^2+\alpha\beta k^2}{\mu^2}\right] + C\,. \end{aligned} \tag{8.3.58}$$

And in this limit

$$\mathcal{G}^{-1}(k^2)_{\mu\nu} = (g_{\mu\nu}k^2 - k_\mu k_\nu)\left(\frac{1}{e^2} - K_{\rm ren}(k^2)\right) - \frac{1}{\xi}k_\mu k_\nu\,. \tag{8.3.59}$$

Two features of this "floating point" method of renormalization with an arbitrary unit of mass μ should be noted. First it is clear from the explicit form of the result that no singularity appears in the massless limit $m^2 \to 0$. Second, the freedom of choosing the mass unit may be used to transfer the large k^2 behavior from $K_{\rm ren}(k^2)$ to the running coupling $1/e^2(k^2)$. That is, on replacing μ^2 by k^2, $K_{\rm ren}(k^2)$ approaches a constant in the large k^2 limit, and this limit is controlled by the running coupling

constant,

$$k^2 \gg m^2: \qquad \frac{1}{e^2(\mu)} - K_{\rm ren}(k^2;\mu) \simeq \frac{1}{e^2(k^2)} + \text{const.} \tag{8.3.60}$$

Conversely, using the explicit form for the large k^2 limit of $K_{\rm ren}(k^2;\mu)$, Eq. (8.3.60) produces the running coupling relation (8.3.55).

We have just shown how the minimal subtraction method and the resulting running coupling is useful for exhibiting the general features of the theory and its high-energy limit. However, to relate the coupling parameter e^2 to the fine structure constant observed in low-energy experiments, "mass-shell" renormalization must be used. This is done by setting $\mu^2 = Am^2$, where the constant A is chosen to cancel the additive constant C in $K_{\rm ren}(k^2)$. This procedure produces

$$K_{\rm phy}(k^2) = \frac{8}{(4\pi)^2}\int_0^1 d\alpha\alpha(1-\alpha)\ln[1+\alpha(1-\alpha)k^2/m^2]\,. \tag{8.3.61}$$

In particular, with this choice

$$K_{\rm phy}(0) = 0\,, \tag{8.3.62}$$

and so for small k^2,

$$k^2 \simeq 0: \qquad \mathcal{G}(k^2)_{\mu\nu} \simeq e^2\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2} - \xi\frac{k_\mu k_\nu}{(k^2)^2}\,. \tag{8.3.63}$$

This small k^2 behavior of the photon propagator determines the nature of an external current-current interaction at large distances. With the mass-shell renormalization, two (elementary) charges widely separated (on the scale set by the electron's Compton wavelength $1/m$) obey Coulomb's law with charge e, confirming that this is the charge measured at low energies, or equivalently, large distances.*

It is convenient to rewrite Eq. (8.3.61) in the form of a dispersion relation as was done for the scalar "bubble" towards the end of Section 5 in Chapter 3. To this end, we use

$$\alpha(1-\alpha) = \frac{d}{d\alpha}\left[\frac{1}{2}\alpha^2 - \frac{1}{3}\alpha^3\right] \tag{8.3.64}$$

and integrate by parts to obtain

$$K_{\rm phy}(k^2) = -\frac{8}{(4\pi)^2}\frac{k^2}{m^2}\int_0^1 d\alpha\,\frac{\alpha^2\,(1/2-\alpha/3)\,(1-2\alpha)}{1+\alpha(1-\alpha)k^2/m^2}\,. \tag{8.3.65}$$

* Conversely, we should note that the short-distance, high k^2 behavior is determined by the running coupling $e^2(k^2)$. In this limit, the discussion above shows that $\mathcal{G}(k^2) \sim e^2(k^2)/k^2$.

This has the dispersion relation form

$$K_{\rm phy}(k^2) = \frac{1}{4\pi}\int_{4m^2}^{\infty} ds\, r(s)\,\frac{k^2}{k^2+s}\,, \tag{8.3.66}$$

with the spectral weight given by

$$r(s) = -\frac{2}{\pi}\int_0^1 d\alpha\,\alpha^2\left(\frac{1}{2}-\frac{\alpha}{3}\right)(1-2\alpha)\delta(s\alpha(1-\alpha)-m^2)\,. \tag{8.3.67}$$

The roots of the δ-function appear at

$$\alpha_\pm = \frac{1}{2}\left[1\pm\sqrt{1-\frac{4m^2}{s}}\right]\,. \tag{8.3.68}$$

The strength of the δ-function contributions is the inverse of the magnitude of the slope $s(1-2\alpha_\pm)$. A little algebra is needed to add up the terms and find that

$$r(s) = \frac{1}{3\pi}\,\frac{1}{s}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)\,. \tag{8.3.69}$$

The square root reflects the two-body phase space associated with the corresponding loop graph. This spectral weight is essentially the same as the weight for the scalar loop given in Eq. (3.5.63) except for the factor $(1+2m^2/s)$ which arises from the electron spin and vector nature of the current correlation loop.

In the approximation which has been made all along,

$$Z[J,0,0] = \langle T\,\exp\{i\int JA\}\rangle \simeq \exp\left\{\frac{i}{2}\int\int J\mathcal{G}J\right\}\,. \tag{8.3.70}$$

This yields the normalized vector potential expectation value

$$\begin{aligned}\langle A_\mu(x)\rangle &= \frac{1}{Z[J,0,0]}\left\langle TA_\mu(x)\,\exp\left\{i\int JA\right\}\right\rangle\\ &= \frac{1}{i}\,\frac{\delta}{\delta J_\mu(x)}\,\ln Z[J,0,0]\\ &= \int(d^4x')\mathcal{G}_{\mu\nu}(x-x')J^\nu(x')\,.\end{aligned} \tag{8.3.71}$$

For an external physical, conserved current, $\partial_\nu J^\nu = 0$, the longitudinal pieces in the Green's function $\mathcal{G}_{\mu\nu}$ may be deleted. Expanding the Green's function in powers of the vacuum polarization correction and keeping only the first term gives

$$\langle A_\mu(x)\rangle = e^2\int(d^4x')\int\frac{(d^4k)}{(2\pi)^4}\left[\frac{1}{k^2}+\alpha\int_{4m^2}^{\infty}ds\,\frac{r(s)}{k^2+s}\right]e^{ik(x-x')}J_\mu(x')\,. \tag{8.3.72}$$

Here $\alpha = e^2/4\pi$ is the fine structure constant. The higher-order terms in the expansion of the vacuum polarization come together with higher powers of α and thus provide smaller corrections.

3 Atomic level shifts

Let us apply this result to compute the electrostatic potential $\phi(\mathbf{r})$ produced by a time-independent charge distribution $\rho(\mathbf{r})$. Using the conventional notation for these, $\phi = A^0/e$, and $\rho = eJ^0$. With a time-independent charge distribution, the x'^0-integral produces $2\pi\delta(k^0)$, and so

$$\begin{aligned}\phi(r) &= \int(d^3\mathbf{r}')\int\frac{(d^3\mathbf{k})}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left[\frac{1}{\mathbf{k}^2}+\alpha\int_{4m^2}^{\infty}\frac{r(s)}{\mathbf{k}^2+s}\right]\rho(\mathbf{r}')\\ &= \int(d^3\mathbf{r}')\left[\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}+\int_{4m^2}^{\infty}dsr(s)\frac{e^{-\sqrt{s}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\right]\rho(\mathbf{r}')\,.\end{aligned} \tag{8.3.73}$$

In particular, for a point charge $\rho(\mathbf{r}) = q\delta(\mathbf{r})$,

$$\phi(\mathbf{r}) = \frac{q}{4\pi|\mathbf{r}|}\left[1+\alpha\int_{4m^2}^{\infty}dsr(s)e^{-\sqrt{s}|\mathbf{r}|}\right]\,. \tag{8.3.74}$$

The large distance limit of the potential, $m|\mathbf{r}|\gg 1$, probes the threshold behavior of the weight $r(s)$. Using

$$s\simeq 4m^2\,:\qquad r(s)\simeq\frac{1}{16\pi m^3}\sqrt{s-4m^2}\,, \tag{8.3.75}$$

and writing

$$s = 4m^2+4\xi^2 m/|\mathbf{r}|\,, \tag{8.3.76}$$

this long-distance limit is given by

$$\begin{aligned}m|r|\gg 1\,:\qquad \phi(r) &\simeq \frac{q}{4\pi|\mathbf{r}|}\left[1+\frac{\alpha}{\pi}\,\frac{e^{-2m|\mathbf{r}|}}{(m|\mathbf{r}|)^{3/2}}\int_0^\infty d\xi\xi^2e^{-\xi^2}\right]\\ &\simeq \frac{q}{4\pi|\mathbf{r}|}\left[1+\frac{\alpha}{4\sqrt{\pi}}\,\frac{e^{-2m|\mathbf{r}|}}{(m|\mathbf{r}|)^{3/2}}\right]\,.\end{aligned} \tag{8.3.77}$$

The vacuum polarization is a small (order $\alpha\approx 1/137$) short range ($\frac{1}{2}$ the electron Compton wavelength $1/m\approx 4\times10^{-11}$ cm) effect. On the other hand, the short distance limit of the potential probes the large s behavior of the spectral weight. Using

$$s\gg 4m^2\,:\qquad r(s)\simeq\frac{1}{3\pi s}\,, \tag{8.3.78}$$

the short-distance limit involves

$$\int_{4m^2}^{\infty} ds r(s) e^{-\sqrt{s}|\mathbf{r}|} \simeq \frac{2}{3\pi} \int_{2m|\mathbf{r}|}^{\infty} \frac{dx}{x} e^{-x} \simeq \frac{2}{3\pi} \left[\ln(1/2m|\mathbf{r}|) - \gamma\right] , \tag{8.3.79}$$

where $\gamma = 0.577$ is Euler's constant. Hence, in the short-distance limit,

$$|\mathbf{r}| \ll m^{-1} : \qquad \phi(\mathbf{r}) \simeq \frac{q}{4\pi|\mathbf{r}|} \left[1 + \frac{2\alpha}{3\pi} \ln\left(\frac{e^{-\gamma}}{2m|\mathbf{r}|}\right)\right] . \tag{8.3.80}$$

The correction diverges in the limit $|\mathbf{r}| \to 0$ reflecting the original divergence in the theory which was removed by charge renormalization. The vacuum polarization correction becomes large, of order 1, and the perturbation series breaks down for distances $|\mathbf{r}| < r_B$, where

$$r_B = m^{-1} e^{-3\pi/2\alpha} . \tag{8.3.81}$$

Since $\exp\{-3\pi/2\alpha\} \approx 10^{-300}$, this is a physically irrelevant length — many new phenomena appear before reaching such a short distance (even the Planck length where gravity becomes strong is only on the order of $10^{-22} m^{-1}$). The short-distance limit (8.3.80) exhibited here is just the result obtained from the running coupling. That is, if we write $q = Ze$, then the short distance limit reads

$$e\phi(\mathbf{r}) = \frac{Ze^2(r)}{4\pi|\mathbf{r}|} , \tag{8.3.82}$$

where $e^2(r)$ is just the first two terms of the expansion of the running coupling relation (8.3.55) with $\mu_1/\mu_2 = 2me^{\gamma}|\mathbf{r}|$.

The effect of vacuum polarization is observed in shifts ΔE of atomic energy levels away from their values given by the Dirac equation. To compute the leading correction for a hydrogen-like atom it suffices to use non-relativistic, Schrödinger wave functions† and the first-order perturbation formula

$$\Delta E = \int (d^3\mathbf{r}) |\psi(\mathbf{r})|^2 \Delta V(r) . \tag{8.3.83}$$

With a nucleus of charge Ze, the vacuum polarization correction to the potential is given by

$$\Delta V(r) = -\frac{Z\alpha}{r} \alpha \int_{4m^2}^{\infty} ds r(s) e^{-\sqrt{s}r} . \tag{8.3.84}$$

The correction potential $\Delta V(r)$ vanishes exponentially unless $r \lesssim m^{-1}$. On the other hand, the scale of the radial variation of the wave function is

† The use of Dirac wave functions would give additional higher-order corrections in $Z\alpha$ that are beyond our order of calculation

set by the Bohr radius $a_0 = 1/Z\alpha m$. Hence we may set $|\psi(\mathbf{r})|^2 = |\psi(0)|^2$ in energy shift formula; expanding the wave function about $r = 0$ would give a first correction of order $Z\alpha$ which we neglect. Thus we write

$$\Delta E = |\psi(0)|^2 \int (d^3\mathbf{r}) \Delta V(r) . \tag{8.3.85}$$

This entails

$$\int (d^3\mathbf{r}) \frac{1}{r} e^{-\sqrt{s}r} = 4\pi \int_0^{\infty} dr r e^{-\sqrt{s}r} = \frac{4\pi}{s} , \tag{8.3.86}$$

and so

$$\Delta E = -\alpha(Z\alpha)|\psi(0)|^2 4\pi \int_{4m^2}^{\infty} ds \frac{r(s)}{s} . \tag{8.3.87}$$

Referring to the dispersion relation (8.3.66), we see that

$$\int_{4m^2}^{\infty} ds \frac{r(s)}{s} = 4\pi \frac{dK_{\text{phy}}(k^2)}{dk^2}\bigg|_{k^2=0} . \tag{8.3.88}$$

This is most easily evaluated by using the original parametric representation (8.3.65),

$$\begin{aligned} \int_{4m^2}^{\infty} ds \frac{r(s)}{s} &= -\frac{2}{\pi m^2} \int_0^1 d\alpha \, \alpha^2 \left(\frac{1}{2} - \frac{\alpha}{3}\right) (1 - 2\alpha) \\ &= \frac{1}{15\pi m^2} , \end{aligned} \tag{8.3.89}$$

which yields

$$\Delta E = -\alpha(Z\alpha)|\psi(0)|^2 \frac{4}{15m^2} . \tag{8.3.90}$$

The wave function at the origin vanishes unless the hydrogen-like atom is in an S state. The vacuum polarization correction for states of higher angular momentum l whose wave functions behave as r^l near the origin is of order $(Z\alpha)^{2l}$ relative to that for an S-state. These are negligible corrections. For an S state with principal quantum number n,

$$|\psi_{nS}(0)|^2 = \frac{1}{\pi} \left(\frac{Z\alpha m}{n}\right)^3 , \tag{8.3.91}$$

and

$$\Delta E_{nS} = -\frac{4\alpha}{15\pi} \frac{(Z\alpha)^4}{n^3} m . \tag{8.3.92}$$

Note that even for hydrogen with $Z = 1$, it is worthwhile to distinguish between α and $Z\alpha$. A factor of α corresponds to a radiative correction of order α. The order of $Z\alpha$, on the other hand, gives the order of the binding potential that is involved in a correction. The frequency shift is

$\Delta\nu = \Delta E/2\pi$. Let us work this out for hydrogen, $Z = 1$. The Rydberg energy $E_R = \alpha^2 m/2$, the ionization energy of hydrogen, corresponds to a frequency

$$\nu_R = \alpha^2 m/4\pi \simeq 3.291 \times 10^{15}\ \text{sec}^{-1}. \tag{8.3.93}$$

In these units,

$$\Delta\nu_{nS} = -\frac{8}{15\pi}\frac{\alpha^3}{n^3}\nu_R \simeq -6.597 \times 10^{-8}\frac{\nu_R}{n^3}. \tag{8.3.94}$$

Thus,

$$\Delta\nu_{1S} = -217.1\,\text{MHz}, \tag{8.3.95}$$

$$\Delta\nu_{2S} = -27.14\,\text{MHz}, \tag{8.3.96}$$

and so forth. To test this prediction, we note that the $2S_{1/2}$ and $2P_{1/2}$ energy levels are degenerate not only in the Schrödinger approximation, but also for the Dirac equation with a Coulomb potential. The vacuum polarization contribution lifts this degeneracy and splits the levels with $\Delta\nu_{2S-2P} = -27$ MHz. Experimentally, it is found that the two levels are separated by $+1058$ MHz. We shall conclude this chapter by explaining the discrepancy.

The vacuum polarization effect is, however, easily observed in heavy muonic atoms. Muons in a low energy beam may be captured in a heavy atom where they cascade down their atomic levels by photon emission. Since the muon mass m_μ is some 207 times larger than the electron mass m, the muon Bohr radius is 207 times smaller than that of the electron, and at one stage in the cascade process the muon is well inside the inner electronic K-shell but yet still well outside the nucleus radius. In this region of intermediate energy levels, the muon experiences — to a very good accuracy — a simple Coulomb potential corrected by the effect of vacuum polarization. The average radius for a muon in a level with principal quantum number n is on the order of $n^2/Z\alpha m_\mu$. In lead, for example, where $Z = 82$, this radius is much smaller than the Compton wavelength of the electron m^{-1}, even for $n \sim 5$. Thus the muon is well inside the range of the vacuum polarization potential, and its effects are much larger than in the hydrogen atom. The muonic $5g_{9/2} \to 4f_{7/2}$ transition in lead emits a 431 keV gamma ray, of which the vacuum polarization contributes some 2 keV. Here the vacuum polarization correction is about $\frac{1}{2}$ % in sharp contrast with hydrogen where the correction is only about one part in 10^8.

4 Current correlation structure and e^+e^- annihilation

We have calculated and discussed the current correlation function of free, non-interacting electron fields. The basic structures which we found for this free theory also hold for a fully interacting theory. In general, the current correlation function is defined by

$$K_{\mu\nu}(x - x') = \langle iTj_\mu(x)j_\nu(x')\rangle - C.T., \tag{8.3.97}$$

where $j_\mu(x)$ is the complete current operator to which the photon vector potential is coupled. This current operator is composed of all the lepton fields — electron, muon, and tau, all the quark fields, and all other charged fields. The time evolution of the current is governed by the Hamiltonian of the fully interacting theory. Since the current operator $j_\mu(x)$ is conserved, it suffers no renormalization, and it is a finite (composite) operator.[‡] Nonetheless, the time-ordered product of the two current operators may need a subtraction to render it finite. The situation here is akin to that of the two-point function of the two composite operators $\langle[\phi^2](x)[\phi^2(x')]\rangle$ discussed in Chapter 5, Section 5.2. This subtraction is indicated by the $-C.T.$ in Eq. (8.3.97). As we shall see shortly, this subtraction is needed because the time-ordered product involves the step functions $\theta(x^0 - x'^0)$, $\theta(x'^0 - x^0)$ multiplying singular functions. The unordered current function and the θ-functions are well-defined in the distribution theory sense. But the product of these two generalized functions is not well defined.

We thus turn to first analyze the unordered function

$$\begin{aligned} K^{(+)}_{\mu\nu}(x - x') &= \langle j_\mu(x)j_\nu(x')\rangle \\ &= \int \frac{(d^4k)}{(2\pi)^4} K^{(+)}_{\mu\nu}(k)e^{ik(x-x')}. \end{aligned} \tag{8.3.98}$$

This function has a spectral form, a Lehmann representation. This structure is derived using the methods explained in Chapter 6, Section 1. We briefly review these methods using a more concise operator notation. Except for the vacuum state, the energy-momentum operator P^μ has a spectrum of positive energies whose mass-squared $(-P^2)$ eigenvalues start at some $M^2 > 0$. Since the intermediate vacuum state does not contribute

[‡] More detail: The electromagnetic current is a dimension 3 operator. In general, there are no other dimension 3 operators of the same character (parity, internal symmetry group behavior) that can mix with it. Hence, the electromagnetic current can at most suffer a simple multiplicative renormalization. Current conservation then implies that there is no renormalization because the Ward identities relate the divergence of the insertion of the current in any Green's function to a sum of δ-functions times the Green's function without the current, and these relations would be upset by an additional multiplicative factor. The situation here is akin to that of the stress-energy tensor considered in Section 5.5 of Chapter 5.

to the current correlation, one has the effective operator equality

$$\int_{M^2}^{\infty} ds \int (d^4k)\theta(k^0)\delta(k^2+s)\delta^{(4)}(k-P) = 1\,, \tag{8.3.99}$$

and

$$K^{(+)}_{\mu\nu}(x-x') = \int_{M^2}^{\infty} ds \int (d^4k)\theta(k^0)\delta(k^2+s)e^{ik(x-x')}\langle j_\mu(0)\delta^{(4)}(k-P)j_\nu(0)\rangle\,. \tag{8.3.100}$$

Lorentz invariance and current conservation imply that

$$\langle j_\mu(0)\delta^{(4)}(k-P)j_\nu(0)\rangle = (g_{\mu\nu}k^2 - k_\mu k_\nu)\frac{k^2}{(2\pi)^3}\frac{r(-k^2)}{4\pi}\,. \tag{8.3.101}$$

The matrix element involves a symmetrical product of Hermitian operators. Hence the spectral weight is real. Moreover, the matrix element is a positive-definite structure. Examining the $\mu = 0 = \nu$ component and bearing in mind that $-k^2 > 0$ with $k^2 = \mathbf{k}^2 - k^{02}$, one sees that the spectral weight is a non-negative function,

$$r(s) \geq 0\,. \tag{8.3.102}$$

Writing

$$K^{(+)}_{\mu\nu}(k) = (g_{\mu\nu}k^2 - k_\mu k_\nu)K^{(+)}(k^2)\,, \tag{8.3.103}$$

Eqs. (8.3.100) and (8.3.101) imply that

$$K^{(+)}(k^2) = -\frac{1}{4\pi}\int_{M^2}^{\infty} ds\, s\, r(s)\,\theta(k^0)2\pi\delta(k^2+s)\,. \tag{8.3.104}$$

The time-ordered correlation (8.3.97) is constructed from the unordered functions (8.3.98) by multiplying with the appropriate θ-functions in the time difference. The Fourier transform of Eq. (8.3.104) produces a superposition of $\Delta^{(+)}(x-x';s)$ functions which, when multiplied by the θ-functions, become the $\Delta_+(x-x';s)$ function. The analogous manipulations for a scalar field were discussed in detail in Chapter 6, Section 1, and so we do not repeat the details here. The result is that the Fourier transform of the time-ordered correlation has the conserved form of Eq. (8.3.103) but with

$$K(k^2) = -\frac{1}{4\pi}\int_0^{\infty} ds\frac{sr(s)}{k^2+s}\,. \tag{8.3.105}$$

Going carefully through the steps which have been outlined, one realizes that in the Fourier transform $g_{\mu\nu}k^2 - k_\mu k_\nu$ is replaced by the differential operator combination $g_{\mu\nu}(-\partial^2) + \partial_\mu\partial_\nu$. The $\theta(x^0 - x'^0)$ and $\theta(x'^0 - x^0)$ functions must be commuted with this differential operator so that

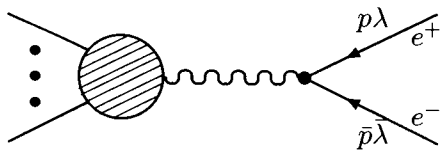

Fig. 8.9. Amplitude for the production of n particles in $e^+ - e^-$ annihilation.

they appear directly multiplying the $\Delta^{(+)}$ to yield the Δ_+ function. The commutator produces terms involving $\delta(x^0 - x'^0)$ which was omitted. When the calculational dust settles, these omitted terms must involve $\delta^{(4)}(x-x')$ so as to achieve Lorentz invariance. We have discarded them because they may be combined with the subtraction denoted by the *C.T.* in Eq. (8.3.97). We have yet to obtain a well-defined result, however. Writing $s = (s+k^2) - k^2$ in Eq. (8.3.105) gives

$$K(k^2) = -\frac{1}{4\pi}\int_{M^2}^{\infty} dsr(s) + \frac{k^2}{4\pi}\int_{M^2}^{\infty} ds\frac{r(s)}{k^2+s}\,. \tag{8.3.106}$$

For the case of the free electron field, $r(s) \sim 1/s$ for large s [Eq. (8.3.78)], and so the first spectral weight integral diverges logarithmically. It is a term independent of k and so Fourier transforms into $\delta^{(4)}(x-x')$ (times the log divergence). Such a divergence will generally appear, but it is removed by the *C.T.* subtraction in Eq. (8.3.97). A finite remainder giving $K(0) \neq 0$ may persist which depends upon how the subtraction is performed. We shall again adopt a "mass-shell" renormalization which requires that $K(k^2)$ vanish at $k^2 = 0$ and yields

$$K_{\text{phys}}(k^2) = \frac{k^2}{4\pi}\int_{M^2}^{\infty} ds\frac{r(s)}{k^2+s-i\epsilon}\,. \tag{8.3.107}$$

Here the proper $i\epsilon$ prescription has been included which implies that

$$\text{Im}K(-s) = -(s/4)r(s)\,. \tag{8.3.108}$$

The spectral weight $r(s)$ is related to the total cross section observed in e^+e^- collisions. This relation obtains if the electromagnetic interaction is treated in lowest order, but no approximation is made with regard to the strong interactions. The total cross section is the sum of all the partial cross sections for the production of a specific final state whose amplitude corresponds to the graph shown in Fig. 8.9. Among these amplitudes is the μ-pair amplitude discussed in Section 2. In this case, the "blob" in the graph above is a simple point vertex of the virtual photon coupled to the $\mu^-\mu^+$ pair. In general, this "blob" contains complex strong interactions such as in the process $e^- + e^+ \to \gamma^* \to \pi^+ + \pi^+ + \pi^- + \pi^-$. Following the work of Section 2, we see that that the amplitude for the production of an

arbitrary final state labeled by n which contains a total four momentum p_n^μ is given by

$$\begin{aligned}\langle n+|p\lambda\bar{p}\bar{\lambda}\rangle &= ie^2\int(d^4x)(d^4x')D_{\mu\nu}(x-x')\langle n+|j^\mu(x)|0\rangle\langle 0|j^\nu(x')|p\lambda\bar{p}\bar{\lambda}\rangle \\ &= i(2\pi)^4\delta^{(4)}(p_n-p-\bar{p})T_n\,, \end{aligned} \tag{8.3.109}$$

where

$$T_n = \frac{e^2}{k^2}\langle n+|j^\mu(0)|0\rangle\bar{v}_{\bar{\lambda}}(\bar{p})\gamma_\mu u_\lambda(p)\,, \tag{8.3.110}$$

with $k = p_n = p+\bar{p}$. The total cross section for unpolarized beams appears as

$$\sigma = \sum_n\int\cdots(2\pi)^4\delta^{(4)}(p_n-p-\bar{p})\frac{1}{4}\sum_{\lambda\bar{\lambda}}|T_n|^2\frac{1}{4p^0\bar{p}^0v_{\rm rel}}\,. \tag{8.3.111}$$

We shall neglect the electron mass, $m_e^2\to 0$. In this case the flux factor becomes $4p^0\bar{p}^0v_{\rm rel} = 2s = -2k^2$. The sum of squared amplitudes involves the final state factor

$$\begin{aligned}&\sum_n\int\cdots(2\pi)^4\delta^{(4)}(p_n-k)\langle 0|j^\nu(0)|n+\rangle\langle n+|j^\mu(0)|0\rangle \\ &\quad = (2\pi)^4\langle 0|j^\nu(0)\delta^{(4)}(P-k)j^\mu(0)|0\rangle \\ &\quad = \frac{1}{2}(g^{\mu\nu}k^2-k^\mu k^\nu)k^2r(-k^2)\,. \end{aligned} \tag{8.3.112}$$

The spin-averaged sum of the initial state factors is the tensor $\frac{1}{4}t_{\mu\nu}(\bar{p},p)$ defined in Section 2. Therefore, since this tensor is conserved, $k^\mu t_{\mu\nu}=0$,

$$\sigma(s) = \left(\frac{e^2}{s}\right)^2\frac{1}{2}s^2r(s)\frac{1}{4}t^\mu{}_\mu(\bar{p},p)\frac{1}{2s}\,. \tag{8.3.113}$$

Since, with $m_e^2=0$, $\frac{1}{4}t^\mu{}_\mu(\bar{p},p) = -2\bar{p}p = s$,

$$\sigma(s) = 4\pi^2\alpha^2r(s)\,. \tag{8.3.114}$$

This result can be compared to the cross section $e^+ + e^-\to\mu^+ + \mu^-$ calculated in Section 2, which for $m_e^2=0$ reads

$$\sigma_{\mu\bar{\mu}} = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left(1+\frac{2m_\mu^2}{s}\right)\,. \tag{8.3.115}$$

The comparison gives a spectral weight for the muon loop contribution to the current-current correlation which is identical, as it must be, to the electron loop contribution given in Eq. (8.3.69).

The relationship between the total cross section in e^+e^- annihilation $\sigma(s)$ and the spectral weight of the current-current correlation function

$r(s)$ given in Eq. (8.3.114) has several applications. These applications use Eq. (8.3.114) to relate the total cross section for the production of hadrons $\sigma_{\rm had}(s)$ to the hadronic contribution to the current-current correlation function which is expressed as a dispersion relation with the weight $r_{\rm had}(s)$. Since the hadronic interactions are strong at low energies, $r_{\rm had}(s)$ cannot be reliably computed in this region from first principles. Instead, the experimentally measured total hadronic cross section and Eq. (8.3.114) may be used to evaluate $r_{\rm had}(s)$. The hadronic contribution to the current correlation function and thus to the vacuum polarization is, for example, a significant contribution to the accurate computation of the magnetic moment of the muon. The lowest-order computation of the muon moment is exactly the same as that for the electron moment which will be presented in Section 5.2. The hadronic vacuum polarization correction to the photon propagator modifies this result in a significant way for the muon. The scale for the momenta running around the loop graph for the muon moment is set by the muon mass $m_\mu\simeq 105$ MeV. Thus only modest energies are needed for the determination of $r_{\rm had}(s)$, $s\lesssim 4\ {\rm GeV}^2$. On the other hand, at high energies the hadronic interaction becomes weak. Thus the hadronic contribution to the current correlation function $K_{\rm had}(k^2)$ at large $|k^2|$ may be computed from the first few terms of the perturbative expansion of the QCD theory. The imaginary part of $K_{\rm had}(k^2)$ is proportional to $r_{\rm had}(s)$ and thus the cross section $\sigma_{\rm had}(s)$. The lowest-order term is the sum of all the simple quark loops. This was the contribution discussed in Section 2. The first correction arises from an additional gluon exchange in these loops. Its energy dependence is given by running strong coupling $\alpha_S(s)\sim 1/\ln(s/\Lambda^2)$. Higher-order perturbative corrections involve higher powers of $\alpha_S(s)$ and they are suppressed at high energies, albeit only by powers of $1/\ln(s/\Lambda^2)$. The first-order correction in $\alpha_S(s)$ is computed in Section 6.2. The nonperturbative contributions responsible for the binding of the (unobservable) quarks and gluons into hadrons give inverse powers of s, and these corrections are not significant at high energy.

5 *Photon states; massless electrons*

We have yet to discuss the construction of single-photon states in the interacting theory. In general, with the "mass-shell" renormalization explained in the discussion of Eq. (8.3.61), the electric charge is fixed in terms of the fine structure constant, $\alpha = e^2/4\pi$. With this normalization, the small k^2 behavior of the photon propagator is given by

$$k^2\simeq 0: \qquad \mathcal{G}(k^2)_{\mu\nu}\simeq e^2\frac{g_{\mu\nu}}{k^2}+\cdots\,, \tag{8.3.116}$$

where the ellipsis stand for the unimportant longitudinal terms. The photon propagator has a pole at $k^2 = 0$ corresponding to the presence of a massless, single-particle state, the photon state $|k\epsilon\rangle$. The wave function is given by

$$\langle 0|A_\mu(x)|k\epsilon\rangle = e\epsilon_\mu e^{ikx} . \tag{8.3.117}$$

The usual reduction method may be used to construct amplitudes involving photons in the initial and final states: An additional photon in the state is obtained from the corresponding Green's function with an additional photon field. The state is obtained from the residue of the pole at $k^2 = 0$. With our somewhat unconventional conventions, this residue must be divided by the charge e which plays the role here of a "photon wave function renormalization constant". Finally, the result is to be contracted with ϵ_μ for an initial state or ϵ^*_μ for a final state. As in the previous discussion around Eq. (8.1.36), the physical states correspond to the two physical polarization vectors ϵ_μ with $\epsilon_0 = 0$, $\mathbf{k}\cdot\vec{\epsilon} = 0$, and $\vec{\epsilon}^{\,2} = 1$. However, as explained in this previous discussion, the photon field is coupled to conserved currents in observable amplitudes, and nothing is changed by a gauge transformation of the polarization vectors $\epsilon_\mu \to \epsilon_\mu + \lambda k_\mu$. This gauge invariance implies that in computing the probability for an unpolarized photon, one can make the replacement

$$\sum\nolimits_\epsilon \epsilon^*_\mu \epsilon_\nu = g_{\mu\nu} . \tag{8.3.118}$$

The photon state construction which has just been described entails the theory renormalized at $n = 4$. Although this method may always be applied in the QED theory, another method is more convenient for the calculation of high-energy processes. Moreover, this alternative procedure is very useful in perturbative calculations in the analogous QCD theory of strong interactions. The leading terms at high energy are obtained by setting the electron mass to zero. However, mass divergencies may appear in intermediate stages of calculation with the massless theory. To see how the massless theory works out, let us return to the lowest order vacuum polarization presented in Eq. (8.3.40). Taking $m^2 = 0$, the parameter integral becomes the standard integral representation of the beta function, giving

$$K(k^2) = -2\frac{(\mathrm{tr}1)}{(4\pi)^2}\Gamma\Big(2-\frac{n}{2}\Big)\frac{\Gamma(\frac{n}{2})^2}{\Gamma(n)}\left(\frac{k^2}{4\pi}\right)^{\frac{n}{2}-2} . \tag{8.3.119}$$

This is the function appearing in the inverse photon propagator

$$\mathcal{G}^{-1}(k)_{\mu\nu} = (g_{\mu\nu}k^2 - k_\mu k_\nu)\left(\frac{1}{e_0^2} - K(k^2)\right) - \frac{1}{\xi_0}k_\mu k_\nu . \tag{8.3.120}$$

With the ultraviolet divergence at $n = 4$ removed by the charge renormalization (8.3.50), the renormalized function $K_{\mathrm{ren}}(k^2)$ is finite at $n = 4$. But in the massless limit, $K_{\mathrm{ren}}(k^2)$ is proportional to $\ln k^2$, and the photon propagator no longer has a simple pole at $k^2 = 0$. An infrared divergence has been brought into the photon propagator by the massless limit.

This difficulty is circumvented by keeping the dimensionality n away from $n = 4$ and by not yet performing the charge renormalization. The momentum space graphs for the photon propagator are well-defined for $n < 4$ and $k^2 > 0$. Once they have been calculated, the result may be continued to arbitrary values of n. The situation here is analogous to the usual integral representation of the gamma function. It defines $\Gamma(z)$ only for $\mathrm{Re}\, z > 0$, but $\Gamma(z)$ may be analytically continued into the entire complex z plane. In the same way, we compute the photon propagator for $n < 4$ and then analytically continue it to arbitrary n values without performing the charge renormalization. Taking $n > 4$ now regulates the infrared divergence. In fact, it follows from Eq. (8.3.119) that with $n > 4$ the vacuum polarization $K(k^2)$ vanishes for $k^2 = 0$ and so

$$n > 4, \quad k^2 \simeq 0 : \qquad \mathcal{G}(k^2)_{\mu\nu} \simeq e_0^2\frac{g_{\mu\nu}}{k^2} + \cdots , \tag{8.3.121}$$

This behavior of the massless theory persists to all orders in the coupling e_0^2 as is easily seen from simple dimensional analysis. The polarization function $K(k^2)$ has the dimension $(\mathrm{mass})^{n-4}$ while the charge has the dimension $(\mathrm{mass})^{4-n}$. Hence $K(k^2)$ has the functional form $(k^2)^{\frac{n}{2}-2}F((k^2)^{\frac{n}{2}-2}e_0^2)$ which vanishes at $k^2 = 0$ when $n > 4$. Thus the reduction formalism can generally be applied to construct photon states if we continue to $n > 4$ and divide by the "wave function" e_0, the bare rather than the mass-shell renormalized charge e. Only after all calculations of a physical process have been made is the limit $n \to 4$ to be taken with, of course, all bare charges expressed in terms of a renormalized charge e^2. The overall basic energy (s) dependence of the process will be exhibited if the renormalized charge is the running coupling evaluated at this energy, $e^2(s)$. Writing the running coupling in terms of the fine structure constant α entails the relation between $e^2(s)$ and $e^2(\mu)$, where μ is a numerical multiple of the electron mass m. This relation produces logarithms of s/m^2.

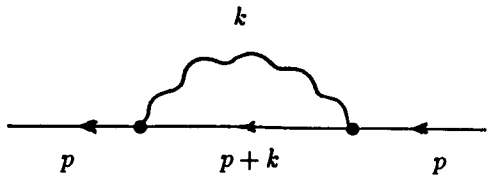

Fig. 8.10. Lowest-order electron self-energy graph.

8.4 Electron self energy

1 Feynman gauge

The electromagnetic interaction alters the electron propagator away from its free form. We write the inverse propagator in momentum space as

$$G_0^{-1}(p) = \gamma p + m_0 + \Sigma_0(p)\,. \tag{8.4.1}$$

This is the bare field propagator which is generated by the unrenormalized sources $\eta_0\,, \bar{\eta}_0$, and m_0 is the bare mass which will be renormalized. We consider here the first-order correction which corresponds to the Feynman graph of Fig. 8.10. This correction is obtained in the usual way of expanding the functional integral representation for the electron propagator, identifying Σ_0, and Fourier transforming. Since similar calculations have been done so often, we shall give no details but simply write down the result after the transformation to n-dimensional Euclidean space:

$$\Sigma_0(p) = -\int \frac{(d_E^n k)}{(2\pi)^n}\,\gamma^\mu\,\frac{m-\gamma(p+k)}{(p+k)^2+m^2}\,\gamma^\nu\left[\frac{e_0^2}{k^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)-\xi_0\frac{k_\mu k_\nu}{(k^2)^2}\right]. \tag{8.4.2}$$

Here $\xi_0 = \xi\mu^{4-n}$ and, for this first-order correction, $e_0^2 = e^2\mu^{4-n}$. The longitudinal terms in the photon propagator, the terms involving $k_\mu k_\nu$, give a gauge-dependent contribution which we shall consider later. For now we shall omit them, which corresponds to making the Feynman gauge choice $\xi = -e^2$ (to lowest order in e^2). The by now very familiar method of exponentiating the denominators, completing the square, performing the k-integral, and changing variables yields

$$\Sigma_0\,(p) = -e^2\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}}\int_0^1 d\lambda\gamma^\mu\left[m-\gamma p(1-\lambda)\right]\gamma_\mu\left[\frac{\lambda m^2+\lambda(1-\lambda)p^2}{\mu^2}\right]^{\frac{n}{2}-2}. \tag{8.4.3}$$

Since the γ-matrix algebra is n-dependent,

$$\gamma^\mu\gamma_\mu = -n\,, \tag{8.4.4}$$

and

$$\gamma^\mu\gamma p\gamma_\mu = \gamma p(n-2) \tag{8.4.5}$$

must be used before taking the physical limit $n \to 4$. Thus we write

$$\Sigma_0(p) = \frac{e^2}{(4\pi)^2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 d\lambda[mn+\gamma p(1-\lambda)(n-2)]\left\{\left[\frac{\lambda m^2+\lambda(1-\lambda)p^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}-1\right\} - \frac{e^2}{(4\pi)^2}\Gamma\left(3-\frac{n}{2}\right)(2m+\gamma p)+\frac{e^2}{(4\pi)^2}\Gamma\left(2-\frac{n}{2}\right)(4m+\gamma p)\,. \tag{8.4.6}$$

Only the last term has a divergence when $n \to 4$. This divergence is given by

$$\Sigma_{\rm div}(p) = -\frac{e^2}{(4\pi)^2}\,\frac{2}{n-4}\,(4m+\gamma p)\,. \tag{8.4.7}$$

Combining it with the bare propagator gives, to our order,

$$G_{\rm div}^{-1}(p) = \gamma p + m_0 + \Sigma_{\rm div}(p) = \left[1-\frac{e^2}{(4\pi)^2}\,\frac{2}{n-4}\right]\left\{\gamma p + m_0\left[1-\frac{e^2}{(4\pi)^2}\,\frac{6}{n-4}\right]\right\}. \tag{8.4.8}$$

With minimal subtraction, the mass is renormalized by writing

$$m_0 = m\left[1+\frac{e^2}{(4\pi)^2}\,\frac{6}{n-4}\right]. \tag{8.4.9}$$

This, of course, defines a running mass parameter m. Remembering that in lowest order $\mu de^2/d\mu = (n-4)e^2$, this minimally subtracted mass obeys

$$\mu\frac{dm}{d\mu} = \delta(e^2)m\,, \tag{8.4.10}$$

in which

$$\delta(e^2) = -6\frac{e^2}{(4\pi)^2}\,. \tag{8.4.11}$$

Writing

$$G_0(p) = z^2 G(p)\,, \tag{8.4.12}$$

where $G(p)$ is the finite, renormalized Green's function, the minimally subtracted wave-function renormalization constant is given by

$$z^2 = 1 + \frac{e^2}{(4\pi)^2}\,\frac{2}{n-4}\,. \tag{8.4.13}$$

The renormalized Green's function contains the renormalized fields $\psi\,,\bar{\psi}$ which are generated by the renormalized sources $\bar{\eta}, \eta$, with

$$\eta_0 = \eta z^{-1}\,, \qquad \bar{\eta}_0 = \bar{\eta} z^{-1}\,. \tag{8.4.14}$$

As with all minimally renormalized parameters, they also depend upon the mass unit μ, with

$$\mu\frac{d}{d\mu}\left\{\begin{array}{c}\eta\\ \bar{\eta}\end{array}\right\} = \gamma(e^2)\left\{\begin{array}{c}\eta\\ \bar{\eta}\end{array}\right\}\,. \tag{8.4.15}$$

In leading order,

$$\gamma(e^2) = \frac{e^2}{(4\pi)^2}\,. \tag{8.4.16}$$

We have now defined a minimally subtracted, renormalized propagator

$$G^{-1}(p) = \gamma p + m + \Sigma_{\rm ren}(p)\,, \tag{8.4.17}$$

in which

$$\Sigma_{\rm ren}(p) = \Sigma_0(p) - \Sigma_{\rm div}(p)\,. \tag{8.4.18}$$

The limit $n \to 4$ yields

$$\Sigma_{\rm ren}(p) = -\frac{e^2}{(4\pi)^2}\int_0^1 d\lambda[4m + 2(1-\lambda)\gamma p]\ln\left[\frac{\lambda m^2 + \lambda(1-\lambda)p^2}{4\pi\mu^2}\right] - \frac{e^2}{(4\pi)^2}[2(1+2\gamma)m + (1+\gamma)\gamma p]\,. \tag{8.4.19}$$

The physical electron mass $m_{\rm phy}$ is identified by the position of the pole in the propagator

$$-\gamma p = m_{\rm phy}\ : \qquad G^{-1}(p) = 0\,, \tag{8.4.20}$$

which gives, to our order,

$$\begin{aligned} m_{\rm phy} &= m + \Sigma_{\rm ren}(p)|_{\gamma p = -m} \\ &= m\left\{1 - \frac{e^2}{(4\pi)^2}\left[3\ln\left(\frac{m^2}{4\pi\mu^2}\right) - 4 + 3\gamma\right]\right\}\,. \end{aligned} \tag{8.4.21}$$

As a check on the calculation, we note that the renormalization group equations (8.4.10) and (8.4.11) for the scale dependence of the running mass imply that

$$\mu\frac{dm_{\rm phy}}{d\mu} = 0\,, \tag{8.4.22}$$

as must be the case. Expressing the propagator in terms of the the physical mass,

$$G^{-1}(p) = \gamma p + m_{\rm phy} + \bar{\Sigma}(p)\,, \tag{8.4.23}$$

and performing a little calculation shows that

$$\begin{aligned} \bar{\Sigma}(p) = &-\frac{e^2}{(4\pi)^2}\int_0^1 d\lambda[4m + 2(1-\lambda)\gamma p]\ln\left[\frac{m^2 + (1-\lambda)p^2}{\lambda m^2}\right] \\ &-\frac{e^2}{(4\pi)^2}(\gamma p + m)\left[\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 2\right]\,. \end{aligned} \tag{8.4.24}$$

2 Infrared divergence of wave function renormalization

The analytic nature of the self-energy function is best described by expressing it as a dispersion relation. This is done in several steps. The integral involving $\ln(1/\lambda)$ in Eq. (8.4.24) is easily computed. To deal with the remaining integral, we change variables to $\kappa = 1 - \lambda$, write

$$[4m + 2\kappa\gamma p] = \frac{d}{d\kappa}[4m\kappa + \kappa^2\gamma p]\,, \tag{8.4.25}$$

integrate by parts, and again change vanishes to $s = m^2/\kappa$. Expressing the logarithm in the surface term which appears in the partial integration as

$$\ln\left[\frac{m^2 + p^2}{m^2}\right] = \int_{m^2}^{\infty}\frac{ds}{s}\,\frac{p^2}{p^2+s}\,, \tag{8.4.26}$$

we find that

$$\begin{aligned} \bar{\Sigma}(p) = &-\frac{e^2}{(4\pi)^2}\left\{4m + \frac{3}{2}\gamma p + (\gamma p + m)\left[\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 2\right]\right. \\ &\left. + \int_{m^2}^{\infty}\frac{ds}{s}\,\frac{p^2}{p^2+s}\left[1 - \frac{m^2}{s}\right]\left[4m + \left(1 + \frac{m^2}{s}\right)\gamma p\right]\right\}\,. \end{aligned} \tag{8.4.27}$$

We write

$$\frac{p^2}{p^2+s}\left[1 - \frac{m^2}{s}\right] = \frac{p^2 + m^2}{p^2+s} - \frac{m^2}{s}\,, \tag{8.4.28}$$

and do the integrals associated with the $-m^2/s$ term. They cancel the $4m + 3\gamma p/2$ terms, and so we arrive at

$$\bar{\Sigma}(p) = -\frac{e^2}{(4\pi)^2}(\gamma p + m)F(\gamma p)\,, \tag{8.4.29}$$

with

$$F(\gamma p) = (m - \gamma p)\int_{m^2}^{\infty}\frac{ds}{s}\left[4m + \left(1 + \frac{m^2}{s}\right)\gamma p\right]\frac{1}{p^2 + s} + \ln\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 2\,. \tag{8.4.30}$$

The electron propagator may now be written as

$$G^{-1}(p) = (\gamma p + m_{\text{phy}})\left[1 - \frac{e^2}{(4\pi)^2}F(\gamma p)\right]. \tag{8.4.31}$$

To obtain the correct normalization for the construction of single-particle electron states, a further renormalization must be made so that the new propagator behaves as $1/(\gamma p + m_{\text{phy}})$ for $-\gamma p$ near m_{phy}. Fields used for the creation of physical states must have propagator poles with unit residues. A further wave function renormalization is needed with

$$Z = 1 + \frac{e^2}{(4\pi)^2}F(\gamma p)\bigg|_{-\gamma p = m}. \tag{8.4.32}$$

This is the usual procedure when we make the intermediate, "floating", renormalization with the minimal subtraction which removes all the *ultraviolet* divergences. For the case at hand it does not work. The function $F(\gamma p)$ has an *infrared* divergence. This infrared divergence appears because the photon is massless. The minimum squared center-of-mass energy of the intermediate $e + \gamma$ state is the squared electron mass, m^2. The dispersion integral in Eq. (8.4.30) starts at $s = m^2$. This places the beginning of the branch line, the branch point, on top of the pole of the propagator as shown in Fig. 8.11. This overlap of pole and branch point would cause no trouble if the behavior at the branch point was "soft" — if the cut contribution vanished at the branch point. But this is not the case here. With $-\gamma p \simeq m$, the singular behavior appears at $s = m^2$ and so the integral in the dispersion relation (8.4.30) gives

$$-\gamma p \simeq m: \qquad F(\gamma p) \simeq -4\ln\left(\frac{p^2 + m^2}{m^2}\right) + \text{const.} \tag{8.4.33}$$

When $-p^2 \to m^2$, $F(\gamma p)$ has a logarithmic infrared divergence.

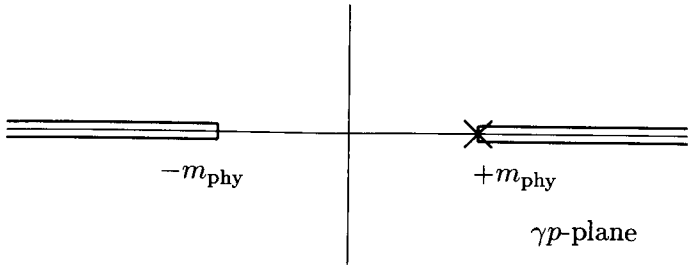

Fig. 8.11. Analytic structure of the Green's function as a function of γp. Since $(\gamma p)^2 = -p^2$, it may be considered as a square root of $-p^2$ and treated as a number (which it becomes when acting on an eigenspinor).

The traditional cure for the infrared divergence is to introduce a mass term into the Lagrange function giving the photon a finite mass. This mass removes the infrared divergence. After a physically observable quantity has been computed, all the singular terms involving the photon mass cancel, and it is taken to vanish. Although the photon mass breaks gauge invariance, it causes no real problem for the Abelian QED theory. This is not the case for a non-Abelian theory such as QCD where giving the gauge fields a mass does cause trouble. A uniform treatment of the infrared problem for non-Abelian as well as Abelian gauge theories can be done by using dimensional continuation to regulate the infrared divergence as well as the ultraviolet divergence. Infrared divergences arise from the *small* momentum behavior in integrals such as

$$\int (d^n k)\,\frac{1}{(k^2)^2}\,,$$

and they are removed with $n > 4$.

In regulating the infrared divergence by dimensional subtraction, it is best not to make the intermediate, minimal subtraction renormalization which we have done. Instead, we return to

$$\Sigma_0(p) = \frac{e^2}{(4\pi)^2}\Gamma\left(2 - \frac{n}{2}\right)\int_0^1 d\lambda[mn + \gamma p(1-\lambda)(n-2)]\left[\frac{\lambda m^2 + \lambda(1-\lambda)p^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}. \tag{8.4.34}$$

With p^2 off the mass shell, $-p^2 \neq m^2$, the integrals defining the self energy $\Sigma_0(p)$ are well defined for $n < 4$. Having calculated $\Sigma_0(p)$, we may now analytically continue the entire result to $n > 4$. With $n > 4$, the intrusion of the branch cut into the pole of $G_0(p)$ causes no difficulty — the discontinuity is "soft" — and the self energy can be expanded in

powers of $(\gamma p + m_{\rm phy})$. The term of zero order gives the physical mass renormalization while the first-order term provides the "physical" wave function renormalization, albeit for a space-time of dimensionality $n > 4$. The higher-order terms in the expansion do not contribute to the pole, and they may be discarded. The limit $n \to 4$ may be taken only after all the pieces have been put together to compute a physically measurable quantity. Thus we write

$$\Sigma_0(p) = \Delta m + (\gamma p + m_{\rm phy})A, \tag{8.4.35}$$

so that

$$G_0^{-1}(p) \simeq (\gamma p + m_{\rm phy})(1 + A), \tag{8.4.36}$$

with

$$m_{\rm phy} = m_0 + \Delta m. \tag{8.4.37}$$

This identifies the physically normalized wave function renormalization,

$$Z_1^{-1} = 1 + A. \tag{8.4.38}$$

For our first order in e^2 self-energy function, we may set $m = m_0 = m_{\rm phy}$ and the expansion in powers of $(\gamma p + m_{\rm phy})$ is equivalent to an expansion in $(\gamma p + m)$. Hence to our order

$$\begin{aligned}\Delta m &= m\frac{e^2}{(4\pi)^2}\Gamma\Big(2-\frac{n}{2}\Big)\int_0^1 d\lambda\,[2+(n-2)\lambda]\left[\frac{\lambda^2 m^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\\ &= m\frac{e^2}{(4\pi)^2}\,\Gamma\Big(2-\frac{n}{2}\Big)\,\frac{n-1}{n-3}\left[\frac{m^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2},\end{aligned} \tag{8.4.39}$$

and

$$\begin{aligned}A &= \frac{e^2}{(4\pi)^2}\Gamma\Big(2-\frac{n}{2}\Big)\int_0^1 d\lambda\Big\{(n-4)[2+(n-2)\lambda]\frac{1-\lambda}{\lambda}\\ &\qquad +(1-\lambda)(n-2)\Big\}\left[\frac{\lambda^2 m^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\\ &= \frac{e^2}{(4\pi)^2}\,\Gamma\Big(2-\frac{n}{2}\Big)\,\frac{n-1}{n-3}\left[\frac{m^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}.\end{aligned} \tag{8.4.40}$$

The $n \to 4$ limit of Δm just reproduces the previous, complete mass renormalization.

Just as in the minimal subtraction scheme, the mass renormalization Δm vanishes in the massless case $m = 0$. But we also see that with $n > 4$, there is no wave function renormalization when $m = 0$. To see how this

massless case works out in detail, we set $m = 0$ to compute

$$\begin{aligned}\Sigma_0(p) &= \gamma p\frac{e^2}{(4\pi)^2}\,\Gamma\Big(2-\frac{n}{2}\Big)(n-2)\int_0^1 d\lambda\,\lambda^{\frac{n}{2}-2}(1-\lambda)^{\frac{n}{2}-1}\left[\frac{p^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\\ &= \gamma p\frac{e^2}{(4\pi)^2}\,\Gamma\Big(2-\frac{n}{2}\Big)\,\frac{2\Gamma(\frac{n}{2})^2}{\Gamma(n-1)}\left[\frac{p^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}.\end{aligned} \tag{8.4.41}$$

With $n > 4$, the coefficient of γp vanishes when $p^2 = 0$, and so there is no wave function renormalization. The situation here is the same as that discussed in Section 3.5 above for the photon propagator in the theory with massless electrons. And just as in that discussion, it is easy to see from simple dimensional analysis that with $n > 4$ there is no wave function renormalization for the electron propagator to all orders in the electric charge.

3 Gauge dependence

We have yet to discuss the gauge-dependent part of the self-energy function (8.4.2),

$$\Sigma_{\rm gauge}(p) = (e^2+\xi)\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\left(\frac{1}{k^2}\right)^2\gamma k\frac{1}{\gamma(p+k)+m}\gamma k. \tag{8.4.42}$$

To exhibit the structure of this contribution, the zeroth-order Ward identity

$$\gamma k = [\gamma(p+k)+m] - [\gamma p + m] \tag{8.4.43}$$

may be used to write

$$\begin{aligned}\Sigma_{\rm gauge}(p) = (e^2+\xi)\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\left(\frac{1}{k^2}\right)^2\\ \left\{-[\gamma p+m]+[\gamma p+m]\frac{1}{\gamma(p+k)+m}[\gamma p+m]\right\}.\end{aligned} \tag{8.4.44}$$

Here we have deleted a term involving γk since it vanishes in the symmetrical k-integration. Note that since this gauge part of the electron self-energy contains a overall factor of $\gamma p + m$, it does not alter the mass renormalization. This must be the case for the physical mass $m_{\rm phy}$ — it is an observable quantity which must be gauge invariant. But we also learn that the intermediate, finite running mass m is also gauge invariant.

The structure of Eq. (8.4.44) is clarified by noting that the self-energy function in coordinate space corresponds to the Green's function alteration $G^{(1)}$ via

$$\Sigma(x-x') = -\Big(\gamma\frac{1}{i}\partial + m\Big)G^{(1)}(x-x')\Big(-\gamma\frac{1}{i}\overleftarrow{\partial}' + m\Big). \tag{8.4.45}$$

The gauge dependent part of the self-energy function (8.4.44) corresponds to the correction

$$G^{(1)}_{\text{gauge}}(x-x') = -(e^2+\xi)\Lambda(x-x')G^{(0)}(x-x')\,, \tag{8.4.46}$$

where

$$\Lambda(x-x') = \mu^{4-n}\int \frac{(d^n_E k)}{(2\pi)^n}\left(\frac{1}{k^2}\right)^2\left[e^{ik(x-x')}-1\right], \tag{8.4.47}$$

and $G^{(0)}(x-x')$ is the free propagator. As shown in Problem 2, this is just the lowest order form of the general result for the dependence on the gauge-fixing parameter ξ. Green's functions calculated with different values of the gauge-fixing parameter are related by

$$G^{(\xi')}(x-x') = \exp\left\{-(\xi'-\xi)\Lambda(x-x')\right\}G^{(\xi)}(x-x')\,. \tag{8.4.48}$$

Although Eq. (8.4.44) exhibits the structure of the gauge part of the electron propagator, the explicit calculation is most easily done by returning to the original form (8.4.42). Rationalizing the electron propagator in the integrand, we encounter

$$\begin{aligned}-\gamma k[m-\gamma(p+k)]\gamma k = k^2(\gamma p+m)+\gamma k(p^2+m^2)\\ -\gamma k\left[(k+p)^2+m^2\right].\end{aligned} \tag{8.4.49}$$

The terms in the second line give an integrand which is odd in k_μ; they give a vanishing integral. Hence

$$\Sigma_{\text{gauge}}(p) = -(\gamma p+m)\,(e^2+\xi)\mu^{4-n}\left[F_1+F_2\right], \tag{8.4.50}$$

in which

$$F_1 = \int \frac{(d^n_E k)}{(2\pi)^n}\frac{1}{k^2}\frac{1}{(p+k)^2+m^2}\,, \tag{8.4.51}$$

and

$$F_2 = (m-\gamma p)\int \frac{(d^n_E k)}{(2\pi)^n}\frac{\gamma k}{(k^2)^2}\frac{1}{(p+k)^2+m^2}\,. \tag{8.4.52}$$

This decomposition is convenient because F_1 has an ultraviolet divergence, but no infrared divergence. On the other hand, F_2 has no ultraviolet divergence, but it does have an infrared divergence when $p^2+m^2\to 0$.

The calculation of the simple loop integral in F_1 has been done several times, and it will not be repeated here. The divergent part of F_1 produces

$$\Sigma_{\text{gauge, div}} = -(\gamma p+m)\frac{e^2+\xi}{(4\pi)^2}\frac{2}{n-4}\,. \tag{8.4.53}$$

The minimally subtracted wave function renormalization (8.4.13) is now

altered to read

$$z^2 = 1-\frac{\xi}{(4\pi)^2}\frac{2}{n-4}\,. \tag{8.4.54}$$

Clearly, the wave function renormalization is gauge-dependent. It is gauge-dependent because the field ψ creates a charged state from the uncharged vacuum state. This process does not correspond to a conserved current. We also learn that the minimal subtraction electron wave function renormalization vanishes to first order if $\xi=0$. This is called the "Landau gauge". In this gauge, the photon propagator has only a transverse part. As is shown in Problem 2, the gauge dependence of the minimally-subtracted wave function renormalization is given to all orders by the overall factor

$$z^2_{\text{gauge}} = \exp\left\{-\frac{\xi}{(4\pi)^2}\frac{2}{n-4}\right\}, \tag{8.4.55}$$

which multiplies a gauge-invariant remaining factor of the wave function renormalization.

The F_1 contribution provides only a wave function renormalization and terms which are regular on the mass shell $p^2+m^2=0$. Hence the infrared part of the renormalized self-energy function is given by the $n\to 4$ limit of F_2. This is easily calculated by the standard methods, and one finds

$$\Sigma_{\text{gauge, ren}}(p) = (\gamma p+m)\,2\frac{e^2+\xi}{(4\pi)^2}\left\{\ln\left(\frac{p^2+m^2}{m^2}\right)+\ldots\right\}. \tag{8.4.56}$$

Remembering Eq. (8.4.33), we now see that the complete first-order renormalized self energy of Eq. (8.4.29) becomes

$$\bar{\Sigma}(p) = (\gamma p+m)\left\{2\frac{\xi+3e^2}{(4\pi)^2}\ln\left(\frac{p^2+m^2}{m^2}\right)+\ldots\right\}. \tag{8.4.57}$$

Note that the logarithmic singularity disappears for the (lowest-order) gauge choice $\xi=-3e^2$. This is the Fried-Yennie gauge. As shown in Problem 2, the leading infrared contributions to the electron self energy in all orders combine to give the exponential correction

$$G^{-1}(p) \simeq (\gamma p+m_{\text{phy}})\exp\left\{2\left(\frac{\xi}{(4\pi)^2}+\frac{3\alpha}{4\pi}\right)\ln\left(\frac{p^2+m^2_{\text{phy}}}{m^2_{\text{phy}}}\right)\right\}. \tag{8.4.58}$$

8.5 Vertex function

The vertex function Γ^μ describes the single-particle irreducible, electron-electron-photon interaction as "dressed" by interactions, with

$$\langle -T\psi(x)\bar{\psi}(y)A_\mu(z)\rangle = \int (dx')(dy')(dz')\mathcal{G}_{\mu\nu}(z-z') G(x-x')\,\Gamma^\nu(x',y',z')\,G(y'-y)\,. \quad (8.5.1)$$

The reduction formula relates this to the electron-electron matrix element of the vector potential. With

$$G(p) = \frac{Z_1^{(\mathrm{phy})}}{\gamma p + m_{\mathrm{phy}}} + \ldots\,, \quad (8.5.2)$$

this matrix element is given by

$$\langle p'\lambda'|A_\mu(0)|p\lambda\rangle = \mathcal{G}_{\mu\nu}(p'-p)\,\bar{u}_{\lambda'}(p')\,\Gamma^\nu_{\mathrm{phy}}(p',p)\,u_\lambda(p)\,, \quad (8.5.3)$$

where

$$\Gamma^\nu_{(\mathrm{phy})}(p',p) = Z_1^{\mathrm{phy}}\,\Gamma^\nu(p',p)\,. \quad (8.5.4)$$

The construction of each of the two physical states removes a factor $\sqrt{Z_1^{(\mathrm{phy})}}/(\gamma p + m_{\mathrm{phy}})$ and so there remains one factor of $Z_1^{(\mathrm{phy})}$. The wave-function renormalization constant has both ultraviolet and infrared divergences. The normalization of the vertex function is revealed by examining the Ward identity

$$(p'-p)_\mu\Gamma^\mu(p',p) = G^{-1}(p') - G^{-1}(p)\,. \quad (8.5.5)$$

Differentiating the Ward identity with respect to p'_μ and then setting $p' = p$ yields

$$\Gamma^\mu(p,p) = \frac{\partial G^{-1}(p)}{\partial p_\mu}\,. \quad (8.5.6)$$

Since

$$G^{-1}(p) = Z_1^{(\mathrm{phy})-1}(\gamma p + m_{\mathrm{phy}}) + \mathrm{Order}\,(\gamma p + m_{\mathrm{phy}})^2\,, \quad (8.5.7)$$

Equations (8.5.4) and (8.5.6) tell us that on the mass shell and with no momentum transfer

$$\bar{u}_{\lambda'}(p)\,\Gamma^\mu_{\mathrm{phy}}(p,p)\,u_\lambda(p) = \bar{u}_{\lambda'}(p)\gamma^\mu u_\lambda(p)\,. \quad (8.5.8)$$

The work of Problem 6 of Chapter 7 shows that this is just the correct normalization to ensure that the electron carries unit total charge. In this section we shall compute the first-order correction to the bare vertex function. With the neglect of vacuum polarization, the mass-shell

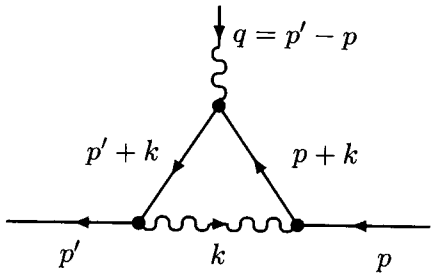

Fig. 8.12. Lowest-order vertex graph.

function gives the current matrix element

$$\langle p'\lambda'|j^\mu(0)|p\lambda\rangle = \bar{u}_{\lambda'}(p')\,\Gamma^\mu_{\mathrm{phy}}(p',p)\,u_\lambda(p)\,. \quad (8.5.9)$$

1 First order correction

The first-order correction to the vertex function is described by the Feynman graph in Fig. 8.12. We shall compute this correction in Feynman gauge where it becomes

$$\Gamma^\mu(p',p) = \gamma^\mu + e^2\mu^{4-n}\int \frac{(d_E^n k)}{(2\pi)^n}\frac{1}{k^2}\,\gamma^\sigma\frac{1}{\gamma(p'+k)+m}\,\gamma^\mu\frac{1}{\gamma(p+k)+m}\,\gamma_\sigma\,. \quad (8.5.10)$$

The Ward identity (8.5.5) serves as a useful check that the signs and overall factors have been done correctly. Using the lowest-order Ward identity

$$\gamma(p'-p) = [\gamma(p'+k)+m] - [\gamma(p+k)+m] \quad (8.5.11)$$

it is easy to check that Eq. (8.5.10) does indeed obey the Ward identity with the electron self-energy function given by Eq. (8.4.2) in Feynman gauge. The usual manipulations produce

$$\Gamma^\mu(p',p) = \Gamma_1^\mu(p',p) + \Gamma_2^\mu(p',p)\,, \quad (8.5.12)$$

with

$$\Gamma_1^\mu(p',p) = \gamma^\mu + \frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{1}{2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 d\alpha d\beta d\gamma\delta\,(1-\alpha-\beta-\gamma)\left[\alpha(p'^2+m^2) + \beta(p^2+m^2) - Q^2\right]^{\frac{n}{2}-2}\gamma^\sigma\gamma^\rho\gamma^\mu\gamma_\rho\gamma_\sigma\,, \quad (8.5.13)$$

and

$$\Gamma_2^\mu(p',p) = \frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\,\Gamma\Big(3-\frac{n}{2}\Big)\int_0^1 d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)$$
$$\Big[\alpha(p'^2+m^2)+\beta(p^2+m^2)-Q^2\Big]^{\frac{n}{2}-3}$$
$$\gamma^\sigma\,[m-\gamma p'+\gamma Q]\,\gamma^\mu\,[m-\gamma p+\gamma Q]\,\gamma_\sigma\,, \qquad (8.5.14)$$

where

$$Q = \alpha p' + \beta p\,. \qquad (8.5.15)$$

Only $\Gamma_1^\mu(p',p)$ contains an ultraviolet divergence. To evaluate this function, we recall that

$$\gamma^\rho\gamma^\mu\gamma_\rho = \gamma^\mu(n-2)\,, \qquad (8.5.16)$$

from which follows

$$\gamma^\sigma\gamma^\rho\gamma^\mu\gamma_\rho\gamma_\sigma = \gamma^\mu(n-2)^2\,. \qquad (8.5.17)$$

Thus, since

$$\int_0^1 d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma) = \frac{1}{2}\,, \qquad (8.5.18)$$

the divergent term occurs in

$$\Gamma^\mu_{\rm div}(p',p) = \gamma^\mu\left[1-\frac{e^2}{(4\pi)^2}\,\frac{2}{n-4}\right]. \qquad (8.5.19)$$

The factor which appears here is, as it must be, the inverse of the minimal subtraction wave function renormalization of the electron propagator given in Eq. (8.4.13). Subtracting this pole, the limit $n \to 4$ may now be taken. We shall also take the mass-shell limit and subtract a further constant to get

$$\Gamma^\mu_{{\rm phy},1}(p',p) = \gamma^\mu\left\{1-2\frac{e^2}{(4\pi)^2}\int_0^1 d\alpha d\beta d\gamma\delta(1-\alpha-\beta-\gamma)\right.$$
$$\left.\ln\left[\frac{-Q^2}{(1-\gamma)^2m^2}\right]\right\}. \qquad (8.5.20)$$

It is convenient to use the momentum transfer variable

$$q = p'-p\,. \qquad (8.5.21)$$

In terms of this variable

$$-Q^2 = \alpha\beta q^2+(\alpha+\beta)^2m^2 = \alpha\beta q^2+(1-\gamma)^2m^2 \qquad (8.5.22)$$

and we see that the correct subtraction has been made to obtain the proper normalization at $q^2=0$,

$$\Gamma^\mu_{{\rm phy},1}(p,p) = \gamma^\mu\,. \qquad (8.5.23)$$

Writing $\alpha = x(1-\gamma)$, $\beta = y(1-\gamma)$ and performing the trivial y and γ integrals gives

$$\Gamma^\mu_{{\rm phy},1}(p',p) = \gamma^\mu\left\{1-\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\ln\left[\frac{m^2+x(1-x)q^2}{m^2}\right]\right\}. \qquad (8.5.24)$$

The integral which appears here is just the "bubble-graph" integral which was extensively discussed at the end of Section 5 of Chapter 3. It may be expressed as a dispersion integral (whose threshold is at $4m^2$), and in fact may be evaluated in terms of logarithms. This analysis will not be repeated here.

Although $\Gamma_2^\mu(p',p)$ has no ultraviolet divergence, it does have an infrared divergence when both p' and p are put on the mass shell. This divergence appears when both α and β are small, making $Q^\mu \to 0$. The infrared problem is contained in the mass shell limit

$$\Gamma^\mu_{2,\rm red}(p',p) = \frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\,\Gamma\Big(3-\frac{n}{2}\Big)\int_0^1 d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)$$
$$\Big[-Q^2\Big]^{\frac{n}{2}-3}\,\gamma^\sigma(m-\gamma p')\,\gamma^\mu(m-\gamma p)\,\gamma_\sigma\,, \qquad (8.5.25)$$

which is the result of placing $Q^\mu = 0$ in the numerator in Eq. (8.5.14). We shall work out only the function which appears between the spinors $\bar u_{\lambda'}(p')\cdots u_\lambda(p)$. In this case we can delete a factor of $m+\gamma p'$ when it appears on the extreme left or a factor of $m+\gamma p$ when it appears on the extreme right. Thus, anticommuting $-\gamma p'$ past γ^σ and $-\gamma p$ past γ_σ, we may make the replacement

$$\gamma^\sigma(m-\gamma p')\,\gamma^\mu(m-\gamma p)\,\gamma_\sigma \to 4p'p\,\gamma^\mu\,. \qquad (8.5.26)$$

To explain the structure of the result, we return to the original momentum-space integral. The replacement above gives

$$\Gamma^\mu_{2,\rm red}(p',p) = e^2\mu^{4-n}\int\frac{(d_E^nk)}{(2\pi)^n}\,\frac{1}{k^2}\,4p'p\,\frac{1}{2p'k+k^2}\,\frac{1}{2pk+k^2}\,\gamma^\mu\,. \qquad (8.5.27)$$

Except for the final factor of γ^μ, this momentum-space integral corresponds to a triangle graph with the spin $\frac{1}{2}$ electron replaced by a charged scalar field. The point to be made is that the infrared contribution to the vertex correction involves soft photons whose coupling are not sensitive to the particle spin. The coupling of a scalar field to the photon is given by $A^\mu\frac{1}{i}(\phi^*\partial_\mu\phi-\partial_\mu\phi^*\,\phi)$, with the bare vertex $2p^\mu+k^\mu$. That is, a scalar

triangle graph would be obtained by the further replacement

$$4p'p \to (2p'+k)(2p+k)\,. \tag{8.5.28}$$

We need also to note that a further subtraction must be made to obtain the physical normalization so that the vertex correction vanishes when $p'=p$. Thus we are led to define

$$R(q^2) = -\frac{1}{2}e^2\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\left(\frac{2p'^\rho+k^\rho}{2p'k+k^2}-\frac{2p^\rho+k^\rho}{2pk+k^2}\right)\frac{g_{\rho\sigma}}{k^2}$$
$$\left(\frac{2p'^\sigma+k^\sigma}{2p'k+k^2}-\frac{2p^\sigma+k^\sigma}{2pk+k^2}\right). \tag{8.5.29}$$

Several remarks need to be made. This is a gauge invariant expression since

$$k_\rho\left(\frac{2p'^\rho+k^\rho}{2p'k+k^2}-\frac{2p^\rho+k^\rho}{2pk+k^2}\right)=0\,. \tag{8.5.30}$$

That is, $R(q^2)$ is not altered if longitudinal terms are added to the photon propagator $g_{\rho\sigma}/k^2$. This same property shows that $R(q^2)$ has no ultraviolet divergence, since

$$R(q^2) = e^2\mu^{4-n}\int\frac{(d_E^n k)}{(2\pi)^n}\frac{1}{k^2}$$
$$\left\{\frac{4p'p+(p'+p)k}{(2p'k+k^2)(2p'k+k^2)}-\frac{4p^2+2pk}{(2pk+k^2)^2}\right\}. \tag{8.5.31}$$

Subtracting $R(q^2)\gamma^\mu$ (except for its $p=p'$ subtraction terms) from Γ_2^μ gives the remainder (on the mass shell)

$$\Gamma_2'^\mu(p',p) = \frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\Gamma\left(3-\frac{n}{2}\right)\int_0^1 d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)[-Q^2]^{\frac{n}{2}-3}$$
$$\left\{\gamma^\sigma\left[\gamma Q\gamma^\mu(m-\gamma p)+(m-\gamma p')\gamma^\mu\gamma Q+\gamma Q\gamma^\mu\gamma Q\right]\gamma_\sigma\right.$$
$$\left.+(p'+p)Q\gamma^\mu\right\}. \tag{8.5.32}$$

The translation involved in completing the square effectively replaces a factor of k^α in the numerator by $-Q^\alpha$, and so the final term here, $(p'+p)\,Q\gamma^\mu$, compensates for the addition of the $(p'+p)k$ term in $R(q^2)$.

We shall work out the vertex $\Gamma_2'^\mu(p',p)$ when it is placed between spinors $\bar{u}_{\lambda'}(p')\ldots u_\lambda(p)$ with p' and p on the mass shell. In this case, noting also that $(p'-p)(p+p)=0$, the curly brackets in Eq. (8.5.32) may be replaced by

$$\{\cdots\}\to 2\gamma p\gamma Q\gamma^\mu+2\gamma^\mu\gamma Q\gamma p'+\gamma^\sigma\gamma Q\gamma^\mu\gamma Q\gamma_\sigma+\frac{1}{2}(p'+p)^2(\alpha+\beta)\gamma^\mu\,. \tag{8.5.33}$$

Here we may write

$$2\gamma p\gamma Q\gamma^\mu\to -4pQ\gamma^\mu+4\gamma Qp^\mu-2m\gamma Q\gamma^\mu$$
$$\to -4pQ\gamma^\mu-4m(\alpha+\beta)p^\mu-2m\gamma Q\gamma^\mu\,, \tag{8.5.34}$$

and similarly

$$2\gamma^\mu\gamma Q\gamma p'\to -4p'Q\gamma^\mu-4m(\alpha+\beta)p'^\mu-2m\gamma^\mu\gamma Q\,, \tag{8.5.35}$$

and so

$$\{\cdots\}\to -\frac{3}{2}(p'+p)^2(\alpha+\beta)\gamma^\mu-4m(\alpha+\beta)(p'+p)^\mu+4mQ^\mu+\gamma^\sigma\gamma Q\gamma^\mu\gamma Q\gamma_\sigma\,. \tag{8.5.36}$$

Next we note that α,β appear symmetrically in the remainder of the integration so that we may replace

$$4mQ^\mu\to 2m(\alpha+\beta)(p'+p)^\mu\,. \tag{8.5.37}$$

More algebra gives

$$\gamma^\sigma\gamma Q\gamma^\mu\gamma Q\gamma_\sigma=(n-2)[\gamma^\mu Q^2-2Q^\mu\gamma Q]\,, \tag{8.5.38}$$

and taking now $n=4$,

$$\{\cdots\}\to$$
$$-\frac{3}{2}(p'+p)^2(\alpha+\beta)\gamma^\mu-2m(\alpha+\beta)(p'+p)^\mu+2Q^2\gamma^\mu+2m(\alpha+\beta)^2(p'+p)^\mu\,. \tag{8.5.39}$$

Finally we use (Problem 6, Chapter 7)

$$(p'+p)^\mu\to 2m\gamma^\mu+i\sigma^{\mu\nu}q_\nu \tag{8.5.40}$$

to secure

$$\Gamma_2'^\mu(p',p)=\frac{e^2}{(4\pi)^2}\int_0^1 d\alpha d\beta d\gamma\delta(1-\alpha-\beta-\gamma)\frac{1}{(1-\gamma)^2m^2+\alpha\beta q^2}$$
$$\left\{(1-\gamma)\left[-\frac{3}{2}(p'+p)^2-4m^2\gamma\right]\gamma^\mu-2m\gamma(1-\gamma)i\sigma^{\mu\nu}q_\nu\right\}. \tag{8.5.41}$$

Here we have omitted the term $2Q^2\gamma^\mu$ from the curly brackets since the Q^2 cancels the denominator and gives a constant term which is removed by the subtractions that must be made to obtain the properly normalized physical function. Changing variables with $\alpha=x(1-\gamma)$, $\beta=y(1-\gamma)$, noting that $-(p'+p)^2=q^2+4m^2$, and subtracting a constant produces

$$\Gamma_2'^\mu(p',p)=\frac{e^2}{(4\pi)^2}\int_0^1 dx\frac{1}{m^2+x(1-x)q^2}$$
$$\left\{\left[\frac{3}{2}-4x(1-x)\right]q^2\gamma^\mu-mi\sigma^{\mu\nu}q_\nu\right\}. \tag{8.5.42}$$

The proper subtraction has been made since this correction vanishes at zero momentum transfer.

2 Form factors, magnetic moment

The general structure of the current matrix element was examined at length in Problem 6 of Chapter 7, where it was shown that

$$\langle p'\lambda'|j^\mu(0)|p\lambda\rangle = \bar{u}_{\lambda'}(p')\left\{\gamma^\mu F_1(q^2) - \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2)\right\}u_\lambda(p)\,. \quad (8.5.43)$$

With the results (8.5.24) and (8.5.42) in hand, we have now calculated, to first order,

$$\begin{aligned} F_1(q^2) =& 1 - \frac{\alpha}{4\pi}\int_0^1 dx \ln\left[\frac{m^2 + x(1-x)q^2}{m^2}\right] \\ &+ \frac{\alpha}{4\pi}\int_0^1 dx\left[\frac{3}{2} - 4x(1-x)\right]\frac{q^2}{m^2 + x(1-x)q^2} + R(q^2)\,, \end{aligned} \quad (8.5.44)$$

and

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dx \frac{m^2}{m^2 + x(1-x)q^2}\,. \quad (8.5.45)$$

The integrals that appear here can all be written in dispersion form with thresholds at $4m^2$. Moreover, they can all be done in terms of fairly simple logarithmic functions. But there is no need to do this analysis here; all we do need for future reference is the small q^2 limit

$$F_1(q^2) \simeq 1 + \frac{\alpha}{4\pi}\frac{2}{3}\frac{q^2}{m^2} + R(q^2)\,. \quad (8.5.46)$$

Of particular interest is the electron magnetic moment anomaly given by

$$\frac{1}{2}(g-2) = F_2(0) = \frac{\alpha}{2\pi}\,. \quad (8.5.47)$$

The experimental value $1/\alpha = 137.0360$ gives $(g-2)/2 = 0.001\,161\,410$. The anomaly has now been computed through order α^4. To get an idea of the amount of calculation that this involves, we note that the order α^4 anomaly requires the evaluation of about 80 graphs, with each contribution typically a 11-dimensional parameter integral whose integrand at an intermediate stage of the calculation may have as many as 2,000 algebraic terms. The integrals are evaluated by a Monte Carlo method that requires on the order of *one year* CPU time on a supercomputer.

The current theoretical result is

$$\frac{1}{2}(g-2)\times 10^{12} = 1\,159\,652\,140\,(28)\,. \quad (8.5.48)$$

The parentheses enclose the error in the last digits which comes almost entirely from the experimental error in the fine structure constant. The current experimental result is of even greater precision:

$$\frac{1}{2}(g-2)\times 10^{12} = 1\,159\,652\,185.5\,(4.0)\,. \quad (8.5.49)$$

3 Infrared part

We have yet to evaluate the infrared piece, Eq. (8.5.31). It is given by

$$\begin{aligned} R(q^2) =& \frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\Gamma\left(3-\frac{n}{2}\right)\int_0^1 dxdy\delta(1-x-y)[m^2+xyq^2]^{\frac{n}{2}-3} \\ &\int_0^1 d\gamma(1-\gamma)^{n-5}\left[4p'p - \frac{1}{2}(1-\gamma)(p'+p)^2\right] - (p'=p \text{ limit})\,. \end{aligned} \quad (8.5.50)$$

Here, of course, we are taking $n > 4$ so as to regulate the infrared divergence. This infrared divergence must cancel when a physically measurable process is computed. An illustration of how this cancellation works out will be presented in the next section. The piece involving $(1-\gamma)(p'+p)^2$ is regular in the limit $n \to 4$. Writing $-(p'+p)^2 = q^2 + 4m^2$ and subtracting the $q^2 = 0$ term, this piece becomes

$$R_1(q^2) = \frac{\alpha}{4\pi}\int_0^1 dx \frac{1}{2}\left[1 - 4x(1-x)\right]\frac{q^2}{m^2 + x(1-x)q^2}\,. \quad (8.5.51)$$

Using $4p'p = -2(q^2 + 2m^2)$ and subtracting out the $q^2 = 0$ limit with

$$\begin{aligned} &\frac{-4m^2}{n-4}\left\{[m^2+xyq^2]^{\frac{n}{2}-3} - m^{n-6}\right\} \\ &= \frac{4xyq^2}{n-4}[m^2+xyq^2]^{\frac{n}{2}-3} - \frac{4}{n-4}m^{n-4}\left\{\left[\frac{m^2+xyq^2}{m^2}\right]^{\frac{n}{2}-2} - 1\right\} \\ &\to \frac{4xyq^2}{n-4}[m^2+xyq^2]^{\frac{n}{2}-3} - 2\ln\left[\frac{m^2+xyq^2}{m^2}\right]\,, \end{aligned} \quad (8.5.52)$$

another finite function is encountered

$$R_2(q^2) = -\frac{\alpha}{4\pi}2\int_0^1 dx \ln\left[\frac{m^2 + x(1-x)q^2}{m^2}\right]\,. \quad (8.5.53)$$

Thus we have the decomposition

$$R(q^2) = R_1(q^2) + R_2(q^2) + R_3(q^2)\,, \quad (8.5.54)$$

with the remaining singular part

$$R_3(q^2) = -\frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{\Gamma\left(3-\frac{n}{2}\right)}{n-4}2q^2\int_0^1 dx[1-2x(1-x)]\left[m^2+x(1-x)q^2\right]^{\frac{n}{2}-3}. \tag{8.5.55}$$

Again we record the small q^2 limit

$$R(q^2) \simeq -\frac{\alpha}{4\pi}\left\{\frac{1}{6}+\frac{4}{3}\frac{\Gamma\left(3-\frac{n}{2}\right)}{n-4}\left[\frac{m^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\right\}\frac{q^2}{m^2}. \tag{8.5.56}$$

It is worth comparing this result of regulating the infrared divergence by dimensional continuation with that which is obtained by introducing a small photon mass λ. Going back to the beginning, making the replacement

$$\frac{1}{k^2} \to \frac{1}{k^2+\lambda^2}, \tag{8.5.57}$$

and tracing through the steps, it is not hard to see that with a finite photon mass the limit $n \to 4$ may be taken since the denominator is now regulated. The only divergence that appears when $\lambda^2 \to 0$ is contained in [c.f. Eq. (8.5.50)]

$$R(q^2) = \frac{e^2}{(4\pi)^2}\int_0^1 dxdy\delta(1-x-y)\int_0^1 d\gamma\frac{1-\gamma}{\gamma\lambda^2+(m^2+xyq^2)(1-\gamma)^2}$$
$$\left[4p'p-\frac{1}{2}(1-\gamma)(p'+p)^2\right]-(p'=p \text{ limit}). \tag{8.5.58}$$

Since we are interested only in the $\lambda^2 = 0$ region, where a singularity appears at $\gamma = 1$, we may replace $\gamma\lambda^2$ by λ^2 and use the integral

$$\int_0^1 d(1-\gamma)\frac{1-\gamma}{\lambda^2+(m^2+xyq^2)(1-\gamma)^2} \simeq \frac{1}{2}\frac{1}{m^2+xyq^2}\ln\left[\frac{m^2+xyq^2}{\lambda^2}\right]. \tag{8.5.59}$$

This we need for the $4p'p$ term in the square brackets in Eq. (8.5.58). The other term with the factor $(1-\gamma)$ needs no regulation. Hence

$$R(q^2) = \frac{e^2}{(4\pi)^2}\int_0^1 dx\frac{1}{m^2+x(1-x)q^2}\left\{2p'p\ln\left[\frac{m^2+x(1-x)q^2}{\lambda^2}\right]\right.$$
$$\left.-\frac{1}{2}(p'+p)^2\right\}-(p'=p \text{ limit}). \tag{8.5.60}$$

Yet again we record the small q^2 limit

$$R(q^2) \simeq -\frac{\alpha}{4\pi}\left\{\frac{2}{3}\ln\left(\frac{m^2}{\lambda^2}\right)+\frac{1}{6}\right\}\frac{q^2}{m^2}. \tag{8.5.61}$$

4 Massless limit

At high energies with large momentum transfer q^2, the mass of the electron may be neglected. It is not convenient to take the massless limit of our results because we have previously employed a mass-shell renormalization which trades the unit of mass μ for the electron mass m. It is necessary to keep μ in the massless limit. So we return to the decomposition (8.5.12) and set $p'^2 = p^2 = 0$ and $m = 0$. In this limit Eq. (8.5.13) becomes, with the aid of Eq. (8.5.17),

$$\Gamma_1^\mu(p',p) = \gamma^\mu\left\{1+\frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{1}{2}\Gamma\left(2-\frac{n}{2}\right)\right.$$
$$\left.(n-2)^2\int_0^1 d\alpha d\beta d\gamma\delta(1-\alpha-\beta-\gamma)\left[\alpha\beta q^2\right]^{\frac{n}{2}-2}\right\}. \tag{8.5.62}$$

The parameter integration is of the form

$$\int_0^1 d\alpha_1\ldots d\alpha_N\,\delta(1-\alpha_1-\ldots-\alpha_N)\alpha_1^{z_1-1}\cdots\alpha_N^{z_N-1}$$
$$=\frac{\Gamma(z_1)\cdots\Gamma(z_N)}{\Gamma(z_1+\ldots+z_N)}. \tag{8.5.63}$$

This formula is easily established. The standard integral representation of the Γ-function gives

$$\Gamma(z_1)\cdots\Gamma(z_N) = \int_0^\infty ds_1\cdots ds_N\, e^{-(s_1+\ldots+s_N)}s_1^{z_1-1}\cdots s_N^{z_N-1}. \tag{8.5.64}$$

Making the familiar variable change by inserting

$$1 = \int_0^\infty dS\delta(S-s_1-\ldots-s_N) \tag{8.5.65}$$

in the integrand with $s_a = \alpha_a S$ and performing the resulting S integral proves the formula.

Using Eq. (8.5.63),

$$\Gamma_1^\mu(p',p) = \gamma^\mu\left\{1+\frac{e^2}{(4\pi)^2}\frac{1}{2}(n-2)\Gamma(2-\frac{n}{2})\frac{\Gamma(\frac{n}{2}-1)^2}{\Gamma(n-2)}\left[\frac{q^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\right\}. \tag{8.5.66}$$

The derivation of this formula requires that the space-time dimension lie in the interval $2 < n < 4$. When $n \to 2$ an infrared divergence appears; when $n \to 4$ there is an ultraviolet divergence. However, as discussed in the previous section on the electron self energy, once the calculation has been done, we are free to analytically continue to arbitrary n. Moreover, when the continuation is performed with $n > 4$, there is no physical wave function renormalization for the massless theory. We do this here, and

for later convenience use $\Gamma\left(\frac{n}{2}-1\right)=\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-2\right)$ to write

$$\Gamma_1^\mu(p',p)=\gamma^\mu\Big\{1 - \frac{e^2}{(4\pi)^2}\Big(\frac{n}{2}-1\Big)\Big(\frac{n}{2}-2\Big)\frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-2)^2}{\Gamma(n-2)}\Big[\frac{q^2}{4\pi\mu^2}\Big]^{\frac{n}{2}-2}\Big\}. \tag{8.5.67}$$

To evaluate the second piece given in Eq. (8.5.14), we note that the previous manipulations show that, within spinors for massless particles,

$$\gamma^\sigma[-\gamma p'+\gamma Q]\gamma^\mu[-\gamma p+\gamma Q]\gamma_\sigma \to -\gamma^\mu q^2[2(1-\alpha-\beta)+(n-2)\alpha\beta]. \tag{8.5.68}$$

Hence

$$\begin{aligned}\Gamma_2^\mu(p',p) &= -\gamma^\mu\frac{e^2}{(4\pi)^2}\Gamma\Big(3-\frac{n}{2}\Big)\int_0^1 d\alpha d\beta d\gamma\delta(1-\alpha-\beta-\gamma)\\ &\qquad (\alpha\beta)^{\frac{n}{2}-3}\Big[2\gamma+(n-2)\alpha\beta\Big]\Big[\frac{q^2}{4\pi\mu^2}\Big]^{\frac{n}{2}-2}\\ &= -\gamma^\mu\frac{e^2}{(4\pi)^2}\frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-2)^2}{\Gamma(n-2)}\Big[2+\Big(\frac{n}{2}-2\Big)^2\Big]\Big[\frac{q^2}{4\pi\mu^2}\Big]^{\frac{n}{2}-2}.\end{aligned} \tag{8.5.69}$$

The validity of this calculation requires that we start from $n>4$; an infrared divergence appears when $n\to 4$. In fact, a *double pole* appears at $n=4$. As will be explained in more detail in the next section, there are two types of infrared divergences with massless particles. One arises from the *soft* photon region $k^\mu\to 0$. The other, *collinear* divergence appears because if k^μ and p^μ are the momenta of two (on-shell) massless particles, then $kp=0$ when both spatial momenta are parallel. This gives rise to a divergence in an angular integration at $n=4$. The double pole corresponds to the confluence of these two separate divergences.

Adding the two pieces gives

$$\begin{aligned}\Gamma^\mu(p',p) =& \gamma^\mu\Big\{1-\frac{e^2}{(4\pi)^2}\frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-2)^2}{\Gamma(n-2)}\\ &\times\Big[\frac{n}{2}+2\Big(\frac{n}{2}-2\Big)^2\Big]\Big[\frac{q^2}{4\pi\mu^2}\Big]^{\frac{n}{2}-2}\Big\}.\end{aligned} \tag{8.5.70}$$

8.6 $e^+ + e^- \to \mu^+ + \mu^- + \gamma$

We turn now to extend the previous calculation of muon pair production in high-energy electron-positron collisions of Section 2 to include the production of an extra photon. We shall restrict the discussion to the case in

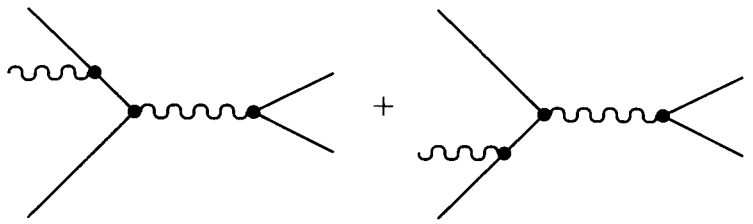

Fig. 8.13. Amplitude for additional photon production in $e^+ - e^-$ annihilation to muon pairs, in which the photons are radiated by muons.

which the photon is emitted by the muons as described by the two Feynman graphs shown in Fig. 8.13. The photons may also be radiated from the electron and positron in the initial state. Indeed, this is the much larger contribution. The electron is very light, it is easily accelerated, and it radiates more copiously than does the muon. There are, however, two reasons for considering only the radiation from the muons. First, this process is somewhat easier to compute, and our intent is to describe the physics of photon emission with a simple example. Second, as was explained in Section 2, quark pair production at high energies is essentially the same as muon pair production. The additional photon corresponds to an additional gluon radiated from a quark.

It is a straightforward matter to follow the method described in Section 2 to derive the amplitude T for the photon production process. The result is:

$$T=\epsilon_\mu^* T^\mu = e^3\epsilon_\mu^* T^{\mu\nu}\frac{1}{-s}L_\nu, \tag{8.6.1}$$

where ϵ_μ is the photon polarization vector,

$$T^{\mu\nu}=\bar{u}_{\lambda'}(p')\Big\{\gamma^\mu\frac{1}{\gamma(p'+k')+m}\gamma^\nu+\gamma^\nu\frac{1}{m-\gamma(\bar{p}'+k')}\gamma^\mu\Big\}v_{\bar{\lambda}'}(\bar{p}'), \tag{8.6.2}$$

and

$$L_\nu=\bar{v}_{\bar{\lambda}}(\bar{p})\gamma_\nu u_\lambda(p). \tag{8.6.3}$$

We now use m to denote the mass of the muon. As always, it is worthwhile to check the calculation to confirm the gauge invariance

$$k'_\mu T^{\mu\nu}=0=T^{\mu\nu}(p'+\bar{p}'+k')_\nu. \tag{8.6.4}$$

This is easily done with the simple Ward identities for γ^μ and γ^ν as was done previously in checking the Ward identity for the vertex function [c. f. Eq. (8.5.11)]. The cross section has an infrared divergence for the production of soft photons. We shall regulate this divergence by continuing to a larger number of space-time dimensions, $n>4$. Thus the partial

cross section appears as

$$\Delta\sigma_\gamma = \int_\Delta \frac{(d^{n-1}p')}{(2\pi)^{n-1}} \frac{1}{2p'^0} \int \frac{(d^{n-1}\bar{p}')}{(2\pi)^{n-1}} \frac{1}{2\bar{p}'^0} \int \frac{(d^{n-1}k')}{(2\pi)^{n-1}} \frac{1}{2k'^0}$$
$$(2\pi)^n \delta^{(n)}(p' + \bar{p}' + k' - p - \bar{p}) T^*_\mu T^\mu \frac{1}{4p^0\bar{p}^0 v_{\rm rel}} . \quad (8.6.5)$$

Here we have summed over the polarizations of the produced photons by writing $T^*_\mu T^\mu$. If the spins of the electrons and muons are not measured, additional spin sums and averages are to be made.

We shall first use this cross section to illustrate the nature of the infrared problem in quantum electrodynamics. We shall then go on to describe how infrared divergences cancel when a suitably "coarse-grained" measurement is performed. In particular, we shall calculate the cross section for the angular distribution of energy, the "antenna pattern" of the produced energy. This is a result that may be directly transferred to the case of quark and gluon production, as we shall discuss.

1 *Soft photons*

Rationalizing the Fermion propagators and noting that the produced particles are on their mass shells gives

$$T^{\mu\nu} = \bar{u}_{\lambda'}(p') \left\{ \gamma^\mu \frac{m - \gamma(p' + k')}{2p'k'} \gamma^\nu + \gamma^\nu \frac{m + \gamma(\bar{p}' + k')}{2\bar{p}'k'} \gamma^\mu \right\} v_{\bar{\lambda}'}(\bar{p}') . \quad (8.6.6)$$

In the soft photon limit, the photon four momentum k' may be omitted in the numerators. Anticommuting $\gamma p'$ to the left past γ^μ and $\gamma\bar{p}'$ to the right and using the Dirac spinor equation gives the soft photon limit

$$T^{\mu\nu} \to T^{\mu\nu}_{\rm soft} , \quad (8.6.7)$$

in which

$$T^{\mu\nu}_{\rm soft} = \left(\frac{p'^\mu}{p'k'} - \frac{\bar{p}'^\mu}{\bar{p}'k'} \right) \bar{u}_{\lambda'}(p') \gamma^\nu v_{\bar{\lambda}'}(\bar{p}') . \quad (8.6.8)$$

The soft photon amplitude is a simple multiple of the amplitude with no photon emission. Moreover, the multiplying factor is the same factor that one would obtain if the Fermion were replaced with a scalar particle.

We shall not present a detailed calculation of the photon emission, but only demonstrate that the divergence arising from soft photon production is canceled by the infrared divergence of the vertex function exhibited in the previous section. The vertex correction modifies the muon pair production amplitude as shown in Fig. 8.14. The point to be made is that the additional production of very low energy photons cannot be detected experimentally. Any scattering or production process involving

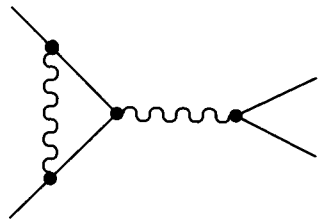

Fig. 8.14. Vertex correction to muon pair production.

charged particles is accompanied by the production of an arbitrarily large number of very soft photons which carry off such a small amount of energy and momentum that they escape detection. Here we are working to the lowest, non-trivial order in the electric charge so that we need to deal with the emission of only a single photon. To this order, the experimentally observed cross section $\Delta\sigma_{\rm exp}$ for the production process $e^+ + e^- \to \mu^+ + \mu^-$ with the possibility of an additional undetected photon is given by

$$\Delta\sigma_{\rm exp} = \Delta\sigma_{\mu\bar{\mu}} + \Delta\sigma^{\rm (soft)}_\gamma . \quad (8.6.9)$$

We shall now show that the infrared divergent vertex correction in the cross section with no photon production $\Delta\sigma_{\mu\bar{\mu}}$ cancels the soft photon divergence in the photon emission cross section $\Delta\sigma^{\rm (soft)}_\gamma$.

To exhibit the cancellation, we note that the addition of the very soft photon momentum and energy k'_μ can be neglected in the energy-momentum conserving δ-function in the production cross section (8.6.5). Therefore, in this leading approximation, the soft photon cross section factorizes into

$$\Delta\sigma^{\rm (soft)}_\gamma \simeq I \Delta\sigma^{(0)}_{\mu\bar{\mu}} , \quad (8.6.10)$$

where $\Delta\sigma^{(0)}_{\mu\bar{\mu}}$ is the lowest-order production cross section and where, in view of Eq. (8.6.8),

$$I = e^2\mu^{4-n} \int_\delta \frac{(d^{n-1}k')}{(2\pi)^{n-1}} \frac{1}{2k'^0} \left(\frac{p'}{p'k'} - \frac{\bar{p}'}{\bar{p}'k'} \right)^2 . \quad (8.6.11)$$

Here we have inserted the proper mass unit μ which was previously neglected. The subscript δ on the integral sign signifies that the integration is limited to a small domain of soft momenta. According to the work of the previous section, the infrared part of the vertex correction involves a function $R(q^2)$ multiplying the bare vertex γ^μ. Here the momentum transfer variable $q^2 = (p' - p)^2$ must be analytically continued into $-s = (p' + \bar{p}')^2$. Thus the infrared part of the vertex correction to the production cross section $\Delta\sigma_{\mu\bar{\mu}}$ is obtained by multiplying the lowest order amplitude by the factor $[1 + R(-s)]$, and the potential infrared divergent part of the

experimentally observable cross section is given by

$$\Delta\sigma_{\text{exp, infra}} = [2Re\, R(-s) + I]\, \Delta\sigma^{(0)}_{\mu\bar{\mu}} . \tag{8.6.12}$$

Twice the real part of $R(-s)$ appears from the order e^2 cross term in $|1 + R(-s)|^2$. The analytic continuation of Eq. (8.5.29) yields

$$R(-s) = -\frac{1}{2}e^2\mu^{4-n}\int \frac{(d^n_E k)}{(2\pi)^n}\frac{1}{k^2}\left(\frac{2p'+k}{2p'k+k^2} - \frac{2\bar{p}'-k}{2\bar{p}'k-k^2}\right)^2 . \tag{8.6.13}$$

Transforming back to Minkowiski space-time, it is easy to see that the infrared divergencies in R and I cancel. Namely, performing the k^0-integration by closing it in the lower half plane, the pole at $k^0 = |k|$ in the $1/k^2$ factor in the integrand of Eq. (8.6.13) contributes an expression that is identical in form to the soft photon emission factor I except for an overall minus sign. Thus the infrared divergence is canceled. The remaining spatial integration in R, however, does not have the restriction to small values as is indicated by the δ in the integration for I. The spatial momentum integration in the piece of R which we are considering — the piece from the k^0 integration of the $1/k^2$ pole — in fact diverges at large values (for $n = 4$). This ultraviolet divergence is canceled when one adds the contributions of the other poles encountered in the initial k^0 integration. Although the non-covariant method that we have just described does easily exhibit the cancellation of the infrared divergencies, it is a cumbersome method with which to compute the finite remainder. For this we turn to a covariant treatment.

For ease of calculation, we shall use an exponential cutoff to limit the emitted photon energy to small values by writing

$$I = e^2\mu^{4-n}(-2p'\bar{p}')\int \frac{(d^{n-1}k')}{(2\pi)^{n-1}}\frac{1}{2k'^0}e^{k'Z}\frac{1}{p'k'\bar{p}'k'} - (p' = \bar{p}'\,\text{limit}) . \tag{8.6.14}$$

Here Z is chosen to be a large time-like vector so that $k'Z < 0$, and the exponential is a damping factor. The familiar machinery can now be brought into operation. Exponentiating the denominators and making the usual variable change gives

$$I = e^2\mu^{4-n}(-2p'\bar{p}')\int_0^1 dx\int_0^\infty u du\int \frac{(d^{n-1}k')}{(2\pi)^{n-1}}\frac{1}{2k'^0}\exp\{k'(Z+uQ)\} - (p' = \bar{p}'\,\text{limit}) , \tag{8.6.15}$$

where

$$Q = xp' + (1-x)\bar{p}' . \tag{8.6.16}$$

To compute the momentum integral, we note that the massless, scalar-field unordered 2-point function in n dimensions is given by

$$\langle\phi(x)\phi(0)\rangle = \int \frac{(d^{n-1}k)}{(2\pi)^{n-1}}\frac{1}{2k^0}e^{ikx} . \tag{8.6.17}$$

For a space-like coordinate, $x^2 > 0$, this is the same as the Euclidean 2-point function,

$$\langle\phi(x)\phi(0)\rangle = \int \frac{(d^n_E k)}{(2\pi)^n}\frac{1}{k^2}e^{ikx} . \tag{8.6.18}$$

Exponentiating the denominator, completing the square and performing the resulting Gaussian integral, and then performing the parameter integration yields

$$\langle\phi(x)\phi(0)\rangle = \frac{\Gamma\left(\frac{n}{2}-1\right)}{(4\pi)^{n/2}}\left(\frac{4}{x^2}\right)^{\frac{n}{2}-1} . \tag{8.6.19}$$

The integral we need is a trivial analytic continuation of this result and so

$$I = \frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\Gamma\left(\frac{n}{2}-1\right)2^{n-2}(-2p'\bar{p}')\int_0^1 dx \int_0^\infty du\frac{u}{[-u^2Q^2 - 2uQZ - Z^2]^{\frac{n}{2}-1}} - (p' = \bar{p}'\,\text{limit}) . \tag{8.6.20}$$

The soft photon divergence corresponds to the region of large u. This divergence is isolated by writing the numerator as

$$u = \frac{1}{-2Q^2}\left[(-2uQ^2 - 2QZ) + 2QZ\right] . \tag{8.6.21}$$

The last $+2QZ$ term gives a well-defined $n = 4$ limit, and it can be neglected. The combination in the parenthesis combines with the denominator in Eq. (8.6.20) to form a total derivative, and so the soft photon piece is given by

$$I_{\text{div}} = \frac{e^2}{(4\pi)^2}\frac{4\Gamma\left(\frac{n}{2}-1\right)}{n-4}\left[\frac{1}{4\pi\mu^2(-Z^2/4)}\right]^{\frac{n}{2}-2} \int_0^1 dx\frac{-2p'\bar{p}'}{-[p'x + \bar{p}'(1-x)]^2} - (p' = \bar{p}'\,\text{limit}) . \tag{8.6.22}$$

Using $-2p'\bar{p}' = -(p'+\bar{p}')^2 - 2m^2 = s - 2m^2$, and explicitly subtracting the $p' = \bar{p}'$ limit gives

$$I_{\text{div}} = \frac{e^2}{(4\pi)^2}\frac{4\Gamma\left(\frac{n}{2}-1\right)}{n-4}\left[\frac{1}{4\pi\mu^2(-Z^2/4)}\right]^{\frac{n}{2}-2}(s-4m^2) \int_0^1 dx\frac{1-2x(1-x)}{m^2 + x(1-x)(s-4m^2)} . \tag{8.6.23}$$

The infrared divergence in this expression when $n \to 4$ is canceled by the infrared divergence in the vertex correction, the divergence contained in the $R_3(q^2)$ of Eq. (8.5.55). To exhibit this cancellation, we first rearrange $R_3(q^2)$ in the form

$$R_3(q^2) = -2\frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{\Gamma\left(3-\frac{n}{2}\right)}{n-4}\int_0^1 dx\left\{\left[q^2+2m^2\right]\left[m^2+x(1-x)q^2\right]^{\frac{n}{2}-3} -2\left[m^2+x(1-x)q^2\right]^{\frac{n}{2}-2}\right\}. \tag{8.6.24}$$

Next, we change variables to $x = (1+y)/2$, with $-1 < y < 1$. Since the integrand is even in y, we double the result and integrate only in the interval $0 < y < 1$. Making the necessary analytic continuation $q^2 \to -s$ now yields

$$R_3(-s) = -2\frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{\Gamma\left(3-\frac{n}{2}\right)}{n-4}\int_0^1 dy\left\{\left[2m^2-s\right]\left[m^2-i\epsilon-\frac{s}{4}+y^2\frac{s}{4}\right]^{\frac{n}{2}-3} -2\left[m^2-i\epsilon-\frac{s}{4}+y^2\frac{s}{4}\right]^{\frac{n}{2}-2}\right\}. \tag{8.6.25}$$

Here the prescription $m^2 \to m^2 - i\epsilon$ has been included so as to obtain the correct continuation. We write the integration over the first term in Eq. (8.6.25) as the integral from 0 to ∞ minus the integral from 1 to ∞, using

$$\Gamma\left(3-\frac{n}{2}\right)\int_0^\infty dy[ay^2+b]^{\frac{n}{2}-3} = \int_0^\infty \frac{dt}{t}t^{3-\frac{n}{2}}e^{-bt}\int_0^\infty dy e^{-ay^2t} = \sqrt{\frac{\pi}{4a}}b^{\frac{1}{2}(n-5)}\Gamma\left(\frac{5-n}{2}\right), \tag{8.6.26}$$

to produce the decomposition

$$R_3(-s) = C(s) + S(s), \tag{8.6.27}$$

with

$$C(s) = \frac{\alpha}{\pi}\sqrt{\pi}\Gamma\left(\frac{5-n}{2}\right)\frac{s-2m^2}{[s(4m^2-i\epsilon-s)]^{1/2}}\frac{1}{n-4}\left[\frac{4m^2-i\epsilon-s}{16\pi\mu^2}\right]^{\frac{1}{2}(n-4)}, \tag{8.6.28}$$

and

$$S(s) = -2\frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{\Gamma\left(3-\frac{n}{2}\right)}{n-4}\left\{[s-2m^2]\int_1^\infty dy\left[m^2-\frac{s}{4}+\frac{y^2s}{4}\right]^{\frac{n}{2}-3} -2\int_0^1 dy\left[m^2-i\epsilon-\frac{s}{4}+\frac{y^2s}{4}\right]^{\frac{n}{2}-2}\right\}. \tag{8.6.29}$$

The function $C(s)$ is the relativistic generalization of the Coulomb scattering correction in the final mu pair state. In the non-relativistic limit,

the graph shown in Fig. 8.14 corresponds to the first-order correction brought about by the Coulomb force between the produced mu pair. Including all the higher orders of this non-relativistic Coulomb interaction replaces the plane-wave final state with a Coulomb wave function. In the non-relativistic limit, the μ^+ and μ^- are produced at the same spatial point, and these final state corrections correspond to multiplying the amplitude by the Coulomb wave function at zero relative coordinate, $\psi^*(0)$. These Coulomb corrections contain, of course, only a part of the full relativistic amplitude. The $-i\epsilon$ requires that $4m^2 - i\epsilon - s = e^{-\pi i}(s-4m^2)$. Remembering the discussion of the kinematics of the incident flux which is involved with a cross section computation,

$$\sqrt{s(s-4m^2)} = \frac{1}{2}sv_{\rm rel}, \tag{8.6.30}$$

where $v_{\rm rel}$ is the relative velocity of two outgoing particles. Hence, the $n \to 4$ limit may be written as

$$\begin{aligned} C(s) &= i\frac{\alpha}{v_{\rm rel}}\left(\frac{2s-4m^2}{s}\right)\left\{\frac{1}{n-4}+\frac{1}{2}\ln\left[e^{-\pi i}\left(\frac{s-4m^2}{4\pi\mu^2}\right)\right]+\text{const.}\right\} \\ &= i\frac{\alpha}{v_{\rm rel}}\left(\frac{2s-4m^2}{s}\right)\left\{\frac{1}{n-4}+\frac{1}{2}\ln\left(\frac{s-4m^2}{4\pi\mu^2}\right)+\text{const.}\right\} \\ &\quad +\frac{\pi\alpha}{2v_{\rm rel}}\left(\frac{2s-4m^2}{s}\right). \end{aligned} \tag{8.6.31}$$

Note that in the non-relativistic limit, $(2s-4m^2)/s \to 1$. The divergent pole piece in $n-4$ corresponds to the first-order term in α of the divergent phase encountered in the Coulomb wave function. The function initially defined to be well defined near the origin contains the phase factor $\exp\{i(\alpha/v_{\rm rel})\ln kr\}$ at large distances which results from the long-range nature of the Coulomb potential. But the proper Coulomb wave function must asymptotically match a plane wave. This requires that a divergent phase $[\ln kr\,, kr \to \infty]$ be removed from the function. The divergent logarithm is replaced by $1/(n-4)$ in the dimensional regularization. The real part of $C(s)$ corresponds to the non-relativistic Coulomb correction to the value of the wave function at the origin, $|\psi(0)|^2 = (\text{other factors})\exp\{\pi\alpha/v_{\rm rel}\}$. Note that this is an unusually large correction; it is π^2 times the usual measure of α/π. The effects that we have just discussed arise when the form factor, originally defined for a space-like momentum transfer, $q^2 > 0$, is analytically continued — "crossed" — into the time-like region, $q^2 = -s < 0$. The crossing operation changes particle scattering into the production of a particle — antiparticle pair. There is an attractive Coulomb force between this produced pair which did not exist in the original scattering situation. In the non-relativistic limit, this

attractive force increases the size of the wave function at the origin which is in accord with the sign which we have found for the $\pi\alpha/v_{\rm rel}$ term. Such physical considerations, of course, help one check that a result is correct.

Since the divergent part of $C(s)$ is imaginary, it does not contribute to the cross section which we are calculating. Turning now to the $S(s)$ part, we replace $y \to 1/y$ in the first integration and add and subtract terms in this integral so as to display an overall factor of $(s-4m^2)$:

$$\begin{aligned} S(s) = & -2\frac{e^2\mu^{4-n}}{(4\pi)^{n/2}}\frac{\Gamma\left(3-\frac{n}{2}\right)}{n-4}\left\{[s-4m^2]\int_0^1 dy y^{4-n}\left[1-\frac{1}{2}(1-y^2)\right]\right. \\ & \left[m^2+\frac{1}{4}(1-y^2)(s-4m^2)\right]^{\frac{n}{2}-3} \\ & +2\int_0^1 dy\left\{y^{4-n}\left[m^2+\frac{1}{4}(1-y^2)(s-4m^2)\right]^{\frac{n}{2}-2}\right. \\ & \left.\left.-\left[m^2-i\epsilon-\frac{1}{4}(1-y^2)s\right]^{\frac{n}{2}-2}\right\}\right\}. \end{aligned} \tag{8.6.32}$$

Reverting to the variable $x=(1+y)/2$ and taking the limit $n \to 4$ gives

$$\begin{aligned} S(s) = & -2\frac{e^2}{(4\pi)^2}\frac{1}{n-4}(s-4m^2)\int_0^1 dx\frac{1-2x(1-x)}{m^2+x(1-x)(s-4m^2)} \\ & \left[\frac{m^2+x(1-x)(s-4m^2)}{4\pi\mu^2}\right]^{\frac{n}{2}-2}+\text{finite}. \end{aligned} \tag{8.6.33}$$

The potential divergent terms in the cross section correction are contained in the factor

$$\begin{aligned} F = & I_{\rm div}+2{\rm Re}\,S_{\rm div}(s) \\ = & (s-4m^2)\frac{\alpha}{2\pi}\int_0^1 dx\frac{1-2x(1-x)}{m^2+x(1-x)(s-4m^2)}\frac{1}{\frac{n}{2}-2} \\ & \left\{\left[\frac{1}{4\pi\mu^2(-Z^2/4)}\right]^{\frac{n}{2}-2}-\left[\frac{m^2+x(1-x)(s-4m^2)}{4\pi\mu^2}\right]^{\frac{n}{2}-2}\right\}, \end{aligned} \tag{8.6.34}$$

but in the $n \to 4$ limit this is the finite quantity,

$$\begin{aligned} F = & -(s-4m^2)\frac{\alpha}{2\pi}\int_0^1 dx\frac{1-2x(1-x)}{m^2+x(1-x)(s-4m^2)} \\ & \ln\left[\frac{m^2+x(1-x)(s-4m^2)}{4/(-Z^2)}\right]. \end{aligned} \tag{8.6.35}$$

The cutoff is essentially a minimum photon energy,

$$\frac{4}{-Z^2} = K^2. \tag{8.6.36}$$

That is, photons with an energy less than K cannot be experimentally detected. To interpret our result we need to state (see Problem 3) that summing to all orders in e^2 gives an exponential, with the cross section for the production of an arbitrary number of undetectable soft photons together with the associated vertex corrections of the form

$$\Delta\sigma_{\rm exp} = e^F\Delta\sigma_0, \tag{8.6.37}$$

where $\Delta\sigma_0$ has no soft photon effects. Since

$$F \sim -\ln\left(\frac{1}{K^2}\right), \tag{8.6.38}$$

we see that $F \to -\infty$ when $K \to 0$. The cross section with the production of no photons *vanishes*. Next we note that F vanishes at threshold, $s \to 4m^2$. At threshold there is no acceleration of the produced charged particles and no radiative effects. Finally, we consider the high-energy limit $s \gg m^2$. The x-parameter integral gives a term proportional to $1/s$ in this limit except for the equivalent singular regions $x \to 1$ and $x \to 0$ where the factor $x(1-x)$ vanishes. The leading term in the parameter integral is contained in

$$2\int_0^1 dx\frac{1}{m^2+xs} = \frac{2}{s}\ln\left(\frac{s}{m^2}\right), \tag{8.6.39}$$

and thus

$$s \gg m^2: \qquad F \simeq -\frac{\alpha}{\pi}\ln\left(\frac{s}{m^2}\right)\ln\left(\frac{s}{K^2}\right). \tag{8.6.40}$$

There is a logarithmic mass singularity when $m^2 \to 0$.

This singularity comes from a *collinear* divergence. The real photon emission involves the integral

$$\int(d^3\mathbf{k})\frac{1}{|\mathbf{k}|}\frac{1}{p'k}\frac{1}{\bar{p}'k}.$$

It has a soft photon divergence from the region $|\mathbf{k}| \to 0$. If the particles are massless, $p'^2 = 0 = \bar{p}'^2$, another divergence arises from the region where $\mathbf{p}'$(or $\bar{\mathbf{p}}'$) is *parallel* to $\mathbf{k}$ since here $-p'k' = |\mathbf{p}||\mathbf{k}| - \mathbf{p}\cdot\mathbf{k} = 0$. Denoting the angle between $\mathbf{p}'$ and $\mathbf{k}$ by θ, this region involves the angular integral

$$\int\frac{d\cos\theta}{1-\cos\theta}$$

which is logarithmically divergent.* The inclusion of a finite mass removes the collinear divergence and replaces it with $\ln m^2$. The logarithmic dependence on the particle mass corresponds to the fact that lighter particles are more easily accelerated and therefore are more strongly coupled to the radiation field. Thus, if we were actually concerned with the dominant radiative corrections to the muon pair production process, we should concentrate on the photons attached to the initial electrons rather than the produced muons as we have done. However, our purpose here has been to describe the qualitative and semi-quantitative features of the infrared divergence problem rather than to provide a detailed calculation. In the following such a calculation will be done.

2 *Antenna pattern*

We turn now to consider the high energy limit of the process $e\bar{e} \to \mu\bar{\mu}\gamma$ where the photon is emitted by one of the final state muons. In the high energy limit there are collinear (mass) singularities as well as soft photon singularities. All these singularities cancel if a suitably coarse-grained measure is computed. We shall show here that the angular distribution of the energy produced in the final state is such a finite measure. This is the angular distribution of the energy carried by both the muons and the photons. We shall compute the first order in α correction to the $1+\cos^2\theta$ angular distribution (8.2.33) of the energy carried by the muons in the basic $e\bar{e} \to \mu\bar{\mu}$ process. As we have remarked before, the result we shall obtain also applies to the high energy limit of the angular distribution of the energy produced in final hadronic states. The basic process here is $e\bar{e} \to q\bar{q}$, and it is corrected by the emission of an additional gluon, $e\bar{e} \to q\bar{q}g$. A simple substitution of coupling constants involving $\alpha \to 4\alpha_S/3$, where α_S is the strong coupling, gives the first-order correction to the angular pattern of hadronic energy. The perturbation calculation for the hadronic process is valid because the fragmentation of the quarks and gluons into the observed hadrons occurs with a limited average transverse momentum $\langle p_T \rangle$ which is independent of the total energy of the process. The angular distribution of the final hadrons follows that of their parent quarks and gluons to corrections on the order of $\langle p_\perp \rangle / W$ where W is the center-of-mass energy, $s = W^2$. The fragmentation correction vanishes in the high-energy limit. This hadronic analog provides a major motivation for considering the $e\bar{e} \to \mu\bar{\mu}\gamma$ process with the photons being radiated

* Dimensional continuation to $n > 4$ removes both divergencies. With $n > 4$ the volume element is increased so that there is obviously no soft photon divergence. As we shall shortly see in detail, increasing the number of dimensions also increases the size of the solid angle element — it puts a factor of $\sin\theta$ to a positive power in the angular integral above. Thus this continuation also removes the collinear divergencies.

only by the final state muons. We shall present the calculation in some detail so as to both display explicitly how the dimensional continuation to $n > 4$ is a useful device for regulating the infrared divergencies and also to display the algebraic complexity of an actual calculation.

We first compute the contribution to the energy-weighted cross section provided by the energy of the produced μ^+ or μ^-. Normalizing to the total energy W, this is given by multiplying the integrand for the ordinary differential cross section for μ^- or μ^+ production by E/W or $\bar{E}/W$, where we now write $E = p'^0$, $\bar{E} = \bar{p}'^0$. In view of the symmetry under the interchange $\mu^- \leftrightarrow \mu^+$, we need calculate only the μ^- contribution and multiply it by a factor 2. Using

$$\int (d^{n-1}k') \frac{1}{2k'^0} = \int (d^n k')\, \delta(k'^2)\theta(k'^0) \tag{8.6.41}$$

and integrating out the photon momentum in Eq. (8.6.5) expresses these contributions to the energy-weighted differential cross section as

$$\frac{d\Sigma_\gamma^{(\mu)}}{d\Omega} = \frac{1}{(2\pi)^{2n-3}} \frac{2}{4W} \int_0^\infty dE\, E^{n-2} \int d^{n-2}\bar{\Omega} \int_0^\infty d\bar{E}\, \bar{E}^{n-3}$$
$$\delta\left[W(W - 2E - 2\bar{E}) + 2E\bar{E}(1 - \cos\chi)\right] T_\mu^* T^\mu \frac{1}{2W^2}. \tag{8.6.42}$$

Here, since we are interested in the high energy limit, we have set the muon mass to zero in computing k'^2 from the energy-momentum conservation law, and labeled the angle between the μ^+ and μ^- directions by χ. We are computing the cross section in the center-of-mass frame where the initial leptons are oppositely directed, $\bar{\mathbf{p}} = -\mathbf{p}$. The calculation of the squared amplitude, averaged over initial spins and summed over final spins, is straightforward but somewhat lengthly. We shall just quote the result of the calculation which makes use of the previous expressions for the amplitude, Eqs. (8.6.1), (8.6.2), (8.6.3):

$$T_\mu^* T^\mu = \frac{8(e^2\mu^{4-n})^3}{W^2} \frac{1}{W - 2E} \frac{1}{W - 2\bar{E}} \Big\{ E^2[1 + (\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}})^2]$$
$$+ \bar{E}^2[1 + (\hat{\bar{\mathbf{p}}}' \cdot \hat{\mathbf{p}})^2] + \frac{1}{2}(n-4)[\omega^2 + (\mathbf{k}' \cdot \hat{\mathbf{p}})^2] \Big\}, \tag{8.6.43}$$

where $\omega = k'^0 = W - E - \bar{E}$ is the energy of the produced photon and $-\mathbf{k}' = \mathbf{p}' + \bar{\mathbf{p}}'$. In writing this result we have omitted an overall n-dependent factor arising from the γ-matrix trace, tr1, setting $\mathrm{tr}1 = 4$, the $n = 4$ limit. There is no need to carry along this n-dependent factor since it multiplies every contribution and the limit $n \to 4$ must be taken after all the singularities at $n = 4$ have canceled.

To perform the integration over the solid angle $d^{n-2}\bar{\Omega}$ of the produced μ^+, we take the direction $\hat{\mathbf{p}}'$ of the μ^- as the generalized "z" direction,

the "north pole" of the hyper-spherical coordinate system. We need to express the solid angle in terms of the angle χ that $\bar{\mathbf{p}}'$ makes with respect to this axis and the remaining $n-3$ dimensional solid angle. To do this, we note that in general the ν-dimensional solid angle is the area of a unit hypersphere in a $\nu+1$ dimensional Euclidean space,

$$\Omega^{(\nu)} = \int (d^{\nu+1}\mathbf{r})\, 2\,\delta(1-\mathbf{r}^2)\,. \tag{8.6.44}$$

Writing $\mathbf{r} = (\cos\chi, \vec{\rho})$, where $\vec{\rho}$ lies in the ν dimensional space orthogonal to the "z"-axis gives

$$\Omega^{(\nu)} = \int_{-1}^{1} d\cos\chi \int (d^{\nu}\,\vec{\rho})\, 2\,\delta\,(\sin^2\chi - \vec{\rho}^{\,2})\,, \tag{8.6.45}$$

and rescaling $\vec{\rho} \to \sin\chi\vec{\rho}$ produces

$$\Omega^{(\nu)} = \int_{-1}^{1} \sin^{\nu-2}\chi\, d\,\cos\chi \int (d^{\nu}\vec{\rho})\, 2\,\delta(1-\vec{\rho}^{\,2})\,. \tag{8.6.46}$$

Hence

$$d^{\nu}\Omega = \sin^{\nu-2}\chi\, d\cos\chi\, d^{\nu-1}\Omega\,. \tag{8.6.47}$$

The total solid angle in arbitrary dimension also will be needed. To compute this, we note that

$$\pi^{\frac{1}{2}(\nu+1)} = \int (d^{\nu+1}\mathbf{r})\, e^{-\mathbf{r}^2} = \int d^{\nu}\Omega \int_0^{\infty} r^{\nu} dr e^{-r^2}\,. \tag{8.6.48}$$

Setting $r = t^{1/2}$, the radial integral becomes the standard representation of the Γ-function, and so

$$\Omega^{(\nu)} = \int d^{\nu}\Omega = \frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))}\,. \tag{8.6.49}$$

Writing

$$\hat{\bar{\mathbf{p}}}' = \cos\chi\,\hat{\mathbf{p}}' + \sin\chi\,\mathbf{p}_{\perp}\,, \tag{8.6.50}$$

where $\mathbf{p}_{\perp}\cdot\hat{\mathbf{p}}' = 0$ and $\mathbf{p}_{\perp}^2 = 1$, the angular average over $\bar{\mathbf{p}}'$ with fixed angle χ gives

$$\begin{aligned}\langle(\hat{\bar{\mathbf{p}}}'\cdot\hat{\mathbf{p}})^2\rangle &= \frac{1}{\bar{\Omega}^{(n-3)}}\int d^{n-3}\bar{\Omega}\left(\hat{\bar{\mathbf{p}}}'\cdot\hat{\mathbf{p}}\right)^2 \\ &= \cos^2\chi\,(\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}})^2 + \sin^2\chi\,\frac{[1-(\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}})^2]}{n-2}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle(\mathbf{k}'\cdot\hat{\mathbf{p}})^2\rangle &= \langle[(\mathbf{p}'+\bar{\mathbf{p}}')\cdot\hat{\mathbf{p}}]^2\rangle \\ &= \langle[(E+\bar{E}\cos\chi)\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}} + \bar{E}\sin\chi\mathbf{p}_{\perp}\cdot\hat{\mathbf{p}}]^2\rangle \\ &= (E+\bar{E}\cos\chi)^2(\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}})^2 + \bar{E}^2\sin^2\chi\frac{[1-(\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}})^2]}{n-2}\,.\end{aligned} \tag{8.6.51}$$

Using the solid angle volume for $\nu = n-3$, and writing $\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}} = \cos\theta$, we now have

$$\begin{aligned}\frac{d\Sigma_{\gamma}^{(\mu)}}{d\Omega} = &\left(\frac{e^2}{4\pi}\right)^3 \frac{2^{n-1}}{\pi\Gamma(\frac{n}{2}-1)}\left(\frac{1}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\frac{1}{W^5}\int_0^{\infty} dE E^{n-2}\int_0^{\infty} d\bar{E}\bar{E}^{n-3} \\ &\int_{-1}^{1}\sin^{n-4}\chi d\cos\chi\,\delta\left[W(W-2E-2\bar{E})+2E\,\bar{E}(1-\cos\chi)\right] \\ &\frac{1}{W-2E}\,\frac{1}{W-2\bar{E}}\Big\{E^2(1+\cos^2\theta)+\bar{E}^2\Big[1+\cos^2\chi\cos^2\theta \\ &+\frac{\sin^2\chi\sin^2\theta}{n-2}\Big]+\frac{1}{2}(n-4)\Big[(W-E-\bar{E})^2 \\ &+(E+\bar{E}\cos\chi)^2\cos^2\theta+\frac{\bar{E}^2\sin^2\chi\sin^2\theta}{n-2}\Big]\Big\}\,.\end{aligned} \tag{8.6.52}$$

The angular integral may be done against the δ-function. It selects

$$1+\cos\chi = \frac{(W-2E)(W-2\bar{E})}{2E\bar{E}}\,, \tag{8.6.53}$$

which gives

$$\sin\chi = \frac{[(W-2E)(W-2\bar{E})W(2E+2\bar{E}-W)]^{1/2}}{2E\bar{E}}\,. \tag{8.6.54}$$

Since $\sin\chi$ must be real, the remaining energy integration is restricted to the region

$$0 \le E \le W/2\,, \qquad 0 \le \bar{E} \le W/2\,, \tag{8.6.55}$$

and

$$2E + 2\bar{E} - W \ge 0\,. \tag{8.6.56}$$

At $n = 4$ the denominators become singular at the end points $E = W/2$, $\bar{E} = W/2$. Either of these end points corresponds to $\cos\chi = -1$; the muons here are produced back-to-back, corresponding to the emission

of a soft photon.† Since the factor $1+\cos\chi$ removes both singularities, the singular parts are isolated by setting $\cos\chi=-1$:

$$\frac{d\Sigma_\gamma^{(\mu)}\text{div}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{4}{\pi\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{1}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\frac{1}{W^5}\int_0^{W/2}dE\,E\int_{W/2-E}^{W/2}d\bar{E}$$
$$\left[(W-2E)(W-2\bar{E})W(2E+2\bar{E}-W)\right]^{\frac{n}{2}-2}\frac{1}{W-2E}\,\frac{1}{W-2\bar{E}}$$
$$\left\{[E^2+\bar{E}^2](1+\cos^2\theta)+\frac{1}{2}(n-4)[(W-E-\bar{E})^2+(E-\bar{E})^2\cos^2\theta]\right\}. \tag{8.6.57}$$

The remaining piece is well behaved and $n=4$ may be taken in its evaluation:

$$\frac{d\Sigma_\gamma^{(\mu)}\text{reg}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{4}{\pi}\frac{1}{W^5}\int_0^{W/2}dE\,E\int_{W/2-E}^{W/2}d\bar{E}$$
$$\frac{1}{4E^2}W(2E+2\bar{E}-W)\left(\frac{1}{2}\sin^2\theta-\cos^2\theta\right)$$
$$=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{16\pi W^2}\left(1-3\cos^2\theta\right). \tag{8.6.58}$$

To evaluate the singular piece, we change variables by writing

$$E=(1-\alpha)W/2\,,\qquad \bar{E}=(1-\beta)W/2\,, \tag{8.6.59}$$

and use

$$\theta(1-\alpha-\beta)=\int_0^1 d\gamma\,\delta(1-\alpha-\beta-\gamma) \tag{8.6.60}$$

to obtain a symmetrical form. Thus

$$\frac{d\Sigma_\gamma^{(\mu)}\text{div}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{8\pi\Gamma(\frac{n}{2}-1)}\left(\frac{W^2}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\left(\frac{1}{W}\right)^{n-2}$$
$$\int_0^1 d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)\,(1-\alpha)(\alpha\beta)^{\frac{n}{2}-3}\gamma^{\frac{n}{2}-2}$$
$$\left[(1-\alpha)^2+(1-\beta)^2+\frac{1}{2}(n-4)(\alpha^2+\beta^2)\right](1+\cos^2\theta)\,. \tag{8.6.61}$$

A term involving $(n-4)\alpha\beta\sin^2\theta$ makes no contribution in the $n\to4$ limit since the $\alpha\beta$ factor cancels the singularities, and it has been omitted in writing this equation. Using the formula (8.5.63), performing some

† The momentum carried by a soft photon may be neglected. Momentum conservation in the center-of-mass frame then requires that the muons are produced back-to-back.

tedious algebra, and taking the limit $n\to4$ in the regular terms yields

$$\frac{d\Sigma_\gamma^{(\mu)}\text{div}}{d^{n-2}\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{8\pi}\left(\frac{W^2}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\left(\frac{1}{W}\right)^{n-2}$$
$$\left[4\frac{\Gamma(\frac{n}{2}-2)^2}{\Gamma(\frac{3n}{2}-3)}+\frac{28}{3}\frac{\Gamma(\frac{n}{2}-2)\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{3n}{2}-3)}+\frac{10}{3}\right](1+\cos^2\theta)\,. \tag{8.6.62}$$

The cross section in which the energy is carried by the photon must be added to the result. Interchanging the roles of p' and k' and integrating out p' as was done previously for k' in Eq. (8.6.42), it is given by

$$\frac{d\Sigma_\gamma^{(\gamma)}}{d\Omega}=\frac{1}{(2\pi)^{2n-3}}\frac{1}{8W^3}\int_0^\infty d\omega\,\omega^{n-2}\int d^{n-2}\bar{\Omega}\int_0^\infty d\bar{E}\bar{E}^{n-3}$$
$$\delta\left[W(W-2\omega-2\bar{E})+2\omega\bar{E}(1-\cos\chi)\right]T_\mu^*T^\mu\,. \tag{8.6.63}$$

Here χ is now the angle between $\mathbf{k}'$ and $\bar{\mathbf{p}}'$. The squared amplitude in the appropriate variables reads

$$T_\mu^*T^\mu=\frac{8\,(e^2\mu^{4-n})^3}{W^2}\frac{1}{2(\omega+\bar{E})-W}\,\frac{1}{W-2\bar{E}}$$
$$\left\{(W-\omega-\bar{E})^2+\left[(\bar{\mathbf{p}}'+\mathbf{k}')\cdot\hat{\mathbf{p}}\right]^2+\bar{E}^2[1+(\hat{\bar{\mathbf{p}}}'\cdot\hat{\mathbf{p}})^2]\right.$$
$$\left.+\frac{1}{2}(n-4)\omega^2[1+(\hat{\mathbf{k}}'\cdot\hat{\mathbf{p}})^2]\right\}. \tag{8.6.64}$$

The angle at which the energy is detected is now given by $\hat{\mathbf{k}}'\cdot\hat{\mathbf{p}}=\cos\theta$. The interchange $\mathbf{p}'\leftrightarrow\mathbf{k}'$ in Eqs. (8.6.51) and (8.6.51) produces

$$\langle(\hat{\bar{\mathbf{p}}}'\cdot\hat{\mathbf{p}})^2\rangle=\cos^2\chi\cos^2\theta+\frac{\sin^2\chi\sin^2\theta}{n-2}\,, \tag{8.6.65}$$

and

$$\langle\left[(\bar{\mathbf{p}}'+\mathbf{k}')\cdot\hat{\mathbf{p}}\right]^2\rangle=(\omega+\bar{E}\cos\chi)^2\cos^2\theta+\frac{\bar{E}^2\sin^2\chi\sin^2\theta}{n-2}\,. \tag{8.6.66}$$

Performing the angular integration using Eqs. (8.6.47) and (8.6.49),

$$\frac{d\Sigma_\gamma^{(\gamma)}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{2^{n-3}}{\pi\Gamma(\frac{n}{2}-1)}\left(\frac{1}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\frac{1}{W^5}$$
$$\int d\omega\,\omega^{n-3}\int d\bar{E}\,\bar{E}^{n-4}\sin^{n-4}\chi\frac{1}{2(\omega+\bar{E})-W}\,\frac{1}{W-2\bar{E}}$$
$$\left\{(W-\omega-\bar{E})^2+(\omega+\bar{E}\cos\chi)^2\cos^2\theta+\frac{\bar{E}^2\sin^2\chi\sin^2\theta}{n-2}\right.$$

$$+\bar{E}^2\left[1+\cos^2\chi\cos^2\theta+\frac{\sin^2\chi\sin^2\theta}{n-2}\right]+\frac{n-4}{2}\omega^2[1+\cos^2\theta]\Bigg\}. \tag{8.6.67}$$

Here

$$\sin\chi=\frac{[(W-2\omega)(W-2\bar{E})W(2\omega+2\bar{E}-W)]^{1/2}}{2\omega\bar{E}}, \tag{8.6.68}$$

and the requirement that $\sin\chi$ be real gives the limits to be applied to the energy integrals.

It follows from Eq. (8.6.68) that

$$1+\cos\chi=\frac{(W-2\omega)(W-2\bar{E})}{2\omega\bar{E}}, \tag{8.6.69}$$

which informs us that

$$2\omega\bar{E}(1-\cos\chi)=\frac{2\omega\bar{E}\sin^2\chi}{1+\cos\chi}=W(2\omega+2\bar{E}-W). \tag{8.6.70}$$

Hence

$$(\omega+\bar{E}\cos\chi)^2=(W-\omega-\bar{E})^2-\bar{E}^2\sin^2\chi. \tag{8.6.71}$$

As before, the singularities appear when $\sin\chi=0$. Therefore, the divergent piece is given by

$$\frac{d\Sigma_\gamma^{(\gamma)}\text{div}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{2}{\pi\Gamma(\frac{n}{2}-1)}\left(\frac{1}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\frac{1}{W^5}\int d\omega\,\omega\int d\bar{E}$$
$$[(W-2\omega)(W-2\bar{E})W(2\omega+2\bar{E}-W)]^{\frac{n}{2}-2}\frac{1}{2(\omega+\bar{E})-W}\frac{1}{W-2\bar{E}}$$
$$\left[(W-\omega-\bar{E})^2+\bar{E}^2+\frac{1}{2}(n-4)\omega^2\right](1+\cos^2\theta). \tag{8.6.72}$$

The limit $n=4$ may be taken in remainder:

$$\frac{d\Sigma_\gamma^{(\gamma)}\text{reg}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{\pi W^4}\int_0^{W/2}\frac{d\omega}{\omega}(W-2\omega)\int_{W/2-\omega}^{W/2}d\bar{E}\left(\frac{1}{2}\sin^2\theta-\cos^2\theta\right)$$
$$=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{8\pi}\frac{1}{W^2}(1-3\cos^2\theta). \tag{8.6.73}$$

As before, we write $\omega=(1-\alpha)W/2$, $\bar{E}=(1-\beta)W/2$ to obtain

$$\frac{d\Sigma_\gamma^{(\gamma)}\text{div}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{16\pi\Gamma(\frac{n}{2}-1)}\left(\frac{W^2}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\left(\frac{1}{W}\right)^{n-2}$$
$$\int_0^1 d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)\,(1-\alpha)\alpha^{\frac{n}{2}-2}(\beta\gamma)^{\frac{n}{2}-3}$$

$$\left[(\alpha+\beta)^2+(1-\beta)^2+\frac{1}{2}(n-4)(1-\alpha)^2\right](1+\cos^2\theta). \tag{8.6.74}$$

Some more tedious algebra reduces this to

$$\frac{d\Sigma_\gamma^{(\gamma)}\text{div}}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^3\frac{1}{16\pi}\left(\frac{W^2}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\left(\frac{1}{W}\right)^{n-2}$$
$$\left[\frac{16}{3}\frac{\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}-2)}{\Gamma(\frac{3n}{2}-3)}+\frac{10}{3}\right](1+\cos^2\theta), \tag{8.6.75}$$

where the limit $n\to 4$ has been taken when possible.

The complete, energy-weighted differential cross section for the process $e\bar{e}\to\mu\bar{\mu}\gamma$, in which the photon is emitted by one of the final state muons, is given by the sum of Eqs. (8.6.58), (8.6.62), (8.6.73), and (8.6.75) :

$$\frac{d\Sigma_\gamma}{d\Omega}=\frac{\alpha^3}{4\pi}\left(\frac{1}{W}\right)^{n-2}\left(\frac{W^2}{4\pi\mu^2}\right)^{\frac{3n}{2}-6}\Bigg\{\frac{3}{4}(1-3\cos^2\theta)$$
$$+\left[2\frac{\Gamma(\frac{n}{2}-2)^2}{\Gamma(\frac{3n}{2}-3)}+6\frac{\Gamma(\frac{n}{2}-2)\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{3n}{2}-3)}+\frac{5}{2}\right](1+\cos^2\theta)\Bigg\}. \tag{8.6.76}$$

This is not the whole story. To this must be added the differential cross section with no photon production, $e\bar{e}\to\mu\bar{\mu}$, but with inclusion of the vertex correction in which a virtual photon is exchanged between the final muon pair. Here the energy weighting does not alter the cross section: The cross section is symmetrical under the interchange $\mu\leftrightarrow\bar{\mu}$, both the muon and antimuon carry the energy $W/2$, and so the sum of the two energy weighted terms, divided by the total energy W, is the simple unweighted cross section.

In the high-energy limit, the vertex correction is the result (8.5.70) for massless particles, which is of the form

$$\Gamma^\mu(p',p)=\gamma^\mu[1+V]. \tag{8.6.77}$$

Since this is a numerical multiple of γ^μ, the corresponding corrected cross section is given by

$$\frac{d\Sigma\,\text{virt}}{d\Omega}=[1+2\text{Re}V]\frac{d\sigma}{d\Omega}. \tag{8.6.78}$$

Here $d\sigma/d\Omega$ is the high-energy limit of the cross section computed in Section 2 except that it is for an n-dimensional space-time. Reviewing the work of Section 2, it is easy to see that this extension reads

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4}\left(\frac{1}{W}\right)^{n-2}\left(\frac{W^2}{4\pi\mu^2}\right)^{n-4}\left(1+\cos^2\theta\right). \tag{8.6.79}$$

The vertex correction entails the analytic continuation from space-like to time-like momentum transfer, $q^2 \to e^{-\pi i} W^2$. Thus, taking the limit $n \to 4$ in Eq. (8.5.70) when possible, we have

$$V(W^2) = -\frac{\alpha}{4\pi} F(n) \left[\frac{e^{-\pi i} W^2}{4\pi\mu^2} \right]^{\frac{n}{2}-2} , \tag{8.6.80}$$

where

$$F(n) = 2\frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-2)^2}{\Gamma(n-2)} + \frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}-2)}{\Gamma(n-2)} + 2 . \tag{8.6.81}$$

Expanding the phase factor

$$e^{-\pi i(\frac{n}{2}-2)} = 1 - \pi i \left(\frac{n}{2}-2\right) - \frac{\pi^2}{2}\left(\frac{n}{2}-2\right)^2 + \ldots \tag{8.6.82}$$

gives, in the $n \to 4$ limit,

$$\mathrm{Re}V(W^2) = -\frac{\alpha}{4\pi} F(n) \left(\frac{W^2}{4\pi\mu^2} \right)^{\frac{n}{2}-2} + \frac{\alpha\pi}{4} . \tag{8.6.83}$$

The experimentally observable energy-weighted differential cross section, the angular distribution or "antenna pattern" of the produced energy, is given — to order α^3 — by the basic process corrected by the emission of a real photon and the exchange of a virtual photon,

$$\frac{d\Sigma}{d\Omega} = \frac{d\Sigma_\gamma}{d\Omega} + \frac{d\Sigma\mathrm{virt}}{d\Omega} . \tag{8.6.84}$$

Using Eq. (8.6.76) and the previous results, this sum is given by

$$\frac{d\Sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\left[1 + \frac{\alpha}{4\pi}C\right] + \frac{3\alpha^3}{16\pi W^2}\left(1 - 3\cos^2\theta\right) , \tag{8.6.85}$$

where

$$\begin{aligned} C =& \Gamma\left(\frac{n}{2}-2\right)^2 4\left[1 - \frac{\Gamma(\frac{3n}{2}-3)\Gamma(3-\frac{n}{2})}{2\Gamma(n-2)}\right] \\ &+\Gamma\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-1\right) 2\left[6 - \frac{\Gamma(\frac{3n}{2}-3)\Gamma(3-\frac{n}{2})}{2\Gamma(n-2)}\right] \\ &+10 - 4 + 2\pi^2 . \end{aligned} \tag{8.6.86}$$

Here the limit $n \to 4$ may be taken in the basic cross section $d\sigma/d\Omega$ since the total result is finite. The $n \to 4$ limit has also been taken in an overall factor of C since C has a finite limit. Using

$$\Gamma(z+\epsilon) = \Gamma(z)\left\{1 + \epsilon\psi(z) + \frac{1}{2}\epsilon^2\left[\psi(z)^2 + \psi'(z)\right] + \ldots\right\} , \tag{8.6.87}$$

with

$$\psi(z+1) = \psi(z) + \frac{1}{z} , \tag{8.6.88}$$

which follows from $\Gamma(z+1) = z\Gamma(z)$, and

$$\psi'(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} , \tag{8.6.89}$$

a final stretch of tedious algebra shows that in the $n \to 4$ limit

$$C = 3 . \tag{8.6.90}$$

At the end of such a calculation one surely has divided emotions. On the one hand, pride that one has gotten through it. On the other hand, a feeling that there must be a better way to do it. Unfortunately, many of the calculations in quantum field theory are of this nature — involved computations that end with a simple number.‡

The result has two aspects. One is an overall change in the size of the basic cross section; the other is a small alteration of the angular distribution of the produced energy whose angular average vanishes. The total energy-weighted cross section is the same as the ordinary total cross section — this follows from energy conservation; the total energy cross section adds up all the produced energy, and dividing by the total energy W gives a unit factor. Thus

$$\sigma_{\mathrm{tot}} = \int d\Omega \frac{d\Sigma}{d\Omega} = \frac{4\pi\alpha^2}{3W^2}\left(1 + \frac{3\alpha}{4\pi}\right) . \tag{8.6.91}$$

As we have discussed before, very similar results apply to the production of hadronic energy via the processes $e\bar{e} \to q\bar{q}$ and $e\bar{e} \to q\bar{q}g$ together with the vertex correction arising from virtual gluon exchange. At this first order in the strong coupling α_S, the non-Abelian nature of the QCD theory plays no role other than to alter the definition of the coupling constant. With the conventional definition for the strong coupling, this amounts to making the (color averaged) replacement $\alpha \to 4\alpha_S(W)/3$. Here we have also used the running strong coupling. This must be done since the QCD theory is asymptotically free — it is only at high energies that this coupling is small and perturbation theory can be applied. Making this replacement for the coupling in Eq. (8.6.85) where the emitted (and virtual) photon has been replaced by a gluon and also summing over

‡ One would think that the energy distribution cross section that we have calculated could be more simply evaluated by relating it to some expectation value of the Poynting vector $T^{0k}(x)$, but this I have never been able to do.

the quark charges and colors gives the hadronic energy distribution

$$\frac{d\Sigma}{d\Omega} = \frac{3\alpha^2}{4W^2}\sum_f Q_f^2 \left\{\left[1+\frac{\alpha_S(W)}{\pi}\right]\left(1+\cos^2\theta\right)+\frac{\alpha_S(W)}{\pi}\left(1-3\cos^2\theta\right)\right\}. \tag{8.6.92}$$

The solid angle integral produces the total cross section. This cross section, divided by the total cross section for muon production, is the R ratio discussed in Section 2,

$$R = 3\sum_f Q_f^2\left[1+\frac{\alpha_S(W)}{\pi}\right]. \tag{8.6.93}$$

In principle, for large W the first-order correction is very small, and the ratio is close to that for point-like particles. In practice, at energies below the Z^0 threshold, the first order correction is not large, but well before it becomes negligible the new effects of the Z^0 production must be taken into account.

8.7 Lamb shift

1 *General formulation*

Radiative corrections shift atomic energy levels. An example of such a shift has already been presented in the discussion of vacuum polarization in Section 3.3. Here we shall work out a general formulation and then apply it to calculate the leading effect. We shall consider only hydrogen-like atoms, atoms with a single electron but with an arbitrary nuclear charge Z. Moreover, we shall work in the approximation in which the nucleus is treated as an infinitely heavy, fixed distribution of charge. Thus we have an electron moving under a prescribed potential $\langle A_\mu\rangle$ which does include all the effects of vacuum polarization. This dynamics is described by the electron's Green's function

$$G(x,x') = \langle 0|iT\psi(x)\bar{\psi}(x')|0\rangle^{(A)}. \tag{8.7.1}$$

Since the system is invariant under time-translations, its time dependence may be exhibited by the Fourier transform

$$G(x,x') = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\,G(\mathbf{r},\mathbf{r}';E)\,e^{-iE(t-t')}. \tag{8.7.2}$$

In momentum space, the self-energy function for a free particle is a function of the momentum; passing to coordinate space, the product of the self-energy function with the Green's function becomes a convolution integral. For a bound particle there is no translational invariance, and the simple product form in momentum space is lost. Thus it is convenient to

introduce an operator notation for the spatial coordinates which makes no reference to a particular set of basis states. We write

$$\Pi^0 = E - \langle A^0\rangle, \qquad \Pi^k = p^k - \langle A^k\rangle, \tag{8.7.3}$$

where p_k is represented by $\frac{1}{i}\partial_k$ when it acts to the left on a coordinate eigenvector. In terms of this notation, the Green's function with self-energy corrections obeys the operator equation

$$[\gamma\Pi + m + \Sigma(E)]\,G(E) = 1. \tag{8.7.4}$$

Here a physical renormalization is used in which m is the experimentally observed electron mass.

Introducing a complete set of intermediate energy eigenstates on the right-hand side of Eq. (8.7.1) and Fourier transforming in the time produces the decomposition

$$G(E) = \sum_n \frac{\psi_n\bar{\psi}_n}{E_n - E - i\epsilon} + \mathcal{C}. \tag{8.7.5}$$

The sum runs over the bound states labeled by n with bound state energies $E_n < m$ and with wave functions

$$\psi_n = \langle 0|\psi|n\rangle, \qquad \bar{\psi}_n = \langle n|\bar{\psi}|0\rangle. \tag{8.7.6}$$

Since the operator ψ destroys the charge of the electron, and the electrons (not the positrons) are in bound states, only the time ordering with $t > t'$ contributes to the bound state sum. The remaining contribution of the continuum states (which arise from both electron and positron intermediate states) is denoted by $\mathcal{C}$. Inserting this decomposition into the operator Green's function equation gives

$$\left[\gamma\Pi + m + \Sigma(E)\right]\left\{\sum_n \frac{\psi_n\bar{\psi}_n}{E_n - E - i\epsilon} + \mathcal{C}\right\} = 1. \tag{8.7.7}$$

Consider the limit in which the energy approaches one of the bound state energy eigenvalues, $E \to E_n$. Since no other singularity appears to compensate the resulting pole, its residue must vanish,

$$\left[-\gamma^0(E_n - \langle A^0\rangle) + \vec{\gamma}\cdot\vec{\Pi} + m + \Sigma(E_n)\right]\psi_n = 0. \tag{8.7.8}$$

This is the generalization of the bound state, energy-eigenvalue equation which follows from the time-independent Dirac equation. It is the generalization which contains all the radiative corrections except for those associated with nuclear motion. Taking E close to E_n, expanding in $(E - E_n)$, and then taking the limit $E \to E_n$ now gives

$$\left[\gamma^0 - \frac{\partial\Sigma(E_n)}{\partial E_n}\right]\psi_n\bar{\psi}_n + \left[-\gamma^0(E_n - \langle A^0\rangle) + \vec{\gamma}\cdot\vec{\Pi} + m + \Sigma(E_n)\right]

$$\times\left\{\sum_{n'\neq n}\frac{\psi_{n'}\bar{\psi}_{n'}}{E_{n'}-E_n}+C\right\}=1\,. \tag{8.7.9}$$

Multiplying this by $\bar{\psi}_n$ on the left — taking the scalar product — produces $\bar{\psi}_n$ on the right-hand side of the resulting equation. We conclude that the normalization of the bound state wave functions must be taken to be

$$\bar{\psi}_n\Big[\gamma^0-\frac{\partial\Sigma(E_n)}{\partial E_n}\Big]\psi_n=1\,, \tag{8.7.10}$$

with $\bar{\psi}_n$ orthogonal to the other terms.

The generalized Dirac equation (8.7.8) yields the energy shifts coming from the radiative corrections. To exhibit these shifts, we compare Eq. (8.7.8) with the "ordinary" Dirac equation

$$\Big[-\gamma^0\Big(E_n^{(0)}-\langle A^0\rangle\Big)+\vec{\gamma}\cdot\vec{\Pi}+m\Big]\psi_n^{(0)}=0\,, \tag{8.7.11}$$

whose Dirac adjoint reads

$$\bar{\psi}_n^{(0)}\Big[-\gamma^0(E_n^{(0)}-\langle A^0\rangle)+\vec{\gamma}\cdot\vec{\Pi}+m\Big]=0\,. \tag{8.7.12}$$

These are the ordinary Dirac equations except that the potentials contain the vacuum polarization correction. The scalar product of Eq. (8.7.8) with the comparison wave function gives

$$\Delta E_n\,\bar{\psi}_n^{(0)}\gamma^0\psi_n=\bar{\psi}_n^{(0)}\Sigma(E_n)\psi_n\,, \tag{8.7.13}$$

where

$$\Delta E_n=E_n-E_n^{(0)}\,. \tag{8.7.14}$$

As has been remarked before, it is convenient to regard α and $Z\alpha$ as separate expansion parameters, even though $Z=1$ for the hydrogen atom. Powers of α count the order of the radiative corrections; powers of $Z\alpha$ count the order of binding corrections. From now on we shall consider only the leading, order α radiative correction. In this case, the self-energy $\Sigma(E)$ is computed to first order, and the wave function and energy in Eq. (8.7.13) may be replaced by their unperturbed values. In this leading order, the normalization condition (8.7.10) becomes $\bar{\psi}_n^{(0)}\,\gamma^0\psi_n^{(0)}=1$, and the energy shift formula (8.7.13) yields

$$\Delta E_n=\bar{\psi}_n^{(0)}\Sigma(E_n^{(0)})\psi_n^{(0)}\,. \tag{8.7.15}$$

2 *Calculation in leading order*

The lowest-order self-energy or mass operator $\Sigma(E)$ is given — up to renormalization subtraction — by

$$\Sigma(E)=ie^2\mu^{4-n}\int\frac{(d^nk)}{(2\pi)^n}D_{\mu\nu}(k)\,\gamma^\mu\frac{1}{\gamma(\Pi+k)+m}\gamma^\nu\,. \tag{8.7.16}$$

This is just the function discussed in Section 4 except that the momentum variable is now replaced by an operator, $p\to\Pi$. Any gauge may be used for the photon propagator $D_{\mu\nu}(k)$. Longitudinal parts of the photon propagator are irrelevant here. Just as was explained at the end of Section 4, a Ward identity shows that they involve an overall factor of $(\gamma\Pi+m)$, and so they do not contribute to the energy shift formula (8.7.15) since $(\gamma\Pi+m)\psi_n^{(0)}=0$. It is not possible to compute the mass operator in closed form. The commutator of different components of Π_μ produce the electromagnetic field-strength tensor,

$$[\Pi_\mu,\Pi_\nu]=i\,F_{\mu\nu}\,. \tag{8.7.17}$$

Since this commutator does not vanish, the manipulations previously done to compute Σ can no longer be performed. They involved shifting the momentum integration variables k_μ by the commuting numbers p_μ. However, an approximate form for $\Sigma(E)$ can be obtained which suffices for the evaluation of the leading correction in $Z\alpha$. The traditional way to derive this approximation is to make a non-covariant partition of the magnitude of the spatial photon momentum, $|\mathbf{k}|\leq K$ and $|\mathbf{k}|>K$. For $|\mathbf{k}|>K$ no infrared, logarithmic divergences appear and a straight forward expansion in powers of $Z\alpha$ can be made. This is not the case for the region $|\mathbf{k}|\leq K$, but here one may choose $K<<m$ so that the electron dynamics can be described by the non-relativistic limit. With this approximation, the self-energy operator is easily evaluated. This traditional method involves a considerable amount of computation, however, since the non-covariant split destroys the inherent simplicity of the basic, covariant theory. We shall compute the level shift using a method that simplifies the traditional technique.

The sometime music critic Mark Twain once remarked that Richard Wagner's music is really much better than it sounds. In a similar vein, we remark that the method we are about to describe is really much simpler than it seems.

Our method is to consider the energy shift as a function of the space-time dimensionality n. With $n>4$ there is no infrared divergence, and the self energy may be expanded in powers of $Z\alpha$. With $n<4$ there is a potential infrared divergence which is avoided only by retaining binding effects and an expansion in powers of $Z\alpha$ is not allowed. However, with $n<4$ the theory has no ultraviolet divergence; the virtual photon exchange is effectively soft, the resulting electron recoil is small, and the electron motion may be approximated by its non-relativistic limit. Since the self-energy operator is an analytic function of the space-time dimensionality n, a comparison of the two limiting forms yields the correct $n=4$ limit which is valid to the leading order in $Z\alpha$, and which includes a non-analytic term involving $\ln Z\alpha$.

3 $n > 4$

We begin by considering $n > 4$ where the self-energy operator may be expanded in powers of $\langle A_\mu \rangle$. After the mass renormalization which identifies the physical electron mass m, the self-energy operator to zero order in $\langle A_\mu \rangle$ is of the form $(\gamma p + m)F(\gamma p)$ displayed in Eq. (8.4.29). Gauge invariance requires that the inclusion of $\langle A_\mu \rangle$ promotes this zero order form to the structure $(\gamma \Pi + m)F(\gamma \Pi)$ plus additional terms which depend upon the gauge-invariant field-strength tensor $F_{\mu\nu}$. Since $\bar{\psi}_n^{(0)}(\gamma \Pi + m) = 0$, the $(\gamma \Pi + m)F(\gamma \Pi)$ structure does not contribute to the order α energy shift (8.7.15) which we are about to compute. Thus it suffices to consider the terms in the self energy which are explicitly of first order in $\langle A_\mu \rangle$. In leading order $\langle A_k \rangle = 0$ and $\langle A^0 \rangle = V$ is the simple Coulomb binding potential (in an n-dimensional space-time). Thus we may approximate, in view of Eq. (8.1.95),

$$\Sigma(x, x') = \int (d^n z)\, \Gamma^0_{(1)}(x, x', z)\, V(z)\,, \tag{8.7.18}$$

where $\Gamma^{(0)}_{(1)}$ is the order α contribution to the vertex function. Here $\Gamma^0_{(1)}(x, x', z)$ is the off-mass-shell function that contains — in the momentum space version — some terms with factors of $(\gamma p' + m)$ and/or $(\gamma p + m)$ on the extreme left and right. As has just been remarked, including higher order terms in the potential V and noting that the result must be gauge invariant, such factors turn into the operator $(\gamma \Pi + m)$. These operators act directly on the unperturbed wave functions in computing the energy shift and thus vanish. The vertex can be replaced by its value on the mass shell in computing the energy shift. Remembering the Fourier transform convention given in Eq. (8.1.88) and the mass shell decomposition (8.5.43) in terms of the form factors $F_1(q^2)$ and $F_2(q^2)$, we have

$$\Sigma(x, x') = \int \frac{(d^n p')}{(2\pi)^n} \frac{(d^n p)}{(2\pi)^n} e^{ip'x - ipx'} \left\{ \gamma^0 [F_1(q^2) - 1] - \frac{i}{2m} \sigma^{0l} q_l F_2(q^2) \right\} \int (d^n z)\, e^{-iqz}\, V(z)\,, \tag{8.7.19}$$

where

$$q = p' - p\,. \tag{8.7.20}$$

The form factors are functions of q^2/m^2. In the energy shift matrix element, q^2 is on the order of the squared momentum in the atomic wave function, $q^2 \simeq (Z\alpha m)^2$. Therefore, the higher order terms in q^2 are higher order in $(Z\alpha)^2$, and they may be neglected. Thus we may approximate

$$F_1(q^2) - 1 \simeq \frac{\alpha}{3\pi} f_1(n) \frac{q^2}{m^2}\,, \tag{8.7.21}$$

and

$$F_2(q^2) \simeq \frac{\alpha}{2\pi}\,. \tag{8.7.22}$$

Here we have used the result (8.5.47) of Section 5 for $F_2(0)$ at $n = 4$. We may immediately take this limit since $F_2(0)$ has no infrared divergence. The results (8.5.46) and (8.5.56) of Section 5 give

$$f_1(n) = \frac{3}{8} - \frac{\Gamma(3 - \frac{n}{2})}{n - 4} \Big(\frac{m^2}{4\pi\mu^2} \Big)^{\frac{n}{2} - 2}\,. \tag{8.7.23}$$

Here the $n \to 4$ limit has been taken in terms which have no infrared divergence, but this may not be done for all the terms defining $F_1(0)$. The pole at $n = 4$ in Eq. (8.7.23) reflects an infrared divergence. From Eq. (8.7.19) it follows that a factor of q_l is equivalent to ∇_l / i operating on V, while effectively $q^0 = 0$ since V is time independent. Using $-i\sigma^{0l} = \alpha^l$, we may write the equivalent operator form as

$$\Sigma(E) = \Sigma_{\rm orb}(E) + \Sigma_{\rm spin}(E)\,, \tag{8.7.24}$$

where

$$\Sigma_{\rm orb}(E) = \gamma^0 \frac{\alpha}{3\pi} f_1(n) \frac{-\nabla^2 V}{m^2}\,, \tag{8.7.25}$$

and

$$\Sigma_{\rm spin}(E) = -\frac{i\alpha}{4\pi m} \vec{\alpha} \cdot \nabla V\,. \tag{8.7.26}$$

The leading non-relativistic reduction of the Dirac equation gives the leading terms in $Z\alpha$. Thus we may set $\gamma^0 = 1$ in Eq. (8.7.25) and use Schrödinger wave functions to compute the energy shift (8.7.15). To cast the result in a form suitable for comparison with the result for $n < 4$, we note that within these non-relativistic states which obey

$$(H - \mathcal{E}_n)\psi_n = 0 = \psi_n^*(H - \mathcal{E}_n)\,, \tag{8.7.27}$$

where

$$H = \frac{p^2}{2m} + V\,, \tag{8.7.28}$$

and $\mathcal{E}_n$ is the non-relativistic energy,

$$E_n = m + \mathcal{E}_n \tag{8.7.29}$$

one has, effectively,

$$\mathbf{p} \cdot (H - \mathcal{E}_n) \cdot \mathbf{p} \to \frac{1}{2}[\mathbf{p}, H] \cdot \mathbf{p} - \frac{1}{2}\mathbf{p}[\mathbf{p}, H] = -\frac{1}{2}[\mathbf{p}, \cdot[\mathbf{p}, V]] = \frac{1}{2}\nabla^2 V\,. \tag{8.7.30}$$

Hence to leading order,

$$\Sigma_{\rm orb}(E) = -\frac{2\alpha}{3\pi} f_1(n) \frac{\mathbf{p}}{m} \cdot (H - \mathcal{E}_n) \cdot \frac{\mathbf{p}}{m} . \tag{8.7.31}$$

To reduce the spin term (8.7.26), we use the Dirac equations

$$(E_n - V)\psi_n = (\vec{\alpha} \cdot \mathbf{p} + \beta m)\psi_n , \tag{8.7.32}$$

and

$$\bar{\psi}_n(E_n - V) = \bar{\psi}_n(-\vec{\alpha} \cdot \mathbf{p} + \beta m) , \tag{8.7.33}$$

to compute the matrix element of the anticommutator

$$\begin{aligned} \bar{\psi}\frac{1}{2}\{(E_n - V), \alpha \cdot (\nabla V)\}\psi &= \frac{1}{2}\bar{\psi}[\alpha \cdot \nabla V, \alpha \cdot \mathbf{p}]\psi \\ &= \frac{1}{4}\bar{\psi}\Big([\alpha_k, \alpha_l]\{\nabla_k V, p_l\} - \{\alpha_k, \alpha_l\}[p_k, \nabla_l V]\Big)\psi \\ &= \frac{1}{2}\bar{\psi}\Big(i\sigma_{kl}\{\nabla_k V, p_l\} + i\nabla^2 V\Big)\psi . \end{aligned} \tag{8.7.34}$$

To leading order, $E_n - V$ may be replaced by m on the left hand side of this equation to obtain the effective operator

$$\Sigma_{\rm spin}(E) = \frac{\alpha}{4\pi m^2}\left\{\vec{\sigma} \cdot \nabla V \times \mathbf{p} + \frac{1}{2}\nabla^2 V\right\} . \tag{8.7.35}$$

Again using Eq. (8.7.30) and the explicit form $V = -Z\alpha/r$ for the Coulomb potential, this may be expressed as

$$\Sigma_{\rm spin}(E) = \frac{\alpha Z\alpha}{4\pi m^2}\frac{1}{r^3}\vec{\sigma} \cdot \mathbf{L} + \frac{\alpha}{4\pi}\frac{\mathbf{p}}{m} \cdot (H - \mathcal{E}_n)\frac{\mathbf{p}}{m} . \tag{8.7.36}$$

4 $n < 4$

Our concern here is only with the infrared behavior of the self energy which is regulated by binding effects and produces a term which ultimately involves $\ln Z\alpha$. All that we need do in the $n < 4$ region is to explicitly exhibit this infrared sensitive term — the remaining terms are determined by the requirement that the extrapolation of the $n < 4$ result to $n > 4$ match the self energy which has been calculated for $n > 4$. The use of the radiation gauge simplifies the extraction of the infrared sensitive term from the self-energy operator (8.7.16). In this gauge, the time component $D_{00}(k)$ is independent of k^0. It describes the instantaneous Coulomb potential, it gives no infrared divergence, and it may be neglected. Taking only the spatial components of the radiation gauge propagator and using the relation $X^{-1} = (\gamma^0 X)^{-1}\gamma^0$ to exhibit the denominator structure gives

$$\Sigma(E) = ie^2\mu^{4-n}\int \frac{(d^n k)}{(2\pi)^n} D_{lm}(k)$$

$$\gamma^l \frac{1}{\vec{\alpha} \cdot (\mathbf{p} + \mathbf{k}) + \beta(m - i\epsilon) + V - E - k^0}\gamma^0\gamma^m , \tag{8.7.37}$$

where the spatial components of the radiation gauge propagator are

$$D_{lm}(k) = \frac{\delta_{lm} - \hat{\mathbf{k}}_l\hat{\mathbf{k}}_m}{\mathbf{k}^2 - k^{0\,2} - i\epsilon} . \tag{8.7.38}$$

Using the translation operator

$$e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}e^{i\mathbf{k}\cdot\mathbf{r}} = \mathbf{p} + \mathbf{k}, \tag{8.7.39}$$

Eq. (8.7.37) may be written as

$$\begin{aligned} \Sigma(E) \simeq ie^2\mu^{4-n}\int \frac{(d^{n-1}k)}{(2\pi)^{n-1}} e^{-i\mathbf{k}\cdot\mathbf{r}}\Bigg\{\int \frac{dk^0}{2\pi} D_{lm}(k) \\ \gamma^l \frac{1}{\vec{\alpha} \cdot \mathbf{p} + \beta(m - i\epsilon) + V - E - k^0}\gamma^0\gamma^m\Bigg\} e^{i\mathbf{k}\cdot\mathbf{r}} . \end{aligned} \tag{8.7.40}$$

The operator $\mathbf{r}$ acting on the bound state wave function produces terms on the order of the Bohr radius $a_0 = 1/Z\alpha m$. As will be seen, the $\ln Z\alpha$ term of interest arises from photon momenta $\mathbf{k}$ on the order of the atomic binding energy, $|\mathbf{k}| \approx (Z\alpha)^2 m$. Hence $\mathbf{k} \cdot \mathbf{r}$ is effectively of order $Z\alpha$, and the translation factors may be replaced by unity. This is the dipole approximation. The analytic structure of the denominator in Eq. (8.7.40), the (operator) electron Green's function, was discussed in Problem 8 of Chapter 7 and shown in Fig. 7.2 of that chapter. In view of this analytic structure, we compute the k^0 integral by closing the contour in the upper-half complex plane. We do this so as to avoid encircling the bound state poles and the right-hand cut of the electron Green's function. The pole in the photon propagator at $k^0 = -k = -|\mathbf{k}|$ is encircled with this propagator contributing a residue $1/2k$. The remaining k^0 integration encloses the left-hand cut of the electron propagator. Here $k^0 < -m$, no infrared sensitive term is produced, and the contribution may be neglected. Thus, to leading order in $Z\alpha$, the infrared sensitive term is contained in

$$\begin{aligned} \Sigma(E) \simeq -e^2\mu^{4-n}\int \frac{(d^{n-1}k)}{(2\pi)^{n-1}} (\delta_{lm} - \hat{\mathbf{k}}_l\hat{\mathbf{k}}_m)\frac{1}{2k} \\ \gamma^l \frac{1}{\vec{\alpha} \cdot \mathbf{p} + \beta(m - i\epsilon) + V - E + k}\gamma^0\gamma^m \end{aligned} \tag{8.7.41}$$

The nonrelativistic limit provides the leading terms on $Z\alpha$. This limit can be obtained by rationalizing the denominator in the electron Green's function, performing some Dirac matrix algebra akin to that done in the previous subsection, and neglecting the higher-order spin interaction. The

result of this procedure is

$$\Sigma(E) \simeq e^2\mu^{4-n} \int \frac{(d^{n-1}k)}{(2\pi)^{n-1}} (\delta_{lm} - \hat{k}_l\hat{k}_m)\frac{1}{2k^2} \frac{p_l}{m}\frac{-k}{H-\mathcal{E}+k-i\epsilon}\frac{p_m}{m}, \quad (8.7.42)$$

where H and $\mathcal{E}$ are the nonrelativistic Hamiltonian and energy identified in Eqs. (8.7.28) and (8.7.29). We have not spelled out the details of this reduction because the result should be clear: In the nonrelativistic limit, the electron-photon interaction is made with the velocity operator $\mathbf{p}/m$ which is the coefficient of the linear term in $\mathbf{A}$ in the modified kinetic energy $(\mathbf{p}-\mathbf{A})^2/2m$. A factor $-k$ has been put in the numerator so that an unwanted term can be removed. We write $-k = -(H-\mathcal{E}+k)+(H-\mathcal{E})$ and note that the first set of terms cancels the denominator, giving a numerical coefficient multiplying $\mathbf{p}^2/m$. We discard this contribution since it does not contribute to the infrared sensitive term. Using the angular average $\langle \hat{k}_l\hat{k}_m\rangle = \delta_{lm}/(n-1)$, we now have

$$\Sigma(E) \simeq e^2\mu^{4-n}\frac{n-2}{n-1}\int \frac{(d^{n-1}k)}{(2\pi)^{n-1}}\frac{1}{2k^2}\frac{\mathbf{p}}{m}\cdot\frac{H-\mathcal{E}}{H-\mathcal{E}+k-i\epsilon}\frac{\mathbf{p}}{m}. \quad (8.7.43)$$

The point to be made is that $\Sigma(E_n)$ must be normalized so that it vanishes when it acts on a free particle state. This is now the case since in this limit $(H-\mathcal{E}_n)$ commutes with the $\mathbf{p}$ operator and vanishes when acting on the state vector.

The remaining integration is easily done by passing to hyper-spherical coordinates. It proves convenient to rewrite the total solid angle given in Eq. (8.6.49) using the Γ-function duplication formula

$$\frac{2^{n-2}}{\sqrt{\pi}}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)\Gamma\Big(\frac{n}{2}\Big) = \Gamma(n-1). \quad (8.7.44)$$

Thus

$$\int \frac{d^{n-2}\Omega}{(2\pi)^{n-1}} = \frac{1}{4\pi^2}\frac{2^{n-2}}{(4\pi)^{\frac{n}{2}-2}}\frac{\Gamma(\frac{n}{2})}{\Gamma(n-1)}. \quad (8.7.45)$$

The remaining radial integral appears in the form of the dispersion integral representation for the function z^{n-4}:

$$\int_0^\infty dk \frac{k^{n-4}}{k+z} = \frac{\pi}{\sin\pi(4-n)}z^{n-4}. \quad (8.7.46)$$

This result is easily confirmed by calculating the discontinuity across the cut on the negative real axis of both sides of the equation. We have now

arrived at

$$\Sigma(E) \simeq \frac{\alpha}{\pi}\frac{n-2}{n-1}\frac{2\Gamma(\frac{n}{2})}{\Gamma(n-1)}\frac{\pi}{\sin\pi(4-n)} \Big(\frac{m^2}{4\pi\mu^2}\Big)^{\frac{n}{2}-2}\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E})\Big[\frac{2(H-\mathcal{E}-i\epsilon)}{m}\Big]^{n-4}\frac{\mathbf{p}}{m}. \quad (8.7.47)$$

The replacement

$$\frac{\pi}{\sin\pi(4-n)} \to -\frac{1}{n-4} \quad (8.7.48)$$

may be made since the difference vanishes in the $n \to 4$ limit. The occurrence of this pole reflects an ultraviolet divergence in the dispersion relation (8.7.46) It results from our non-relativistic treatment of the electron propagator. The full result, of course, has no such divergence, and so this pole must be subtracted. Relativistic corrections to the electron propagator also add an additional term which is well behaved at $n = 4$. We conclude that the full result for $\Sigma_{\rm orb}(E)$ reads as follows:

$$\Sigma_{\rm orb}(E) = -\frac{2\alpha}{3\pi}\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E})\Bigg[\frac{3}{n-1}\frac{\Gamma(\frac{n}{2})}{\Gamma(n-2)}\frac{1}{n-4}\Big(\frac{m^2}{4\pi\mu^2}\Big)^{\frac{n}{2}-2} \Big\{\Big[\frac{2(H-\mathcal{E}-i\epsilon)}{m}\Big]^{n-4}-1\Big\}+C\Bigg]\frac{\mathbf{p}}{m}. \quad (8.7.49)$$

This is now the *exact result* as far as the leading terms in $Z\alpha$ are concerned. The spin-dependent part $\Sigma_{\rm spin}(E)$ that has been computed for $n > 4$, a part which is regular at $n = 4$, must be added to obtain the full self-energy operator.

5 $n = 4$

To determine the as yet uncalculated constant C, we continue the result (8.7.49) to $n > 4$. In this region $(H-\mathcal{E})^{n-4}$ gives a higher order term in $Z\alpha$ and it may be neglected. Comparing Eq. (8.7.49) with Eq. (8.7.31) thus yields

$$f_1(n) = C - \frac{3}{n-1}\frac{\Gamma(\frac{n}{2})}{\Gamma(n-2)}\frac{1}{n-4}\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}, \quad (8.7.50)$$

where the limit $n \to 4$ is understood. Remembering Eq. (8.7.23),

$$C = \frac{3}{8} + \frac{1}{n-4}\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\left[\frac{3}{n-1}\frac{\Gamma(\frac{n}{2})}{\Gamma(n-2)} - \Gamma\left(3-\frac{n}{2}\right)\right]. \quad (8.7.51)$$

In the limit $n \to 4$,

$$\Gamma\left(\frac{n}{2}\right) \simeq 1 + \frac{1}{2}\psi(2)(n-4), \tag{8.7.52}$$

$$\Gamma(n-2) \simeq 1 + \psi(2)(n-4)\,, \tag{8.7.53}$$

and

$$\Gamma\left(3 - \frac{n}{2}\right) \simeq 1 - \frac{1}{2}\psi(1)(n-4). \tag{8.7.54}$$

Therefore, the pole terms cancel as they must, no logarithm of $(m^2/4\pi\mu^2)$ appears, and

$$C = \frac{3}{8} - \frac{1}{3} - \frac{1}{2}[\psi(2) - \psi(1)] \tag{8.7.55}$$

It follows from $\Gamma(z+1) = z\Gamma(z)$ that $\psi(2) = \psi(1) + 1$. Hence

$$C = \frac{3}{8} - \frac{5}{6}\,. \tag{8.7.56}$$

Taking the limit $n \to 4$ in Eq. (8.7.49) yields

$$\Sigma_{\rm orb}(E) = \frac{2\alpha}{3\pi}\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E})\left\{\ln\left[\frac{m}{2(H-\mathcal{E}-i\epsilon)}\right] + \frac{5}{6} - \frac{3}{8}\right\}\frac{\mathbf{p}}{m}\,. \tag{8.7.57}$$

The matrix element of $\Sigma(E_n)$ generally gives a complex energy shift, and so we write

$$\Delta E_n - \frac{i}{2}\gamma_n = \langle n|\Sigma(E_n)|n\rangle\,. \tag{8.7.58}$$

The imaginary part γ_n is the width of the unstable state; it decays according to $|\psi_n|^2 \sim \exp\{-\gamma_n t\}$. This imaginary part arises from the contribution of intermediate states whose binding energy exceeds the binding energy of the initial state, $\mathcal{E}_{n'} < \mathcal{E}_n$. The initial state may decay into such states. For these, the argument of the logarithm is negative; more precisely, the $-i\epsilon$ prescription tells us that we have $\ln[1/\exp\{-\pi i\}] = +\pi i$. Thus, the total width γ_n is the sum of partial widths $\gamma_{n'n}$,

$$\gamma_n = \sum_{n'<n} \gamma_{n'n}\,, \tag{8.7.59}$$

in which

$$\gamma_{n'n} = \frac{4\alpha}{3}\omega_{n'n}\,|\langle n'|\frac{\mathbf{p}}{m}|n\rangle|^2\,, \tag{8.7.60}$$

where

$$\omega_{n'n} = \mathcal{E}_n - \mathcal{E}_{n'} > 0\,. \tag{8.7.61}$$

This is just the result of elementary radiation theory for the decay width. The Lamb shift energy ΔE_n and width γ_n are the real and imaginary parts of the same analytic function. The energy shift ΔE_n is the dispersion associated with the absorption γ_n. Since the calculation of γ_n is elementary one, so is the calculation of the Lamb shift — almost. By counting dimensions one concludes that there is a logarithmic infrared divergence. Given the decay rate formula (8.7.60) and that the shift and width are the real and imaginary parts of the same function — a logarithm — one can immediately write down

$$\Sigma(E_n) = \frac{2\alpha}{3\pi}\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E}_n)\ln\left[\frac{m}{H-\mathcal{E}_n-i\epsilon}\right]\frac{\mathbf{p}}{m} + \cdots\,. \tag{8.7.62}$$

All our work has been to find that here is an extra factor of $\frac{1}{2}$ in the argument of the logarithm plus an additional $\frac{5}{6} - \frac{3}{8}$. The extra spin contribution $\Sigma_{\rm spin}$ is known from the magnetic moment anomaly $(g-2)/2$.

The order of magnitude of $H - \mathcal{E}_n$ is the binding energy $(Z\alpha)^2 m/2$, which is also called the Rydberg energy unit. Using this energy unit to scale the argument of the logarithm and adding together the two results (8.7.57) and (8.7.36) gives

$$\Sigma(E_n) = \frac{2\alpha}{3\pi}\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E}_n)\left\{\ln\left[\frac{(Z\alpha)^2 m/2}{H-\mathcal{E}_n-i\epsilon}\right] + 2\ln\left(\frac{1}{Z\alpha}\right) + \frac{5}{6}\right\}\frac{\mathbf{p}}{m} + \frac{\alpha Z\alpha}{4\pi m^2}\frac{1}{r^3}\vec{\sigma}\cdot\mathbf{L}\,. \tag{8.7.63}$$

For S wave states it is conventional to define an average excitation energy $\mathcal{K}_n$ in Rydberg units by

$$\begin{aligned}\langle n, l=0|\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E}_n)\left\{\ln\left|\frac{(Z\alpha)^2 m/2}{H-\mathcal{E}_n}\right|\right\}\frac{\mathbf{p}}{m}|n, l=0\rangle \\ = \langle n, l=0|\frac{\mathbf{p}}{m}\cdot(H-\mathcal{E}_n)\left\{\ln\frac{1}{\mathcal{K}_n}\right\}\frac{\mathbf{p}}{m}|n, l=0\rangle\,.\end{aligned} \tag{8.7.64}$$

One expects that the pure number $\mathcal{K}_n$ is roughly of order of magnitude unity. Its evaluation requires a detailed atomic physics calculation. Using Eq. (8.7.30) with $\nabla^2 V = 4\pi Z\alpha\delta(\mathbf{r})$, the energy shift for S states becomes

$$\begin{aligned}\Delta E_{nS} &= \frac{4\alpha Z\alpha}{3m^2}\psi_{nS}(0)^2\left\{\ln\frac{1}{\mathcal{K}_n} + 2\ln\left(\frac{1}{Z\alpha}\right) + \frac{5}{6}\right\} \\ &= \frac{4\alpha}{3\pi}\frac{(Z\alpha)^4 m}{n^3}\left\{\ln\frac{1}{\mathcal{K}_n} + 2\ln\left(\frac{1}{Z\alpha}\right) + \frac{5}{6}\right\}\,.\end{aligned} \tag{8.7.65}$$

This shift is of the same $\alpha(Z\alpha)^4$ order as is the vacuum polarization correction discussed in Section 3.3. However, the numerical factor in front of the shift (8.7.65) is 5 times larger than the coefficient in the

vacuum polarization correction (8.3.92). Moreover, $2\ln(1/Z\alpha) \simeq 9.84$ is a large number. For the $2S$ state, $\mathcal{K}_2 \simeq 16.6$, which is a surprisingly large number, giving $\ln(1/\mathcal{K}_2) \simeq -2.81$. Putting in the numbers, one finds that this shift for a $2S$ hydrogen level is about 39 times larger than 27 MHz of the vacuum polarization correction — it is a shift of some 1 000 MHz. An accurate evaluation of the numbers including the much smaller shift for a P-wave gives

$$\Delta\nu_{2S_{1/2}} - \Delta\nu_{2P_{1/2}} = 1\,079.2 \text{ MHz}\,. \tag{8.7.66}$$

To this should be added the vacuum polarization correction (8.3.96),

$$\Delta\nu_{2S} = -27.1 \text{ MHz}\,, \tag{8.7.67}$$

giving the total

$$\Delta\nu = 1\,052.1 \text{ MHz}\,. \tag{8.7.68}$$

This is to be compared to current experimental value

$$\Delta\nu_{\text{exp}} = 1\,057.845\,(9) \text{ MHz}\,. \tag{8.7.69}$$

The calculation we have done is only to the leading order in the binding correction $Z\alpha$. Higher order powers in $Z\alpha$, which also involve powers of $\ln Z\alpha$, have been calculated as well as nuclear recoil contributions and the correction resulting from the finite distribution of charge in the proton. The present theoretical value is

$$\Delta\nu_{\text{th}} = 1\,057.86 \text{ MHz}\,. \tag{8.7.70}$$

The discovery of the $2S_{1/2} - 2P_{1/2}$ level shift in hydrogen by Lamb and Retherford in 1947 along with the observation of anomaly in the magnetic moment of the electron by Foley and Kusch in the same year triggered the development of relativistic quantum field theory. Having described the Lamb shift and the anomaly, it is now an appropriate place to end this book.

Problems

What is it that confers the noblest delight?
What is that which swell a man's breast with pride
above that which any other experience can bring to him?
Discovery!

Mark Twain

1. Electron-Proton Elastic Scattering

In lowest order, the elastic scattering of an electron on a proton is described by the graph of Fig. 7.1 in Problem 6, Chapter 7. Use the result of Problem 7 in Chapter 7 to compute the differential cross section for the scattering of unpolarized electrons on unpolarized protons. Assume that neither the spin of the scattered electron or proton is measured so that these final spins are summed over. Assume that the energy is high on the scale set by the electron mass so that the electron mass may be neglected. Prove that the differential cross section for an electron to scatter into the solid angle $d\Omega$ in the laboratory frame where the initial proton is at rest is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{2E\sin\theta/2}\right)^2 \frac{E'}{E}\left\{\frac{\cot^2\theta/2}{1+k^2/4M^2}\left[G_E(k^2)^2 + \frac{k^2}{4M^2}G_M(k^2)^2\right] + 2\frac{k^2}{4M^2}G_M(k^2)^2\right\}.$$

Here θ is the electron scattering angle, E and E' are the initial and final electron energies, k^2 is the square of the four-momentum transfer, and M is the proton mass.

2. Electron Propagator Gauge and Infrared Behavior

As shown in Section 3.1, the fully interacting electron propagator may be expressed as

$$G_0\,(x - x') = \int [dA]\, G[A; x, x'] e^{iS_{\text{eff}}}\,,$$

where S_{eff} is the action produced by integrating out the electron field. The sources are taken to vanish in S_{eff}. Here $G[A; x, x']$ is the free-particle Green's function in the presence of an external vector potential. It obeys

$$\left[\gamma\left(\frac{1}{i}\partial - A\right) + m_0\right] G[A; x, x'] = \delta(x - x')\,.$$

a) Use the Green's function equation above to prove that under a gauge transformation

$$G[A+\partial\lambda;x,x'] = \exp\{i[\lambda(x)-\lambda(x')]\}G[A;x,x']\,.$$

The gauge dependence of the Green's function is exhibited by writing

$$G[A;x,x'] = \exp\left\{i\int_{x'}^{x} d\bar{x}^\mu A_\mu(\bar{x})\right\}\bar{G}[A;x,x']\,,$$

where the line integral in the exponential is taken along the straight line running from x' to x. Prove that $\bar{G}[A;x,x']$ is indeed not altered by the gauge transformation $A \to A+\partial\lambda$.

The interacting propagator now appears as

$$G_0(x-x') = \int [dA]\,\bar{G}\,[A;x,x']\exp\{iS_{\text{eff}} + i\int JA\}\,,$$

in which

$$J^\mu(z) = (x-x')^\mu \int_0^1 d\alpha\,\delta(z-x'-(x-x')\alpha)\,.$$

Since $\bar{G}$ is gauge invariant, it is a functional of the field strength $F_{\mu\nu}$ rather than the vector potential A_μ. Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the gauge-invariant function $\bar{G}$ contains an extra factor of k_μ in its couplings to a photon of momentum k_μ. Soft photons as well as longitudinal photons do not couple to $\bar{G}$. The gauge and infrared behavior of the propagator is contained in the JA term in the exponential in the functional integral. To make this explicit, we note that any functional of A may be expressed as

$$\begin{aligned} F[A] &= F\left[e^{-i\int jA}\,\frac{1}{i}\frac{\delta}{\delta j}\,e^{i\int jA}\right]\Big|_{j=0} = e^{-i\int jA}\,F\left[\frac{1}{i}\frac{\delta}{\delta j}\right]e^{i\int jA}\Big|_{j=0} \\ &= F\left[\frac{1}{i}\frac{\delta}{\delta j}\right]e^{i\int jA}\Big|_{j=0}\,. \end{aligned}$$

Hence

$$\begin{aligned} G_0(x-x') &= \bar{G}\left[\frac{1}{i}\frac{\delta}{\delta j};x,x'\right]\int [dA]\exp\{iS_{\text{eff}} + i\int (J+j)A\}\Big|_{j=0} \\ &= \bar{G}\left[\frac{1}{i}\frac{\delta}{\delta j};x,x'\right]\exp\{iW[J+j]\}\Big|_{j=0}\,, \end{aligned}$$

where W is the generating functional for the interacting photon Green's functions.

The generating functional has the form

$$W[j] = \frac{1}{2}\int j^\mu \mathcal{G}_{\mu\nu} j^\nu + \bar{W}[j]\,,$$

where $\bar{W}[j]$ contains only terms of fourth and higher order in j^μ. The work of Section 1.3 implies that $\bar{W}[j]$ involves only the transverse part of j^ν. The response of the functional integral to a variation $\delta\xi_0$ of the gauge-fixing parameter involves $\partial_\mu A^\mu$ which can be expressed in terms of a variational derivative with respect to the longitudinal component of j^μ. Bearing this in mind, it is not hard to prove that $\bar{W}[j]$ is independent of ξ_0. The higher order photon-photon interactions described by $\bar{W}[j]$ are of short range, and they do not contribute to the infrared behavior.

b) Show that (in an operator notation)

$$G_0(x,x') = e^F\,\bar{G}\left[\mathcal{G}J + \frac{1}{i}\frac{\delta}{\delta j};x,x'\right]\exp\left\{\frac{i}{2}j\mathcal{G}j + i\bar{W}[J+j]\right\}\Big|_{j=0}\,,$$

where

$$F = \frac{i}{2}J\mathcal{G}J\,.$$

Since $\bar{G}[A;x,x']$ depends only upon the transverse part of A, the entire ξ_0 dependence of $G_0(x,x')$ is contained in the function F. Moreover, in view of the remarks which have been made, F also contains all the leading infrared contributions. Thus

$$G_0(x,x') = \exp\{F(x-x')\}\,\bar{G}_0(x-x')\,,$$

where $\bar{G}_0(x-x')$ is independent of the gauge-fixing parameter ξ_0 and does not contain leading infrared divergencies. The infrared behavior of $F(x-x')$ arises from the $k^2=0$ pole in $\mathcal{G}_{\mu\nu}(k)$. The residue at this pole involves the electric charge with "mass-shell" renormalization, the charge e^2 which appears in the fine structure constant $\alpha = e^2/4\pi$. Thus the infrared (and gauge) behavior is obtained by replacing $\mathcal{G}_{\mu\nu}$ by $e^2\mu^{4-n}D_{\mu\nu}$.

c) Show that

$$\begin{aligned} e^2\mu^{4-n}D_{\mu\nu}(x) = &\frac{i\mu^{4-n}\,\Gamma(\frac{n}{2}-1)}{2(4\pi)^{n/2}} \\ &\left(\frac{4}{x^2}\right)^{\frac{n}{2}-1}\left[(e^2-\xi)g_{\mu\nu} + (e^2+\xi)(n-2)\frac{x_\mu x_\nu}{x^2}\right]. \end{aligned}$$

d) Using this approximation for $\mathcal{G}_{\mu\nu}$, show that

$$F(x-x') = -\left(\xi + e^2\frac{n-1}{n-3}\right)\Lambda(x-x'),$$

in which

$$\Lambda(x) = \frac{\Gamma(\frac{n}{2}-2)}{(4\pi)^2}\left(\frac{1}{\pi\mu^2x^2}\right)^{\frac{n}{2}-2}\,.$$

The function $\Lambda(x-x')$ is the same as that defined in Section 4.3 of the text, and it has now been proven that the propagators in different gauges are indeed related by

$$G^{(\xi')}(x-x') = \exp\{-(\xi'-\xi)\Lambda(x-x')\}G^{(\xi)}(x-x')\,.$$

The physical limit $n \to 4$ yields

$$F(x-x') = -\frac{\xi+3e^2}{(4\pi)^2}\left\{\frac{2}{n-4} - \ln[\mu^2(x-x')^2]\right\} + \text{const}\,.$$

We see that the infrared and gauge-dependent contribution to the minimally subtracted wave function renormalization is given by

$$Z_{1,\,\text{red}} = \exp\left\{-\frac{\xi+3e^2}{(4\pi)^2}\frac{2}{n-4}\right\}.$$

Aside from constant terms, the infrared modification of the electron propagator is given by

$$G(x-x') = C[\mu^2(x-x')^2]^{\nu}\bar{G}(x-x')\,,$$

where

$$\nu = \frac{\xi+3e^2}{(4\pi)^2}\,.$$

As was shown in Section 4.3 for the first-order correction, the infrared corrections alter the pole at $-p^2 = m^2_{\text{phy}}$ in the Fourier transform of the propagator $G(p)$. This singularity in momentum space controls the long-distance behavior in coordinate space. Assuming that the remaining virtual photon corrections in $\bar{G}(p)$ do not change its pole structure, we have the long-distance space-like limit

$$\bar{G}(x-x') \simeq Z\Big(\gamma\frac{1}{i}\partial + m_{\text{phy}}\Big)\, i\int\frac{(d^4_E p)}{(2\pi)^4}\,\frac{e^{ip(x-x')}}{p^2+m^2_{\text{phy}}}\,.$$

e) Consider

$$\Delta_\lambda(x) = (\mu m_{\text{phy}})^\lambda \int\frac{(d^4_E p)}{(2\pi)^4}\,\frac{e^{ipx}}{(p^2+m^2_{\text{phy}})^{1+\lambda}}\,.$$

Prove that

$$\Delta_\lambda(x) = \frac{(\mu m_{\text{phy}})^\lambda}{\Gamma(1+\lambda)}\,\frac{1}{(4\pi)^2}\int_0^\infty ds s^{\lambda-2}\exp\Big\{-sm^2_{\text{phy}} - \frac{x^2}{4s}\Big\}\,.$$

Using the method of steepest descents, prove that for $m^2_{\text{phy}}x^2 \gg 1$:

$$\Delta_\lambda(x) \simeq \text{const.}(\mu^2x^2)^{\lambda/2}m^2_{\text{phy}}(m^2_{\text{phy}}x^2)^{-3/4}\,e^{-m_{\text{phy}}\sqrt{x^2}}\,.$$

Since the derivative acting on a power of x^2 gives a smaller asymptomatic contribution, the asymptomatic form of the previous infrared modification of the electron propagator may be expressed as

$$G(x-x') \simeq CZ\Big(\gamma\frac{1}{i}\partial + m_{\text{phy}}\Big)[\mu^2(x-x')^2]^{\nu} i\Delta_{\lambda=0}\,(x-x')$$

$$\simeq \text{const.}\Big(\gamma\frac{1}{i}\partial + m_{\text{phy}}\Big) i\Delta_{2\nu}(x-x')\,.$$

Accordingly, we infer that the leading infrared corrections modify the pole in the Fourier transform of the electron propagator to the form $p^2 \simeq m^2_{\text{phy}}$:

$$G(p) \simeq \text{const.}\,\frac{m_{\text{phy}} - \gamma p}{(p^2+m^2_{\text{phy}})^{1+2\nu}}\,.$$

The expansion to first order in ν produces the logarithmic correction computed in Section 4.3.

3. Infrared Problem

The purpose of this problem is to illustrate the basic nature of the infrared problem in quantum electrodynamics. It suffices to consider the electromagnetic interactions of a spin zero field described by the Lagrange function

$$\mathcal{L} = -\phi^*\Big[\Big(\frac{1}{i}\partial - A\Big)^2 + m^2\Big]\phi + \ldots\,,$$

where the ellipsis stands for other interactions. The term linear in A_μ in the Lagrange function identifies the photon particle vertex in momentum space as $(p'+p)^\mu$. The quadratic term in A_μ gives rise to a two-photon particle vertex $g^{\mu\nu}$. In view of the shape of the corresponding graph, this two-photon coupling is called a "seagull" graph. The simplest situation appears when the ϕ-particle scatters on some other neutral particle. The leading infrared corrections arise from the photon emissions from the ϕ-particle propagation exterior to the basic scattering as shown in the graph of Fig. 8.15. The two-photon couplings are omitted here because they entail fewer scalar particle propagator denominators (which have the infrared singularities) and thus these couplings do not contribute to the leading infrared behavior. The "emitted" photons shown in Fig. 8.15 may be either "tied together" in pairs so as to form virtual (loop) corrections, or they may be real emitted photons. Since we are concerned with the infrared behavior of the virtual photon contributions, their mass may be taken to vanish ($k^2 = 0$) in computing the ϕ-particle propagator factors in the external lines. Thus the photon emissions from the external lines, with the ϕ-particle initially (and finally) on the mass shell, are given to

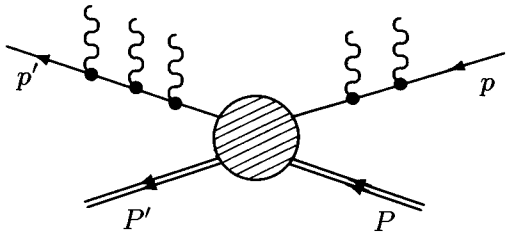

Fig. 8.15. Elastic scattering of a charged and neutral particle with soft-photon emission.

leading order (the small k_μ limit) by the factors

$$\frac{p^{\mu_1}}{pk_1}\,, \qquad \frac{p^{\mu_2}}{p(k_1+k_2)}\,\frac{p^{\mu_1}}{pk_1}\,, \qquad \cdots\,.$$

To deal with this problem, it is convenient to replace the photon interaction by the corresponding interaction with an external vector potential A_μ. This gives a generating functional which may be used to compute the photon interactions and photon production. All of this infrared photon emission is then described by wave functions $\Phi_{\rm in}[x;A]$ and $\Phi^*_{\rm out}[x';A]$ for the incoming and outgoing charged ϕ-particles. It is convenient to write these wave functions in coordinate space — they are to be integrated with the basic scattering amplitude T in coordinate space.

a) Prove that the leading infrared corrections are contained in

$$\Phi\,[x;A] \simeq e^{ipx}\,\exp\left\{\int \frac{(d^4k)}{(2\pi)^4}\,A_\mu(k)\frac{p^\mu}{pk}\,e^{ikx}\right\}\,.$$

That is, show that this is the proper approximate solution to the Klein-Gordon equation. Show that expression in the curly brackets is just the straight-line integral

$$i\int_z^x d\bar{x}^\mu\,A_\mu\,(\bar{x})$$

discussed in Problem 2, but with $z^\mu = p^\mu\tau$ and the limit $\tau \to -\infty$ taken.

Note that the expansion of the infrared wave function gives the external line corrections

$$\frac{p^{\mu_1}}{pk_1}\,, \quad \frac{1}{2}\,\frac{p^{\mu_2}}{pk_2}\,\frac{p^{\mu_1}}{pk_1}\,, \quad \frac{1}{3!}\,\frac{p^{\mu_3}}{pk_3}\,\frac{p^{\mu_2}}{pk_2}\,\frac{p^{\mu_1}}{pk_1}\,,$$

and so forth. This corresponds to the algebraic identity for the properly

symmetrized external line factors

$$\frac{1}{2}\,\frac{1}{p(k_1+k_2)}\left(\frac{1}{pk_1}+\frac{1}{pk_2}\right) = \frac{1}{2}\,\frac{1}{pk_2}\,\frac{1}{pk_1}\,,$$

$$\frac{1}{3!}\sum_{\rm perms}\frac{1}{p(k_1+k_2+k_3)}\,\frac{1}{p(k_1+k_2)}\,\frac{1}{pk_1} = \frac{1}{3!}\,\frac{1}{pk_3}\,\frac{1}{pk_2}\,\frac{1}{pk_1}\,,$$

and so forth. Note that this algebraic identity follows from the the exponential character of the infrared wave function — the identity is logically implied from the previous proof.

We have found that the scattering amplitude with the coupling of soft photons to the external charged particle propagation has the structure indicated in

$$T[A] \sim \int (d^4x)\,(d^4y)\,e^{ipx-p'y}\,e^{i\Theta}\,T\,(x,y,\cdots)\,,$$

where

$$\Theta = \int_{-p^\mu\infty}^{x} d\bar{x}^\mu A_\mu(\bar{x}) + \int_y^{+p'^\mu\infty} d\bar{x}^\mu A_\mu(\bar{x})\,.$$

This is not a gauge-invariant quantity. Gauge invariance is restored if another line integral from x^μ to y^μ is included. The addition corresponds to the emission of soft photons from charge particle lines in the loops that make up the basic scattering amplitude. Since the momentum of such an internal line is integrated over all (off-mass-shell) values, the addition cannot contain the leading $1/pk$ infrared singularity. Indeed

$$i\int_x^y d\bar{x}^\mu A_\mu(\bar{x}) = \int \frac{(d^4k)}{(2\pi)^4}\,A_\mu(k)\frac{p^\mu}{pk}\left[e^{iky}-e^{ikx}\right]$$

has no singularity when $k_\mu \to 0$; the coordinate difference $y^\mu - x^\mu$ is limited by the integration with $T(x,y,\cdots)$. As far as the leading infrared divergences are concerned, the non-locality of the coordinate space version of the scattering amplitude can be neglected with the identification $x = y$ made in Θ and

$$T[A] \sim T\int (d^4x)\,e^{-i(p'+P'-p-P)x}\,e^{i\Theta}\,.$$

In summary, the infrared effects are described by the coupling of the photon field to a classical external current $J^\mu(z)$. This is the current produced by a point particle moving along the trajectory given by the proper time evolution

$$z^\mu(\tau) = x^\mu + v^\mu\tau\,,$$

where

$$\tau<0: \quad v^\mu = p^\mu/m\,, \qquad \tau>0: \quad v^\mu = p'^\mu/m\,.$$

In this approximation, the particle scatters instantaneously from momentum p to momentum p' at proper time $\tau = 0$. The current is given by

$$J^\mu(z) = \int_{-\infty}^{\infty} d\tau\, v^\mu(\tau)\, \delta(z - z(\tau))\,.$$

Making a decomposition into modes, the photon field generated by an external classical current is equivalent to an infinite set of harmonic oscillators driven by external forces. The case of one harmonic oscillator driven by an external force was examined in Problem 4 of Chapter 1. That problem (which is worthwhile to review now) used coherent states to derive the results. A similar method could be employed here, but it is simpler to use another technique. Summing over the unobserved soft photons produces a transition probability of the form

$$\text{Prob.} \sim |T|^2 \int (d^4x)\,(d^4x')\, e^{iQ(x-x')}\, S\,,$$

where Q is the four-momentum transfer which is imparted to the radiation field and

$$S = \sum\nolimits_n \langle 0 - |n+\rangle^{J'} \langle n + |0-\rangle^{J}\,.$$

The sum S involves the radiation field transformation functions driven by external currents, with J' defined by the $p^\mu \to p'^\mu$ scattering event appearing at the space-time point x'^μ rather than at x^μ as it is for J.

b) Compute S by varying J and J' and solving the resulting variational equation. Hints: Note that

$$\delta\langle n + |0-\rangle^J = i \int (d^4z) \langle n + |A_\mu(z)|0-\rangle^J\, J^\mu(z)\,.$$

Use the radiation gauge where A^0 is an explicit function of J^0. For the spatially transverse (T) vector potential define $\omega = \sqrt{-\nabla^2}$ and write

$$\mathbf{A} = \mathbf{A}^{(+)} + \mathbf{A}^{(-)}\,,$$

in which

$$\mathbf{A}^{(\pm)} = \frac{1}{2}\Big(\mathbf{A} \mp \frac{i}{\omega}\,{}^{(T)}\mathbf{E}\Big)\,.$$

Show that

$$(\partial_0 \pm i\omega)\mathbf{A}^{(\pm)} = \pm \frac{ie^2}{2\omega}\,{}^{(T)}\mathbf{J}\,.$$

Integrate this first-order differential equation to express $\mathbf{A}^{(\pm)}(\mathbf{r}, t)$ in terms of $\mathbf{A}^{(\pm)}(\mathbf{r}, \mp\infty)$ and $\mathbf{J}$. That is, integrate it forward in time from $t = -\infty$ for $\mathbf{A}^{(+)}$ and backward in time from $t = +\infty$ for $\mathbf{A}^{(-)}$. Explain why

$$\mathbf{A}^{(+)}(\mathbf{r}, -\infty)|0-\rangle = 0\,.$$

Then use

$$\sum\nolimits_n |n+\rangle\langle n + |\mathbf{A}^{(-)}(\mathbf{r}, +\infty) = \mathbf{A}^{(-)}(\mathbf{r}, +\infty) \sum\nolimits_n |n+\rangle\langle n + |\,,$$

and solve the first-order differential equation to express $\mathbf{A}^{(-)}(\mathbf{r}, +\infty)$ in terms of $A^{(-)}(\mathbf{r}, -\infty)$ and $\mathbf{J}$. Using

$$\langle 0 - |\mathbf{A}^{(-)}(\mathbf{r}, -\infty) = 0\,,$$

$\delta_J S$ has now been calculated. Compute $\delta_{J'} S$ in the same way. Solve these variational equations and make use of the conservation of the currents to show that

$$S = \exp\left\{e^2 \int \frac{(d^3\mathbf{k})}{(2\pi)^3}\,\frac{1}{2k^0}\left[J'_\mu(k)^*\, J^\mu(k) - \frac{1}{2}(|J'(k)|^2 + |J(k)|^2)\right]\right\}\,,$$

where $k^0 = |\mathbf{k}|$.

Inserting the explicit forms for the current, we find that S depends only upon the difference $x - x'$. Hence one of the two space-time integrations in the previous formula for the transition probability produces the usual factor of the space-time volume $(2\pi)^4\delta^{(4)}(0)$ which is canceled when the cross section is evaluated. Thus we find that the cross section involves a phase-space integration over P' and p' with

$$|T|^2 \int (d^4x) e^{iQx}\, e^{\Theta(x)}\,,$$

where now

$$\Theta(x) = e^2 \int \frac{(d^3\mathbf{k})}{(2\pi)^3}\,\frac{1}{2k^0}\left(\frac{p'}{p'k} - \frac{p}{pk}\right)^2 \left(e^{-ikx} - 1\right)\,.$$

Note that a spurious ultraviolet divergence involving the final -1 appears here. It occurs because of the soft photon approximations that have been made in which the classical trajectory abruptly scatters from p to p'. This divergence is removed with a limit on the magnitude of the momenta, $|\mathbf{k}| < m$, which is appropriate to the infrared situation. Note also that expanding the exponential of $\Theta(x)$ and performing the x-integration produces terms involving $\delta^{(4)}(Q - k_1 - \ldots - k_n)$ corresponding to the conservation of energy-momentum in the production of the photon with four momenta $k_1, \ldots, k_n$.

The calculation of the Fourier transform of $e^{\Theta(x)}$ in the general case is a formidable task. The general nature of the infrared problem is revealed, however, in the much simplier case where the target particle (whose momenta are labeled by $\mathbf{P}, \mathbf{P}'$) is very heavy. In this case a slight recoil of the heavy target "soaks up" all the spatial momentum transfer. This equivalent to saying that the spatial momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ (as well as $\mathbf{p}$ and $\mathbf{p}'$) may be omitted from the momentum conserving δ-function or

that we may set $\mathbf{x} = 0$ in $\Theta(\mathbf{x}, t)$. The energy conserving δ-function now depends only upon $(p'^0 + k_1^0 + \ldots + k_n^0 - p^0)$. Integrating the cross section over an energy loss range $0 < p^0 - p'^0 < \Delta E$, the infrared approximation gives

$$\sigma(\Delta E) = \sigma_0 \int_0^{\Delta E} dE \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-iEt} \exp\{\Theta(\mathbf{0}, t)\} .$$

Here $\sigma(\Delta E)$ is the cross section including the emission of an arbitrary number of soft photons whose total energy does not exceed ΔE while σ_0 is the basic elastic cross section. And here

$$\Theta(\mathbf{0}, t) = \frac{\alpha}{\pi} \int_0^m \frac{dk}{k} (e^{ikt} - 1) f(p', p) ,$$

where

$$f(p', p) = \int \frac{d\Omega}{4\pi} \left(\frac{p'}{p'\hat{k}} - \frac{p}{p\hat{k}} \right)^2 ,$$

with $\hat{k}^\mu = (\hat{\mathbf{k}}, 1)$. Expanding the exponential and performing the t-integration, we see that the photon momentum integral associated with e^{ikt} is limited to $k < \Delta E$. Hence we have, effectively,

$$\Theta(0, t) = \frac{\alpha}{\pi} \left\{ \int_0^{\Delta E} \frac{dk}{k} \left(e^{ikt} - 1 \right) - \ln \left(\frac{m}{\Delta E} \right) \right\} f(p', p) .$$

Scaling $E = \bar{E}\Delta E$, $t = \bar{t}/\Delta E$, and $k = \bar{k}\Delta E$ now produces

$$\sigma(\Delta E) = \sigma_0 \exp \left\{ -\frac{\alpha}{\pi} f(p', p) \ln \left(\frac{m}{\Delta E} \right) \right\} F .$$

Here

$$F = \int_0^1 d\bar{E} \int_{-\infty}^{\infty} \frac{d\bar{t}}{2\pi} e^{-i\bar{E}\bar{t}} \exp \left\{ \frac{\alpha}{\pi} \int_0^1 \frac{d\bar{k}}{\bar{k}} \left(e^{i\bar{k}\bar{t}} - 1 \right) f(p', p) \right\}$$

is a well-behaved numerical factor which is independent of the energy loss ΔE. The soft photon correction to the cross section does have the exponential form stated in the text. In particular, the cross section vanishes when the energy resolution vanishes, $\Delta E \to 0$.

c) Prove that

$$F = 1 + \text{Order}(\alpha^2) .$$

d) Prove that $f(p', p)$ may be expressed in the Lorentz invariant form

$$f(p', p) = \int_0^{\infty} d\lambda \frac{-2p'p}{m^2(1 + \lambda^2) + \lambda(-2p'p)} - (p' = p) .$$

Use this result to show that for large momentum transfer

$$f(p', p) \simeq 2 \ln \left(\frac{-p'p}{m^2} \right) .$$

4. Axial Anomaly

Throughout this problem we shall work with the simplest case of a free Dirac field described by

$$\mathcal{L} = -\bar{\psi} \left[\gamma \frac{1}{i} \partial + m \right] \psi .$$

a) This Lagrange function is invariant under global phase rotations. Consider the field variation induced by a local phase rotation.

$$\delta\psi(x) = i\delta\lambda(x)\psi(x) , \quad \delta\bar{\psi}(x) = -i\delta\lambda(x)\bar{\psi}(x) .$$

If the field equations are obeyed, then the action S is stationary against all field variations. In particular, $\delta S = 0$ for the $\delta\lambda$ variation. Show that this action variation implies that

$$\partial_\mu j^\mu = 0 ,$$

where the vector current is given by

$$j^\mu = \bar{\psi}\gamma^\mu\psi .$$

The point to be made is that although $\partial_\mu j^\mu = 0$ follows directly from the field equations, it is simpler and more instructive to establish this current conservation directly from the action. The situation here is an example of a general theorem: If an action is invariant under some infinitesimal global transformation parameterized by $\delta\lambda$, then there is a corresponding conserved current. In general, the global invariance implies that the response to a local transformation must involve $\partial_\mu \delta\lambda(x)$; hence, integrating by parts and noting that the action is stationary when the field equations are obeyed, one finds a conserved current.

b) Prove that the massless theory is invariant under the global chiral rotation

$$\psi \to e^{\gamma_5\theta}\psi , \quad \bar{\psi} \to \bar{\psi}e^{\gamma_5\theta} .$$

Use the method of part a) to show that when $m \neq 0$

$$\partial_\mu j_5^\mu = -2m\bar{\psi}\gamma_5\psi ,$$

where

$$j_5^\mu = \bar{\psi}\gamma^\mu i\gamma_5\psi$$

is the axial current. [This is also an example of a more general theorem.]

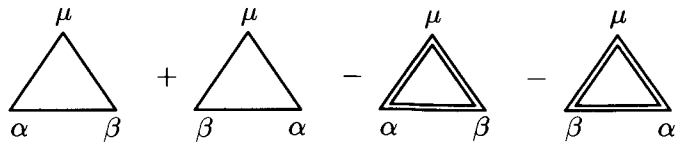

Fig. 8.16. Pauli-Villars regularization of the triangle anomaly.

The vector current remains conserved even for an interacting theory which must be renormalized. This is because the theory may be regulated by a method (such as dimensional continuation) which maintains the conservation at all stages. On the other hand, there is no regularization method which maintains the conservation law of the axial vector current. In particular, dimensional continuation does not maintain the global chiral invariance of the massless Lagrange function: $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ commutes rather than anticommuting with the γ^μ matrices of the extra dimensions. The divergence of the axial current for the interacting theory is, in fact, "anomalous". It has a triangle "anomaly". The divergence of the axial current in the free-field three-point function $\langle T\, j_5^\mu\, j^\alpha\, j^\beta\rangle$ does not give the result one naively expects. This can be shown using dimensional continuation to compute the corresponding triangle graph, with $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ breaking Lorentz invariance when $n \neq 4$. In this particular case, however, another method of regularization — Pauli-Villars regularization — is simpler and more instructive. In this method one introduces an additional Dirac field Ψ of mass M which has a negative norm. The effects of this compensating field is removed by taking the limit $M \to \infty$ after all calculations have been performed. The axial current is now given by

$$j_5^\mu = \bar\psi\gamma^\mu i\gamma_5\psi + \bar\Psi\gamma^\mu i\gamma_5\Psi\,.$$

The negative norm of Ψ produces a *difference* of triangle graphs which is well defined if they are combined to form the integrand of an overall $(n = 4)$ loop momentum integral.

c) With Pauli-Villars regularization, the Fourier transform of the three-point function $\langle T\, j_5^\mu\, j^\alpha\, j^\beta\rangle$ is described by the graphs in Fig. 8.16. Here the double line corresponds to the compensating field Ψ. Write down the corresponding momentum-space loop integral. The divergence $\partial_\mu j_5^\mu$ corresponds to the contraction with $q_\mu = k_\mu + k'_\mu$. Use algebraic identities and shifts of the loop momentum p^μ to prove that the contraction of the loop integral with q_μ corresponds to

$$\partial_\mu j_5^\mu = -2m\bar\psi\gamma_5\psi - 2M\bar\Psi\gamma_5\Psi\,.$$

That is, prove that the contraction with q_μ gives triangle graphs for

$\langle T\,\bar\psi\gamma_5\psi\, j^\alpha\, j^\beta\rangle$, $\langle T\,\bar\Psi\gamma_5\Psi\, j^\alpha\, j^\beta\rangle$ which correspond to this equation. Explain why the regularization is needed to prove this result — show that the separate pieces involve divergent integrals, and shifting p^μ in these separate pieces is not justified.

d) The three-point functions for $\bar\psi\gamma_5\psi$ and $\bar\Psi\gamma_5\Psi$ have the unique tensor structure $\epsilon^{\alpha\beta\mu\nu}\, k_\mu k'_\nu$. Why? Hence $\langle\bar\psi\gamma_5\psi\, j^\alpha\, j^\beta\rangle$ and $\langle\bar\Psi\gamma_5\Psi\, j^\alpha\, j^\beta\rangle$ are separately well-defined. Why? It follows that the limit $M \to \infty$ can be taken in

$$-2M\langle T\bar\Psi\gamma_5\Psi\, j^\alpha\, j^\beta\rangle\,.$$

Compute this limit of the loop integral. Show that the result corresponds to the *anomalous* divergence condition.

$$\partial_\mu j_5^\mu = -2m\bar\psi\gamma_5\psi + \frac{1}{(4\pi)^2}\,{}^dF^{\mu\nu}F_{\mu\nu}\,,$$

where

$${}^dF^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,.$$

Since only the triangle graph is not well defined, this divergence condition holds for the expectation value $\langle j^\mu(x)\rangle^A$ for all orders of the external potential A^μ in the otherwise free field theory. It can, in fact, be shown that the anomalous divergence condition holds even in the presence of interactions.

5. Neutral Pion Decay

a) Let us use $\langle k\epsilon\,, k'\epsilon' + |p\rangle$ to denote the amplitude for a π^0 of momentum p to decay into a pair of photons of momenta and polarization $k^\mu\,, \epsilon^\nu$ and $k'^\mu\,, \epsilon'^\nu$. Explain how $\langle k\epsilon\,, k'\epsilon' + |p\rangle$ is related (to lowest order in e) to

$$M^{\mu\nu}(k,k') = \int (d^4x)(d^4x')e^{-i(kx+k'x')}\langle 0|Tj^\mu(x)j^\nu(x')|p\rangle\,,$$

where j^μ is the electromagnetic current of the fundamental quark fields (including the appropriate charge factors). Note that by Lorentz invariance, current conservation, and the odd parity of the pion, the tensor amplitude has the structure

$$M^{\mu\nu}(k,k') = (2\pi)^4\delta(k+k'-p)\epsilon^{\mu\nu\alpha\beta}k_\alpha k'_\beta B\,,$$

where B is a constant. Derive the formula which relates the decay width Γ of the π^0 to $|B|^2$.

b) Let $A_a^\mu(x)$ be the axial isospin current constructed from the fundamental u and d quark fields. As discussed in Problem 6 of Chapter 6, the pion decay constant F_π is defined by

$$\langle 0|A_a^\mu(x)|bp\rangle = i\delta_{ab}p^\mu F_\pi e^{ipx}\,.$$

An effective pion field $\phi_a(x)$ may be used to create pion states with the normalization $\langle 0|\phi_a(0)|bp\rangle = \delta_{ab}$. Writing

$$A_a^\mu(x) = \bar{A}_a^\mu(x) + F_\pi \partial^\mu \phi_a(x)\,,$$

we see that $\bar{A}_a^\mu(x)$ does not couple the vacuum state to the single pion state. Hence this decomposition may be used to exhibit "pion poles" in amplitudes. Consider, in particular, the amplitude

$$K^{\sigma\mu\nu}(k',q) = \int (d^4x)(d^4x') e^{i(qx-k'x')} \langle 0|T A_3^\sigma(x) j^\mu(0) j^\nu(x')|0\rangle$$

in the (good) approximation in which the pion is taken to be massless. Show that this amplitude has a pole term given by

$$K^{\sigma\mu\nu}(k',q)_{\text{pole}} = -iq^\sigma F_\pi \frac{1}{q^2} \epsilon^{\mu\nu\alpha\beta} k_\alpha k'_\beta B\,,$$

where B is the constant introduced in part a) above.

c) The limit where the pion mass vanishes is the limit where chiral symmetry is good. In this limit the masses of the up and down quarks vanish, $m_u = m_d = 0$. Why? In this limit, the result of the previous Problem 4 tells us that

$$\partial_\mu A_3^\mu(x) = A\, {}^d F^{\mu\nu}(x) F_{\mu\nu}(x)\,.$$

Use the method of this previous Problem to compute the constant A. Use the result of part b) above to compute the $q \to 0$ limit of $q_\sigma K^{\sigma\mu\nu}$ to get an exact relation between A and B. Use the value $F_\pi = 93$ MeV determined from $\pi^\pm$ decay to compute the π^0 decay rate Γ. Compare your result with experiment.

Bibliography

This is a short list of text books (and one article) with which I am acquainted and whose study I think is worth while. The order of the listing corresponds roughly to the order of related topics in the text.

P. A. M. Dirac, *Quantum Mechanics*, 4th Ed., Oxford University Press, London, 1958. This is the classic text on the principles of quantum mechanics, and it should be read by every student of physics. Our notation and we hope our spirit is that of Dirac.

J. J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley, Reading, MA, 1985. This is one of the better books on quantum mechanics at the introductory graduate level.

J. Schwinger, *Quantum Kinematics and Dynamics*, W. A. Benjamin, New York, 1970. Many of the methods used in Chapter 1 were invented by Schwinger. Our treatment of Grassmann numbers in Chapter 2 follows that of this book, to which the reader is referred for more detail. Although this book is somewhat idiosyncratic, it is well worth reading.

R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965. Feynman, taking a hint from Dirac's book, invented the path integral. This book has a complimentary idiosyncrasy to that of Schwinger.

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the scattering amplitude and cross section which is close to the actual experimental situation.

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C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980. This book contains a wealth of information on quantum field theory. It has an extensive discussion of the classic calculations of quantum electrodynamics. It suffers somewhat from an overly historical approach, rather than uniformly using modern methods. The references to the original research which are lacking in our work are, for the most part, cited in this book.

J. Schwinger, *Particles, Sources, and Fields*, Vol. II, Addison-Wesley, Reading, MA., 1973. (Reissued with additional material in Vols. II and III.) This is a difficult book to read. It is written in a rather private language. Nonetheless, it contains many clever and useful techniques applied to the classic problems of quantum electrodynamics.

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